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The Mathematical Life of Nikolai Vasilevski

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Nikolai Vasilevski's creative attitude — not only to mathematics, but to life in general — had already received a great stimulus in Odessa High School 116. This highly selective high school accepted talented children from all over the city, being famous for its selection of quality teachers. A creative, nonstandard, yet highly personal approach to teaching combined with a demanding attitude towards students. The school was also famous for its system of self-government by the students, quite unusual by traditional Soviet standards. Many students that graduated from this school later became well-known scientists and productive researchers.

Upon graduation in 1966, Nikolai became a student at the Department of Mathematics and Mechanics of Odessa State University. In his third year of studies he began his serious mathematical work under the supervision of the well known Soviet mathematician Georgiy Semenovich Litvinchuk. Litvinchuk was a gifted teacher and scientific adviser, with a talent for fascinating his students with problems long interesting and yet up-to-date. The weekly Odessa seminar on boundary value problems, chaired by Prof. Litvinchuk for more than 25 years, deeply influenced Nikolai Vasilevski as well as other students.

Thus Nikolai started to work on the problem of developing Fredholm theory for a class of integral operators with nonintegrable integral kernels. In essence, the integral kernel was the Cauchy kernel multiplied by a logarithmic factor. Integral operators of this type lie in between singular integral operators and those whose kernels have weak (integrable) singularities. A famous Soviet mathematician, F. D. Gakhov, posed this problem in the early 1950s, and it remained open for more than 20 years. Nikolai managed to provide a complete solution in a much more general setting than the original. While working on this problem, Nikolai revealed a key trait of his mathematical talent: his ability to penetrate deeply into the core of the problem, and to see rather unexpected connections between different theories. For instance, in order to solve Gakhov's problem, Nikolai utilized the theory of singular integral operators with coefficients having discontinuities of first kind, and the theory of operators whose integral kernels have fixed singularities — both theories having just appeared at that time. The success of the young mathematician was well recognized by a broad circle of experts working in the area of boundary value problems and operator theory. Nikolai was awarded the prestigious M. Ostrovskii Prize in 1971, given to young Ukrainian scientists with the best research work. Due to his solution of the famous problem, Nikolai quickly entered the mathematical community, and became known to many prominent mathematicians of that time. In particular, he was influenced by regular interaction with outstanding mathematicians such as M. G. Krein and S. G. Mikhlin.

Nikolai graduated from Odessa State University in 1971, obtaining his Master degree. After two years he defended his Ph.D. thesis, and in the same year he became an Assistant Professor at the Department of Mathematics and Mechanics of Odessa State University, where he was later promoted first to the rank of Associate Professor and then to Full Professor.

Having received the degree, Nikolai continued his active mathematical work. He quickly displayed yet another facet of his talent in approaching mathematical problems: his vision and ability to use general algebraic structures in operator theory, which, on the one hand, simplify the problem, and on the other, can be applied to many different problems. We will briefly describe two examples of this.

The first example is the method of orthogonal projections. In 1979, studying the algebra of operators generated by the Bergman projection together with the operators of multiplication by piecewise continuous functions, N. Vasilevski gave a description of the C^*-algebra generated by two self-adjoint elements s and n satisfying the properties $s^2 + n^2 = \mathrm{e}$ and $sn + ns = 0$. A simple substitution $p = (\mathrm{e} + s - n)/2$ and $q = (\mathrm{e} - s - n)/2$ shows that this algebra is also generated by two self-adjoint idempotents (orthogonal projections) p and q (and the identity element e). During the last quarter of the past century, the latter algebra was rediscovered by many authors around the world. Among all algebras generated by orthogonal projections, the algebra generated by two projections is the only tame algebra (excluding the trivial case of the algebra with identity generated by one orthogonal projection). All algebras generated by three or more orthogonal projections are known to be wild, even when the projections satisfy some additional constraints. Many model algebras arising in operator theory are generated by orthogonal projections, and thus any information about their structure essentially broadens the set of operator algebras admitting a reasonable description. In particular, two and more orthogonal projections naturally appear in the study of various algebras generated by the Bergman projection and by piecewise continuous functions having two or more distinct limiting values at a point. Although these projections, say, P, Q_1, $\ldots$, Q_n, satisfy the extra condition $Q_1 + \cdots + Q_n = I$, they still generate, in general, a wild C^*-algebra. However, it was shown that the structure of this algebra is determined by the joint properties of certain positive injective contractions C_k, $k = 1, \ldots, n$, satisfying the identity $\sum_{k=1}^{n} C_k = I$; the structure is therefore determined by the structure of the C^*-algebra generated by the contractions. The principal difference between the case of two projections and the general case of a finite set of projections is now completely clear: for $n = 2$ (with projections P and $Q + (I - Q) = I$) we have only one contraction, and the spectral theorem leads directly to the desired description of the algebra. For $n > 2$ we have to deal with the C^*-algebra generated by a finite set of noncommuting positive injective contractions, which is a wild problem. Fortunately, for many important cases related to concrete operator algebras these projections have yet another special property: the operators $PQ_1P, \ldots, PQ_nP$ commute. This property makes the respective algebra tame, so it has a nice and simple description as the algebra of all $n \times n$ matrix valued functions that are continuous on the joint spectrum Δ of the operators $PQ_1P, \ldots, PQ_nP$, with a certain degeneration on the boundary of Δ.

Another notable example of the algebraic structures used and developed by N. Vasilevski is his version of the Local Principle. The notions of locally equivalent operators and localization theory were introduced and developed by I. Simonenko in the mid-sixties. According to the tradition of that time, the theory was focused on the study of individual operators, and on the reduction of the Fredholm properties of an operator to local invertibility. Later, different versions of the local principle were elaborated by many authors, including G. R. Allan, R. Douglas, I. Ts. Gohberg, N. Ia. Krupnik, A. Kozak, and B. Silbermann. In spite of the fact that many of these versions are formulated in terms of Banach- or C^*-algebras, the main result, as before, reduces invertibility (or the Fredholm property) to local invertibility. On the other hand, at about the same time, several papers on the description of algebras and rings in terms of continuous sections were published by J. Dauns and K. H. Hofmann, M. J. Dupré, J. M. G. Fell, M. Takesaki and J. Tomiyama. These two directions have been developed independently, with no known links between the two series of papers. N. Vasilevski was the one who proposed a local principle which gives the global description of the algebra under study in terms of continuous sections of a certain canonically defined C^*-bundle. This approach is based on general constructions of J. Dauns and K. H. Hofmann, and results of J. Varela. The main contribution consists of a deep re-comprehension of the traditional approach to the local principles unifying the ideas coming from both directions, resulting in a canonical procedure that provides the global description of the algebra under consideration in terms of continuous sections of a C^*-bundle constructed by means of local algebras.

In the eighties and even later, the main direction of the work of Nikolai Vasilevski has been the study of multidimensional singular integral operators with discontinuous coefficients. The main philosophy here is first to study algebras containing these operators, thus providing a solid foundation for the study of various properties (in particular, the Fredholm property) of concrete operators. The main tool has been the version of the local principle described above. This principle was not merely used to reduce the Fredholm property to local invertibility but also for a global description of the algebra as a whole based on the description of the local algebras. Using this methodology, Nikolai Vasilevski obtained deep results in the theory of operators with Bergman's kernel and piece-wise continuous coefficients, the theory of multidimensional Toeplitz operators with pseudodifferential presymbols, the theory of multidimensional Bitsadze operators, the theory of multidimension al operators with shift, etc. N. Vasilevski defended the Doctor of Sciences dissertation in 1988, based on these results, entitled "Multidimensional singular integral operators with discontinuous classical symbols".

Besides being a very active mathematician, N. Vasilevski is an excellent lecturer. His talks are always clear, sparkling, and full of humor, so natural for someone who grew up in Odessa, a city with a longstanding tradition of humor and fun. He was the first at Odessa State University to design and teach a course in general topology. Students enjoyed his lectures in calculus, real analysis, complex analysis, and functional analysis. He was one of the most popular professors at the Department of Mathematics and Mechanics of Odessa State University. Nikolai is a master of presentations, and his colleagues always enjoy his talks at conferences and seminars.

In 1992 Nikolai Vasilevski moved to Mexico. He started his career there as an Investigator (Full Professor) at the Mathematics Department of the Cinvestav (Centro de Investigación y de Estudios Avanzados). His appointment significantly strengthened the Department, which is one of the leading mathematical centers in Mexico. His relocation also visibly revitalized mathematical activity in the country in the field of operator theory. While actively pursuing his own research agenda, Nikolai also served as the organizer of several important conferences. For instance, we mention the (regular since 1998) annual workshop "Análisis: Norte-Sur," and the well-known international conference IWOTA-2009. He facilitated the relocation to Mexico of a number of other experts in operator theory, including Yu. Karlovich and S. Grudsky.

During his career in Mexico, Vasilevski produced a sizable group of students and younger colleagues; six young mathematicians have received a Ph.D. degree under his supervision.

During his life in Mexico, Vasilevski's scientific interests concentrated mainly around the theory of Toeplitz operators on Bergman and Fock spaces. At the end of the 1990s, N. Vasilevski discovered a quite surprising phenomenon in the theory of Toeplitz operators on the Bergman space. Unexpectedly, there exists a rich family of commutative C^*-algebras generated by Toeplitz operators with non-trivial defining symbols. In 1995 B. Korenblum and K. Zhu proved that the Toeplitz operators with radial defining symbols acting on the Bergman space over the unit disk can be diagonalized with respect to the standard monomial basis in the Bergman space. The C^*-algebra generated by such Toeplitz operators is therefore obviously commutative. Four years later Vasilevski also proved the commutativity of the C^*-algebra generated by the Toeplitz operators acting on the Bergman space over the upper half-plane and with defining symbols depending only on $\operatorname{Im} z$. Furthermore, he discovered the existence of a rich family of commutative C^*-algebras of Toeplitz operators. Moreover, it turned out that the smoothness properties of the symbols do not play any role in commutativity: the symbols can be merely measurable. Surprisingly, everything is governed by the geometry of the underlying manifold, the unit disk equipped with the hyperbolic metric. The precise description of this phenomenon is as follows. Each pencil of hyperbolic geodesics determines the set of symbols which are constant on the corresponding cycles, the orthogonal trajectories to geodesics forming the pencil. The C^*-algebra generated by the Toeplitz operators with such defining symbols is commutative. An important feature of such algebras is that they remain commutative for the Toeplitz operators acting on each of the commonly considered weighted Bergman spaces. Moreover, assuming some natural conditions on "richness" of the classes of symbols, the following complete characterization has been obtained: A C^*-algebra generated by the Toeplitz operators is commutative on each weighted Bergman space if and only if the corresponding defining symbols are constant on cycles of some pencil of hyperbolic geodesics. It is also worth mentioning that the proof of this result uses the Berezin quantization procedure in an essential way. Apart from its own beauty, this result reveals an extremely deep influence of the geometry of the underlying manifold on the properties of the Toeplitz operators over the manifold. In each of the mentioned above cases, when the algebra is commutative, a certain unitary operator has been constructed. It reduces the corresponding Toeplitz operators to certain multiplication operators, which also allows one to describe their representations of spectral type. This gives a powerful research tool for the subject, in particular, yielding direct access to the majority of the important properties such as boundedness, compactness, spectral properties, and invariant subspaces of the Toeplitz operators under study. This new approach has enabled the solution of a number of important problems in the theory of Toeplitz operators and related areas.

The results of the research in this direction became a part of the monograph "Commutative Algebras of Toeplitz Operators on the Bergman Space" published by N. Vasilevski in Birkhauser-Verlag in 2008.

The extension of the above result from the unit disk to the unit ball was recently done by Nikolai together with his Mexican colleague Raul Quiroga. Geometry again played an essential role in this study. The commutativity properties of Toeplitz operators here are governed by the so-called Lagrangian pairs, pairs of orthogonal Lagrangian foliations of the unit ball with certain distinguished geometrical properties. The leaves of one of these foliations always turn out to be the orbits of a maximal abelian subgroup of biholomorphisms of the unit ball. The result says that, given any Lagrangian pair, the C^*-algebra generated by Toeplitz operators, whose generating symbols are constant on the leaves being orbits, is commutative on each commonly considered weighted Bergman space on the unit ball.

The program of studying commutative algebras generated by Toeplitz operators as well as the development of various related problems, initiated by N. Vasilevski, is now being carried out by growing groups of mathematicians in different research centers.

During his twenty years at the Cinvestav, Nikolai Vasilevski has consistently applied the best traditions of the Russian mathematical school in his training of young talented Mexican researchers. The constantly growing group of his coauthors, colleagues, and students is an established part of the "Mexican school of Toeplitz operators"—an expression heard more and more at international conferences.

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