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A strategy-based proof of the existence of the value in zero-sum differential games *

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Abstract

The value of a zero-sum differential games is known to exist, under Isaacs' condition, and it is the unique viscosity solution of a Hamilton-Jacobi-Isaacs equation. This approach, in spite of being very effective, does not provide information about the strategies the players should use. In this note we provide a self-contained proof of the existence of the value based on the construction of ε-optimal strategies, which is inspired by the "extremal aiming" method from [5].

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1 Comparison of trajectories

Let U and V be compact subsets of some euclidean space, let $\|\cdot\|$ be the euclidean norm in $\mathbb{R}^n$, and let $f : [0,1] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n$. For each $x \in \mathbb{R}^n$ and $\mathcal{Z} \subset \mathbb{R}^n$, let $D(x, \mathcal{Z}) := \inf_{z \in \mathcal{Z}} \|x - z\|$ be the usual distance from x to the set $\mathcal{Z}$.

Assumption 1.1. *f is uniformly bounded, continuous and there exists $c \geq 0$ such that for all $(u,v) \in U \times V$, $(s,t) \in [0,1]^2$ and $x, y \in \mathbb{R}^n$:*

$$\|f(t,x,u,v) - f(s,y,u,v)\| \leq c\big(|t-s| + \|x-y\|\big).$$

Let $\|f\| := \sup_{(t,x,u,v)} \|f(t,x,u,v)\| < +\infty$.

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The local game. For any $(t,x,\xi)\in[0,1]\times\mathbb{R}^n\times\mathbb{R}^n$ the local game $\Gamma(t,x,\xi)$ is a one-shot game with action sets U and V and payoff function:

$$(u,v)\mapsto\langle\xi,f(t,x,u,v)\rangle.$$

Let $H^-(t,x,\xi)$ and $H^+(t,x,\xi)$ be its maxmin and minmax respectively:

$$\begin{aligned} H^-(t,x,\xi) &:= \max_{u\in U}\min_{v\in V}\langle\xi,f(t,x,u,v)\rangle,\\ H^+(t,x,\xi) &:= \min_{v\in V}\max_{u\in U}\langle\xi,f(t,x,u,v)\rangle. \end{aligned}$$

These functions satisfy $H^-\le H^+$. If the equality

$$H^+(t,x,\xi)=H^-(t,x,\xi)$$

holds, the game $\Gamma(t,x,\xi)$ has a value, and it is denoted by $H(t,x,\xi)$.

Assumption 1.2. *The local game* $\Gamma(t,x,\xi)$ *has a value for all* (t,x,ξ) *in* $[0,1]\times\mathbb{R}^n\times\mathbb{R}^n$.

Assumptions 1.1 and 1.2 hold in the rest of the paper.

1.1 A key Lemma

Introduce the sets of controls:

$$\mathcal{U}=\{\mathbf{u}:[0,1]\to U,\ \text{measurable}\},\quad \mathcal{V}=\{\mathbf{v}:[0,1]\to V,\ \text{measurable}\}.$$

Consider the following dynamical system where $t_0\in[0,1]$, $z_0\in\mathbb{R}^n$ and $(\mathbf{u},\mathbf{v})\in\mathcal{U}\times\mathcal{V}$:

(1) $\qquad \mathbf{z}(t_0)=z_0,\quad \dot{\mathbf{z}}(t)=f(t,\mathbf{z}(t),\mathbf{u}(t),\mathbf{v}(t))\quad \text{a.e. on } [t_0,1].$

Assumption 1.1 ensures the existence of a unique solution to (1), denoted by $\mathbf{z}[t_0,z_0,\mathbf{u},\mathbf{v}]$, in the extended sense: for any $t\in[t_0,1]$,

$$\mathbf{z}[t_0,z_0,\mathbf{u},\mathbf{v}](t):=z_0+\int_{t_0}^t f(s,\mathbf{z}(s),\mathbf{u}(s),\mathbf{v}(s))ds.$$

This result is due to Carathéodory and can be found in [3, Chapter 2]. Elements of U and V are identified with constant controls.

The purpose of this section is to bound the distance between two trajectories: one starting from x_0 and controlled by $(\mathbf{u},v)$, and another one starting from w_0 and controlled by $(u,\mathbf{v})$. The appropriate pair

Local Game: $\Gamma(t_0, x_0, \xi_0)$

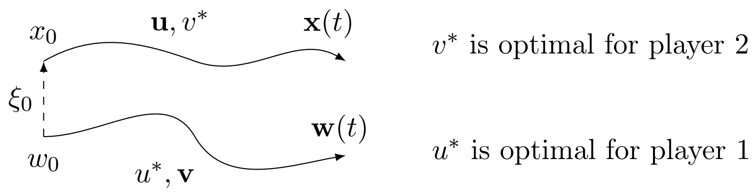

Figure 1: Construction of two trajectories using the local game.

(u, v) is obtained using the existence of the value and of optimal actions in the local game: let u^* (resp. v^*) be optimal for player 1 (resp. 2) in $\Gamma(t_0, x_0, \xi_0)$, where $\xi_0 := x_0 - w_0$. Let $\mathbf{x} := \mathbf{x}[t_0, x_0, \mathbf{u}, v^*]$ and $\mathbf{w} := \mathbf{w}[t_0, w_0, u^*, \mathbf{v}]$ (see Figure 1). The following lemma is inspired by [5, Lemma 2.3.1].

Lemma 1.3. *There exist $A, B \in \mathbb{R}_+$ such that for all $t \in [t_0, 1]$:*

$$\|\mathbf{x}(t) - \mathbf{w}(t)\|^2 \leq (1 + (t - t_0)A)\|x_0 - w_0\|^2 + B(t - t_0)^2.$$

Proof. Let $d_0 := \|x_0 - w_0\|$ and $\mathbf{d}(t) := \|\mathbf{x}(t) - \mathbf{w}(t)\|$. Then:

$$\text{(2)} \quad \mathbf{d}^2(t) = \left\| \xi_0 + \int_{t_0}^{t} [f(s, \mathbf{x}(s), \mathbf{u}(s), v^*) - f(s, \mathbf{w}(s), u^*, \mathbf{v}(s))] ds \right\|^2 .$$

The boundedness of f implies that:

$$\text{(3)} \qquad \left\| \int_{t_0}^{t} [f(s, \mathbf{x}(s), \mathbf{u}(s), v^*) - f(s, \mathbf{w}(s), u^*, \mathbf{v}(s))] ds \right\|^2 \leq$$

$$\leq 4\|f\|^2 (t - t_0)^2.$$

Claim 1.4. *For all $s \in [t_0, 1]$, and for all $(u, v) \in U \times V$:*

$$\text{(4)} \qquad \langle \xi_0, f(s, \mathbf{x}(s), u, v^*) - f(s, \mathbf{w}(s), u^*, v) \rangle \leq 2C(s) d_0 + c d_0^2,$$

where $C(s) := c(1 + \|f\|)(s - t_0)$.

Proof. Assumption 1.1 implies $\|\mathbf{x}(s) - x_0\| \leq (s - t_0)\|f\|$, and then:

$$\|f(s, \mathbf{x}(s), u, v^*) - f(t_0, x_0, u, v^*)\| \leq c\big((s - t_0) + \|f\|(s - t_0)\big) = C(s).$$

From the Cauchy-Schwartz inequality and the optimality of v^* one gets:

$$\begin{aligned}
(5)\qquad \langle \xi_0, f(s,\mathbf{x}(s),u,v^*)\rangle &\leq \langle \xi_0, f(t_0,x_0,u,v^*)\rangle + C(s)d_0,\\
(6)\qquad &\leq H^+(t_0,x_0,\xi_0) + C(s)d_0.
\end{aligned}$$

Similarly, Assumption 1.1 implies $\|\mathbf{w}(s) - x_0\| \leq d_0 + (s-t_0)\|f\|$, and then:

$$\|f(s,\mathbf{w}(s),u^*,v) - f(t_0,x_0,u^*,v)\| \leq C(s) + cd_0.$$

Using the Cauchy-Schwartz inequality and the optimality of u^*:

$$\begin{aligned}
(7)\langle \xi_0, f(s,\mathbf{w}(s),u^*,v)\rangle &\geq \langle \xi_0, f(t_0,x_0,u^*,v)\rangle - (C(s)+cd_0)d_0,\\
(8)\qquad &\geq H^-(t_0,x_0,\xi_0) - C(s)d_0 - cd_0^2.
\end{aligned}$$

The claim follows by substracting the inequalities (6) and (8) and using Assumption 1.2 to cancel $(H^+ - H^-)(t_0,x_0,\xi_0)$. □

In particular, (4) holds for $(u,v) = (\mathbf{u}(s),\mathbf{v}(s))$. Note that

$$\int_{t_0}^t 2C(s)ds \leq (t-t_0)C(t).$$

Thus, integrating (4) over $[t_0,t]$ yields:

$$\int_{t_0}^t \langle \xi_0, f(s,\mathbf{x}(s),\mathbf{u}(s),v^*) - f(s,\mathbf{w}(s),u^*,\mathbf{v}(s))\rangle ds$$
$$\leq (t-t_0)(C(t)d_0 + cd_0^2).$$

Using the estimates (3) and (9) in (2) we obtain:

$$\mathbf{d}^2(t) \leq d_0^2 + 4\|f\|^2(t-t_0)^2 + 2(t-t_0)C(t)d_0 + 2c(t-t_0)d_0^2.$$

Finally, using the relations $d_0 \leq 1 + d_0^2$ and $(t-t_0)C(t) = c(1+\|f\|)(t-t_0)^2$, the result follows with $A := 3c + 2\|f\|$ and $B := 4\|f\|^2 + 2c(1+\|f\|)$. □

1.2 Consequences

We give here three direct consequences of Lemma 1.3. In Section 1.2.1 we use a set of times $\Pi = \{t_0 < t_1 < \cdots < t_N\}$ in $[0,1]$ to construct two trajectories on $[t_0,t_N]$ inductively. Applying Lemma 1.3 to the intervals $[t_m,t_{m+1}]$ for $m = 0,1\ldots,N-1$, we obtain a bound for the distance between the two at time t_N. In particular, if the two trajectories start

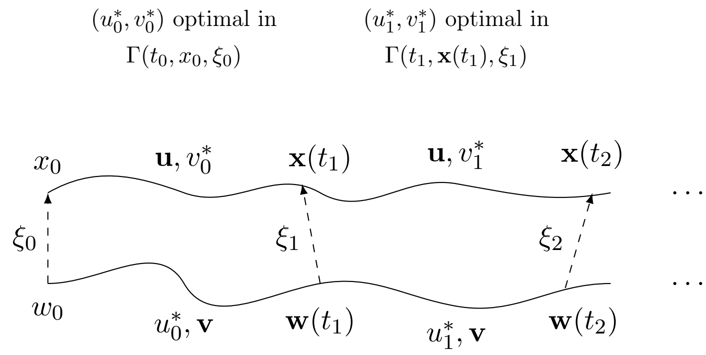

Figure 2: Iterative construction of the two trajectories.

from the same state then their distance at time t_N vanishes as $\|\Pi\| := \max_{1 \le m \le N} t_m - t_{m-1}$ tends to 0. In Section 1.2.2, we replace the distance between two trajectories by the distance between a trajectory and a set. Finally, we combine the two aspects in Section 1.2.3; the result obtained therein is used in Section 2 to prove the existence of the value of zero-sum differential games with terminal payoff.

1.2.1 Induction

Let $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$ be a pair of controls. Define the trajectories $\mathbf{x}$ and $\mathbf{w}$ on $[t_0, t_N]$ inductively: let $\mathbf{x}(t_0) = x_0$ and $\mathbf{w}(t_0) = w_0$ and suppose that $\mathbf{x}(t)$ and $\mathbf{w}(t)$ are defined on $[t_0, t_m]$ for some $m = 0, \dots, N-1$. Consider the local game $\Gamma(t_m, \mathbf{x}(t_m), \xi_m)$, where $\xi_m := \mathbf{x}(t_m) - \mathbf{w}(t_m)$, and let $u_m^* \in U$ and $v_m^* \in V$ be optimal actions for player 1 and 2 respectively. For $t \in [t_m, t_{m+1}]$, set $\mathbf{x}(t) := \mathbf{x}[t_m, \mathbf{x}(t_m), \mathbf{u}, v_m^*](t)$ and $\mathbf{w}(t) := \mathbf{w}[t_m, \mathbf{w}(t_m), u_m^*, \mathbf{v}](t)$ (see Figure 2).

Corollary 1.5. $\|\mathbf{x}(t_N) - \mathbf{w}(t_N)\|^2 \le e^A(\|x_0 - w_0\|^2 + B\|\Pi\|)$.

Proof. For any $0 \le m \le N$, put $d_m := \|\mathbf{x}(t_m) - \mathbf{w}(t_m)\|$. By Lemma 1.3, one has:

$$d_m^2 \le (1 + (t_m - t_{m-1})A)d_{m-1}^2 + B(t_m - t_{m-1})^2.$$

By induction, one obtains:

$$d_N^2 \le \exp\left(A \sum_{m=1}^{N} (t_m - t_{m-1})\right)\left(d_0^2 + B \sum_{m=1}^{N} (t_m - t_{m-1})^2\right).$$

The result follows, since $\sum_{m=1}^{N}(t_m - t_{m-1}) \leq 1$ and $\sum_{m=1}^{N}(t_m - t_{m-1})^2 \leq \|\Pi\|$. □

1.2.2 Distance to a set

Let $\mathcal{W} \subset [t_0, 1] \times \mathbb{R}^n$ be a set satisfying the following properties:

- **P1:** For any $t \in [t_0, 1]$, $\mathcal{W}(t) := \{x \in \mathbb{R}^n \mid (t, x) \in \mathcal{W}\}$ is closed and nonempty.

- **P2:** For any $(t, x) \in \mathcal{W}$ and any $t' \in [t, 1]$:

$$\sup_{u \in U} \inf_{\mathbf{v} \in \mathcal{V}} D(\mathbf{x}[t, x, u, \mathbf{v}](t'), \mathcal{W}(t')) = 0.$$

Equivalent formulations of **P2** were introduced by Aubin [1], although our formulation is inspired by the notion of stable bridge in [5].

Let $x_0 \in \mathbb{R}^n$, and let $w_0 \in \mathrm{argmin}_{\mathcal{W}(t_0)} \|x_0 - w_0\|$ be a point which is the closest to x_0 in $\mathcal{W}(t_0)$ and let v^* be optimal for player 2 in the local game $\Gamma(t_0, x_0, x_0 - w_0)$.

Corollary 1.6. *For every* $t \in [t_0, 1]$ *and* $\mathbf{u} \in \mathcal{U}$*:*

$$D^2(\mathbf{x}[t_0, x_0, \mathbf{u}, v^*](t), \mathcal{W}(t)) \leq (1 + (t - t_0)A) D^2(x_0, \mathcal{W}(t_0)) + B(t - t_0)^2.$$

Proof. Let $\mathbf{u} \in \mathcal{U}$ be fixed and let u^* be optimal in $\Gamma(t_0, x_0, x_0 - w_0)$. By **P2**, for every $\varepsilon > 0$ there exists $\mathbf{v}_{(\varepsilon, u^*)} \in \mathcal{V}$ such that the point $\mathbf{w}_\varepsilon(t) := \mathbf{x}[t_0, w_0, u^*, \mathbf{v}_{(\varepsilon, u^*)}](t)$ satisfies $D(\mathbf{w}_\varepsilon(t), \mathcal{W}(t)) \leq \varepsilon$ (see Figure 3). We use the following abbreviation: $\mathbf{x}_\mathbf{u}(t) := \mathbf{x}[t_0, x_0, \mathbf{u}, v^*](t)$. The triangular inequality gives $D(\mathbf{x}_\mathbf{u}(t), \mathcal{W}(t)) \leq \|\mathbf{x}_\mathbf{u}(t) - \mathbf{w}_\varepsilon(t)\| + \varepsilon$. Taking the limit, as $\varepsilon \to 0$, one has that:

$$D^2(\mathbf{x}_\mathbf{u}(t), \mathcal{W}(t)) \leq \lim_{\varepsilon \to 0} \|\mathbf{x}_\mathbf{u}(t) - \mathbf{w}_\varepsilon(t)\|^2.$$

By Lemma 1.3, $\|\mathbf{x}_\mathbf{u}(t) - \mathbf{w}_\varepsilon(t)\|^2 \leq (1 + (t - t_0)A)\|x_0 - w_0\|^2 + B(t - t_0)^2$ for all $\varepsilon > 0$. The result follows by the choice of w_0. □

1.2.3 A key Corollary

For any $\mathbf{u} \in \mathcal{U}$, define a trajectory $\mathbf{x}_\mathbf{u}$ on $[t_0, t_N]$ inductively: let $\mathbf{x}_\mathbf{u}(t_0) = x_0$ and suppose that $\mathbf{x}_\mathbf{u}$ is defined on $[t_0, t_m]$ for some $m = 0, \ldots, N - 1$.

Let $w_m \in \operatorname{argmin}_{w \in \mathcal{W}(t_m)} \|\mathbf{x}_{\mathbf{u}}(t_m) - w\|$ be a point which is the closest to $\mathbf{x}_{\mathbf{u}}(t_m)$ in $\mathcal{W}(t_m)$, and let v_m^* be optimal for player 2 in the local game

$$\Gamma(t_m, \mathbf{x}_{\mathbf{u}}(t_m), \mathbf{x}_{\mathbf{u}}(t_m) - w_m).$$

Implicitly, we are using two selection rules π_1 and π_2 defined as follows: $\pi_1 : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ assigns to each (t,x) a point which is the closest to x in $\mathcal{W}(t)$; $\pi_2 : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to V$ assigns to each (t,x,ξ) an optimal action for player 2 in the local game $\Gamma(t,x,\xi)$. Thus,

$$v_m^* = \pi_2\big(t_m, \mathbf{x}_{\mathbf{u}}(t_m), \mathbf{x}_{\mathbf{u}}(t_m) - \pi_1(\mathbf{x}_{\mathbf{u}}(t_m))\big).$$

For $t \in [t_m, t_{m+1}]$, put $\mathbf{x}_{\mathbf{u}}(t) := \mathbf{x}[t_m, \mathbf{x}_{\mathbf{u}}(t_m), \mathbf{u}, v_m^*](t)$. Define a control $\beta(\mathbf{u}) \in \mathcal{V}$ inductively by setting $\beta(\mathbf{u}) \equiv v_m^*$ on $[t_m, t_{m+1}]$ for all $0 \le m < N$, so that $\mathbf{x}_{\mathbf{u}}(t) = \mathbf{x}[t_0, x_0, \mathbf{u}, \beta(\mathbf{u})](t)$, for all $t \in [t_0, t_N]$.

Note that the action v_m^* used in the interval $[t_m, t_{m+1}]$ depends only on the current position $\mathbf{x}_{\mathbf{u}}(t_m)$ and on the set $\mathcal{W}(t_m)$. Moreover, the current position depends only on $v_0^*, \dots, v_{m-1}^*$ and on the restriction of $\mathbf{u}$ to the interval $[t_0, t_m]$. In particular, the control $\beta(\mathbf{u})$ is piecewise constant and depends on the set of times Π. Finally, note that for $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$ such that $\mathbf{u}_1 \equiv \mathbf{u}_2$ on $[t_0, t_m]$ for some $0 \le m < N$, the construction described above gives $\beta(\mathbf{u}_1) \equiv \beta(\mathbf{u}_2)$ on $[t_0, t_{m+1}]$. In this sense, $\beta : \mathcal{U} \to \mathcal{V}$ is nonanticipative with delay with respect to the set of times Π.

Putting Corollaries 1.5 and 1.6 together and choosing $x_0 \in \mathcal{W}(t_0)$ yields a useful bound.

Corollary 1.7. *For any* $\mathbf{u} \in \mathcal{U}$, $D^2(\mathbf{x}[t_0, x_0, \mathbf{u}, \beta(\mathbf{u})](t_N), \mathcal{W}(t_N)) \le e^A B \|\Pi\|$.

This result can be interpreted as follows: under **P1-P2** for any control $\mathbf{u} \in \mathcal{U}$ there exists a "reply" $\beta(\mathbf{u}) \in \mathcal{V}$ (which is nonanticipative with delay, and piecewise constant along Π) which keeps a trajectory starting from $\mathcal{W}(t_0)$ at time t_0 arbitrarily close to $\mathcal{W}(t_N)$ at time t_N.

2 Differential Games

Consider now the zero-sum differential game $\mathcal{G}(t_0, x_0)$ played in $[t_0, 1]$ and with the following dynamics in $\mathbb{R}^n$:

$$\mathbf{x}(t_0) = x_0, \quad \dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t)) \quad \text{(a.e. on } [t_0, 1]).$$

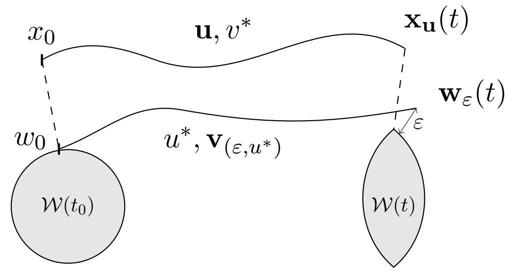

Figure 3: Distance to a set $\mathcal{W} \subset [t_0, 1] \times \mathbb{R}^n$ satisfying **P1** and **P2**.

Definition 2.1. *A strategy for player* 2 *is a map* $\beta : \mathcal{U} \to \mathcal{V}$ *such that, for some finite partition* $s_0 < s_1 < \cdots < s_N$ *of* $[t_0, 1]$*, for all* $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$ *and* $0 \leq m < N$*:*

$$\mathbf{u}_1 \equiv \mathbf{u}_2 \ a.e. \ on \ [s_0, s_m] \implies \beta(\mathbf{u}_1) \equiv \beta(\mathbf{u}_2) \ a.e. \ on \ [s_0, s_{m+1}].$$

These strategies are called nonanticipative strategies with delay (NAD) [2, Section 2.2] in contrast to the classical nonanticipative strategies. The strategies for player 1 are defined in a dual manner. Let $\mathcal{A}$ (resp. $\mathcal{B}$) the set of strategies for player 1 (resp. 2). For any pair of strategies $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$, there exists a unique pair $(\bar{\mathbf{u}}, \bar{\mathbf{v}}) \in \mathcal{U} \times \mathcal{V}$ such that $\alpha(\bar{\mathbf{v}}) = \bar{\mathbf{u}}$ and $\beta(\bar{\mathbf{u}}) = \bar{\mathbf{v}}$ [2, Lemma 1]. This fact is crucial for it allows to define $\mathbf{x}[t_0, x_0, \alpha, \beta] := \mathbf{x}[t_0, x_0, \bar{\mathbf{u}}, \bar{\mathbf{v}}]$ in a unique manner.

The payoff function has two parts: a running payoff $\gamma : \mathbb{R}^n \times U \times V \to \mathbb{R}$ and a terminal payoff $g : \mathbb{R}^n \to \mathbb{R}$. However, the classical transformation of a Bolza problem into a Mayer problem, which gets rid of the running payoff, can also be applied here: enlarge the state space from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, where the last coordinate represents the accumulated payoff; define an auxiliary terminal payoff function $\widetilde{g} : \mathbb{R}^{n+1} \to \mathbb{R}$ as $\widetilde{g}(x, y) = g(x) + y$; we thus obtain an equivalent differential game with no running payoff and dynamic $\widetilde{f} = (f, \gamma)$. Consequently, we can assume without loss of generality that $\gamma \equiv 0$.

Assumption 2.2. *g is Lipschitz continuous.*

Assumption 2.2 holds in the rest of the paper. Introduce the lower and upper value functions:

$$\begin{aligned}
\mathbf{V}^-(t_0, x_0) &:= \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} g\big(\mathbf{x}[t_0, x_0, \alpha, \beta](1)\big), \\
\mathbf{V}^+(t_0, x_0) &:= \inf_{\beta \in \mathcal{B}} \sup_{\alpha \in \mathcal{A}} g\big(\mathbf{x}[t_0, x_0, \alpha, \beta](1)\big).
\end{aligned}$$

The inequality $\mathbf{V}^- \leq \mathbf{V}^+$ holds everywhere. If $\mathbf{V}^-(t_0, x_0) = \mathbf{V}^+(t_0, x_0)$, the game $\mathcal{G}(t_0, x_0)$ has a value, denoted by $\mathbf{V}(t_0, x_0)$. Under Assumption 1.2, usually known as Isaacs' condition, the value exists as the unique viscosity solution of some Hamilton-Jacobi-Isaacs equation with a boundary condition [4]. The functional approach is very effective for it yields the existence and a characterization of the value function. However, it does not tell us much about the strategies the players should use. In this note we focus on the strategies, as in [5], and prove the existence of the value using an explicit construction of ε-optimal strategies. Let us end this section by stating the dynamic programming principle [2, Proposition 2] satisfied by $\mathbf{V}^-$: for all $(t, x) \in [0, 1] \times \mathbb{R}^n$ and all $t' \in [t, 1]$,

$$\mathbf{V}^-(t, x) = \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \mathbf{V}^-\big(t', \mathbf{x}[t, x, \alpha, \beta](t')\big). \tag{9}$$

The dynamic programming principle consists in two inequalities: the $\geq$ (resp. $\leq$) inequality is the superdynamic (resp. subdynamic) programming principle.

2.1 Existence of the value

Let $\phi : [t_0, 1] \times \mathbb{R}^n \to \mathbb{R}$ be a real function satisfying the following properties:

- (i) ϕ is lower semicontinuous.
- (ii) For each $(t, x) \in [t_0, 1] \times \mathbb{R}^n$ and $t' \in [t, 1]$:

$$\phi(t, x) \geq \sup_{u \in U} \inf_{\mathbf{v} \in \mathcal{V}} \phi\big(t', \mathbf{x}[t, x, u, \mathbf{v}](t')\big);$$

- (iii) $\phi(1, x) \geq g(x)$ for all $x \in \mathbb{R}^n$.

Definition 2.3. *For any $\ell \in \mathbb{R}$, define the* ℓ-level set *of ϕ by:*

$$\mathcal{W}_\ell^\phi \;=\; \{(t, x) \in [t_0, 1] \times \mathbb{R}^n \mid \phi(t, x) \leq \ell\},$$

and let

$$\mathcal{W}_\ell^\phi(t) \;=\; \{x \in \mathbb{R}^n \mid \phi(t, x) \leq \ell\}.$$

Lemma 2.4. *For each $\ell \geq \phi(t_0, x_0)$, the ℓ-level set of ϕ satisfies* **P1** *and* **P2***.*

Proof. $x_0 \in \mathcal{W}_\ell^\phi(t_0)$ so that $\mathcal{W}_\ell^\phi(t_0)$ is nonempty. By (i), $\mathcal{W}_\ell^\phi(t)$ is a closed set for all $t \in [0,1]$. The property (ii) implies that for any $t \in [t_0, 1]$, $u \in U$ and $n \in \mathbb{N}^*$ there exists $\mathbf{v}_n \in \mathcal{V}$ such that:

$$\ell \geq \phi(t_0, x_0) \geq \phi\big(t, \mathbf{x}[t_0, x_0, u, \mathbf{v}_n](t)\big) - \frac{1}{n}. \tag{10}$$

The boundedness of f implies that $x_n := \mathbf{x}[t_0, x_0, u, \mathbf{v}_n](t)$ belongs to some compact set. Consider a subsequence $(x_n)_n$ such that

$$\lim_{n\to\infty} \phi(t, x_n) = \liminf_{n\to\infty} \phi(t, x_n),$$

and such that $(x_n)_n$ converges to $\bar{x} \in \mathbb{R}^n$. Take the limit, as $n \to \infty$, in (10). Then by (i) one has:

$$\ell \geq \phi(t_0, x_0) \geq \phi\big(t, \bar{x}).$$

Consequently, $\bar{x} \in \mathcal{W}_\ell^\phi(t) \neq \emptyset$ and $\inf_{n\in\mathbb{N}^*} d\big(\mathbf{x}[t_0, x_0, u, \mathbf{v}_n](t), \mathcal{W}_\ell^\phi(t)\big) = 0$. The proof of these two properties still holds by replacing (t_0, x_0) and $t \in [t_0, 1]$ by any $(t, x) \in \mathcal{W}_\ell^\phi$ and $t' \in [t, 1]$, so that $\mathcal{W}_\ell^\phi$ satisfies **P1** and **P2**. □

2.1.1 Extremal strategies in $\mathcal{G}(t_0, x_0)$

Let $\mathcal{W}^\phi \subset [t_0, 1] \times \mathbb{R}^n$ be the $\phi(t_0, x_0)$-level set of ϕ, i.e.:

$$\mathcal{W}^\phi := \{(t, x) \in [t_0, 1] \times \mathbb{R}^n \mid \phi(t, x) \leq \phi(t_0, x_0)\}.$$

As in Section 1.2.3, let π_1 and π_2 be two selection rules defined as follows: $\pi_1 : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ assigns to each (t, x) a point which is the closest to x in $\mathcal{W}^\phi(t)$; $\pi_2 : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to V$ assigns to each (t, x, ξ) an optimal action for player 2 in the local game $\Gamma(t, x, \xi)$. Finally, let:

$$\pi : [0,1] \times \mathbb{R}^n \to V, \quad (t, x) \mapsto \pi_2(t, x, x - \pi_1(t, x)).$$

Definition 2.5. *An*extremal strategy $\beta = \beta(\phi, \Pi, \pi) : \mathcal{U} \to \mathcal{V}$ *is defined inductively as follows: suppose that* β *is already defined on* $[t_0, t_m]$ *for some* $0 \leq m < N$, *and let* $x_m := \mathbf{x}[t_0, x_0, \mathbf{u}, \beta(\mathbf{u})](t_m)$. *Then set* $\beta(\mathbf{u}) \equiv \pi(t_m, x_m)$ *on* $[t_m, t_{m+1}]$.

These strategies are inspired by the *extremal aiming* method used by Krasovskiĭ and Subbotin in [5, Section 2.4].

Proposition 2.1. *For some* $C \in \mathbb{R}_+$*, and for any extremal strategy* $\beta = \beta(\phi, \Pi, \pi)$*:*

$$g(\mathbf{x}[t_0, x_0, \mathbf{u}, \beta(\mathbf{u})](1)) \leq \phi(t_0, x_0) + C\sqrt{\|\Pi\|}, \quad \forall \mathbf{u} \in \mathcal{U}.$$

Proof. Without loss of generality, $t_N = 1$ so that

$$x_N = \mathbf{x}[t_0, x_0, \mathbf{u}, \beta(\mathbf{u})](1).$$

By Lemma 2.4, $\mathcal{W}^\phi$ satisfies **P1** and **P2**. Thus, by Corollary 1.7:

$$D^2(x_N, \mathcal{W}^\phi(1)) \leq e^A B\|\Pi\|. \tag{11}$$

Using (iii) one obtains that:

$$\mathcal{W}^\phi(1) = \{x \in \mathbb{R}^n \mid \phi(1, x) \leq \phi(t_0, x_0)\} \subset \{x \in \mathbb{R}^n | \; g(x) \leq \phi(t_0, x_0)\}.$$

Let w_N be a point which is the closest to x_N in $\mathcal{W}(1)$ and let κ be the Lipschitz constant of g. Then:

$$\begin{aligned} g(x_N) &\leq g(w_N) + \kappa\|x_N - w_N\|, \\ &\leq \phi(t_0, x_0) + \kappa d(x_N, \mathcal{W}^\phi(1)). \end{aligned}$$

The result follows from (11). □

Theorem 2.6. *The differential game* $\mathcal{G}(t_0, x_0)$ *has a value* $\mathbf{V}$*. Moreover, the extremal strategy* $\beta(\mathbf{V}, \Pi, \pi)$ *is asymptotically optimal for player* 2*, as* $\|\Pi\| \to 0$*.*

Proof. We claim that $\mathbf{V}^-$ satisfies (i), (ii) and (iii) and refer to the Appendix for a proof: $\mathbf{V}^-(1, x) = g(x)$, for all $x \in \mathbb{R}^n$, so that (iii) holds; (ii) can be easily deduced from the superdynamic programming principle (9) (Claim 3.1) or proved directly (Claim 3.3); Assumption 1.1 and 2.2 imply, using Gronwall's lemma, that the map $x \mapsto \mathbf{V}^-(t, x)$ is Lipschitz continuous for all $t \in [t_0, 1]$, so that (i) holds (Claim 3.2). Thus, by Proposition 2.1:

$$\mathbf{V}^+(t_0, x_0) \leq \sup_{\mathbf{u} \in \mathcal{U}} g\big(\mathbf{x}[t_0, x_0, \mathbf{u}, \beta(\mathbf{u})](1)\big) \leq \mathbf{V}^-(t_0, x_0) + C\sqrt{\|\Pi\|}.$$

The existence of the value follows by letting $\|\Pi\|$ tend to 0. Fix now the extremal strategy $\beta = \beta(\mathbf{V}, \Pi, \pi)$ of player 2. Then, to every strategy

$\alpha \in \mathcal{A}$ of player 1 corresponds a unique control $\mathbf{u} \in \mathcal{U}$ so that, by Proposition 2.1:

$$\sup_{\alpha \in \mathcal{A}} g\big(\mathbf{x}[t_0, x_0, \alpha, \beta](1)\big) = \sup_{\mathbf{u} \in \mathcal{U}} g\big(\mathbf{x}[t_0, x_0, \mathbf{u}, \beta(\mathbf{u})](1)\big), \tag{12}$$

$$\leq \mathbf{V}(t_0, x_0) + C\sqrt{\|\Pi\|}. \tag{13}$$

Consequently, for any $\varepsilon > 0$, the strategy $\beta(\mathbf{V}, \Pi, \pi)$ is ε-optimal for sufficiently small $\|\Pi\|$. □

3 Appendix

Claim 3.1. *The superdynamic programming principle* (9) *implies that* $\mathbf{V}^-$ *satisfies* (*ii*).

Proof. Identify every $u \in U$ with a strategy that plays u on $[t_0, 1]$ regardless of $\mathbf{v}$. Then:

$$\sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \mathbf{V}^-\big(t', \mathbf{x}[t_0, x_0, \alpha, \beta](t')\big) \geq \sup_{u \in U} \inf_{\beta \in \mathcal{B}} \mathbf{V}^-\big(t', \mathbf{x}[t_0, x_0, u, \beta(u)](t')\big)$$

$$\geq \sup_{u \in U} \inf_{\mathbf{v} \in \mathcal{V}} \mathbf{V}^-\big(t', \mathbf{x}[t_0, x_0, u, \mathbf{v}](t')\big).$$

The first inequality is clear because $U \subset \mathcal{A}$; the second comes from the fact that $\beta(u) \in \mathcal{V}$ for all $u \in U$. □

Claim 3.2. $\mathbf{V}^-$ *satisfies* (*i*).

Proof. Using Assumption 1.1 and Gronwall's lemma one obtains that, for all $t \in [t_0, 1]$, $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$, and $x, y \in \mathbb{R}^n$:

$$\|\mathbf{x}[t_0, x, \mathbf{u}, \mathbf{v}](t) - \mathbf{x}[t_0, y, \mathbf{u}, \mathbf{v}](t)\| \leq e^{c(t-t_0)}\|x - y\|.$$

Let κ be a Lipschitz constant for g. Then, for all $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$, and for all $x, y \in \mathbb{R}^n$:

$$\big|g\big(\mathbf{x}[t_0, x, \mathbf{u}, \mathbf{v}](1)\big) - g\big(\mathbf{x}[t_0, y, \mathbf{u}, \mathbf{v}](1)\big)\big| \leq \kappa e^c \|x - y\|.$$

Consequently, the map $x \mapsto \mathbf{V}^-(t, x)$ is κe^c-Lipschitz continuous for all $t \in [t_0, 1]$, which is a stronger requirement than (*i*). □

For the sake of completeness, let us end this note by proving that $\mathbf{V}^-$ satisfies (*ii*) directly. The superdynamic programming principle (9) can be proved in the same way.

Claim 3.3. $\mathbf{V}^-$ *satisfies (ii).*

Proof. Let $(t,x) \in [t_0,1] \times \mathbb{R}^n$, let $t' \in [t,1]$ and let $\varepsilon > 0$ be fixed. An ε-optimal strategy for player 1 in $\mathcal{G}(t,x)$ is a strategy $\alpha \in \mathcal{A}$ such that:

$$\sup_{\mathbf{v}\in\mathcal{V}} g\big(\mathbf{x}[t,x,\alpha(\mathbf{v}),\mathbf{v}](1)\big) \geq \mathbf{V}^-(t,x) - \varepsilon.$$

The Lipschitz continuity of $z \mapsto \mathbf{V}^-(t',z)$ implies the existence of some $\delta > 0$ such that any ε-optimal strategy in $\mathcal{G}(t',x')$ remains 2ε-optimal in $\mathcal{G}(t',z)$ for all $z \in B(x',\delta)$ (the euclidean ball of radius δ and center x'). By compactness, $B(x,\|f\|)$ can be covered by some finite family $(E_i)_{i\in I}$ of pairwise disjoint sets such that $E_i \subset B(x_i,\delta)$ for some $x_i \in \mathbb{R}$ $(i \in I)$. Let $\alpha_i \in \mathcal{A}$ $(i \in I)$ be an ε-optimal strategy for player 1 in $\mathcal{G}(t',x_i)$. For any $u \in U$ and $\mathbf{v} \in \mathcal{V}$, put $\mathbf{x}_{u,\mathbf{v}} := \mathbf{x}[x,t,u,\mathbf{v}]$. Note that $\mathbf{x}_{u,\mathbf{v}}(t')$ depends only on the restriction of $\mathbf{v}$ to $[t,t']$. The definition of α_i and E_i $(i \in I)$ ensures that, for all $\mathbf{v}' \in \mathcal{V}$:

$$g\big(\mathbf{x}[t',\mathbf{x}_{u,\mathbf{v}}(t'),\alpha_i,\mathbf{v}'](1)\big)\mathbb{1}_{\{\mathbf{x}_{u,\mathbf{v}}(t')\in E_i\}}$$

$$\geq \mathbf{V}^-\big(t',\mathbf{x}_{u,\mathbf{v}}(t')\big)\mathbb{1}_{\{\mathbf{x}_{u,\mathbf{v}}(t')\in E_i\}} - 2\varepsilon.$$

For each $u \in U$, define a strategy $\alpha_u \in \mathcal{A}$ for player 1 in $\mathcal{G}(t,x)$ as follows. For all $\mathbf{v}' \in \mathcal{V}$:

$$\alpha_u(\mathbf{v}')(s) = \begin{cases} u & \text{if } s \in [t,t'), \\ \alpha_i(\mathbf{v}')(s) & \text{if } s \in [t',1] \text{ and } \quad \mathbf{x}_{u,\mathbf{v}}(t') \in E_i. \end{cases}$$

First, let us check that α_u is a strategy in $\mathcal{G}(t,x)$. Indeed, let $s_1 < \cdots < s_N$ be a common partition of $[t',1]$ for the strategies $(\alpha_i)_i$ – this is possible because the family is finite. Thus, α_u is a strategy with respect to the set of times $t < t' < s_2 < \cdots < s_N$. For any $\mathbf{v}_1,\mathbf{v}_2 \in \mathcal{V}$, let $\mathbf{v}_1 \circ_t \mathbf{v}_2 \in \mathcal{V}$ be the concatenation of the two controls at time t, i.e. $(\mathbf{v}_1 \circ_t \mathbf{v}_2)(s) = \mathbf{v}_1(s)$ if $s \in [0,t]$ and $(\mathbf{v}_1 \circ_t \mathbf{v}_2)(s) = \mathbf{v}_2(s)$ if $s \in [t,1]$. Then, for any $\mathbf{v}'' = \mathbf{v} \circ_{t'} \mathbf{v}' \in \mathcal{V}$:

$$\begin{aligned} g\big(\mathbf{x}[t,x,\alpha_u,\mathbf{v}''](1)\big) &= \sum\nolimits_{i\in I} g\big(\mathbf{x}[t',\mathbf{x}_{u,\mathbf{v}}(t'),\alpha_i,\mathbf{v}'](1)\big)\mathbb{1}_{\{\mathbf{x}_{u,\mathbf{v}}(t')\in E_i\}}, \\ &\geq \sum\nolimits_{i\in I} \mathbf{V}^-\big(t',\mathbf{x}_{u,\mathbf{v}}(t')\big)\mathbb{1}_{\{\mathbf{x}_{u,\mathbf{v}}(t')\in E_i\}} - 2\varepsilon, \\ &= \mathbf{V}^-\big(t',\mathbf{x}_{u,\mathbf{v}}(t')\big) - 2\varepsilon. \end{aligned}$$

Taking the infimum in $\mathcal{V}$ and the supremum in U yields the desired result:

$$\begin{aligned}\mathbf{V}^-(t,x) &\geq \sup_{u\in U}\inf_{\mathbf{v}''\in\mathcal{V}} g\big(\mathbf{x}[t,x,\alpha_u,\mathbf{v}''](1)\big),\\ &\geq \sup_{u\in U}\inf_{\mathbf{v}\in\mathcal{V}} \mathbf{V}^-\big(t',\mathbf{x}_{u,\mathbf{v}}(t')\big)-2\varepsilon.\end{aligned}$$

Conclude by letting ε tend to 0. □

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