

# A Stringy Product on Twisted Orbifold K-theory \*

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## Abstract

In this paper we define an associative stringy product for the twisted orbifold  $K$ -theory of a compact, almost complex orbifold  $\mathcal{X}$ . This product is defined on the twisted  $K$ -theory  ${}^\tau K_{orb}(\wedge\mathcal{X})$  of the inertia orbifold  $\wedge\mathcal{X}$ , where the twisting gerbe  $\tau$  is assumed to be in the image of the inverse transgression  $H^4(B\mathcal{X}, \mathbb{Z}) \rightarrow H^3(B \wedge \mathcal{X}, \mathbb{Z})$ .

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## 1 Introduction

Over the last twenty years, there has been a general trend towards the infusion of physical ideas into mathematics. One of the successful examples in the last few years is the subject of twisted  $K$ -theory. Interest in it originates from two different sources in physics, the consideration of a D-brane charge on a smooth manifold by Witten [30] and the notion of discrete torsion on an orbifold by Vafa [29]. In mathematics, there have been important developments connected to this. On the one hand, it inspired a new subject often referred to as *stringy orbifold theory*. On the

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\*Invited article.

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other hand, it revitalized and re-established connections to many classical topics such as equivariant K-theory, groupoids, stacks and gerbes. For smooth manifolds, the mathematical foundation of twisted K-theory has been worked out and for any cohomology class  $\alpha \in H^3(X, \mathbb{Z})$ , one can associate a twisted K-theory  ${}^\alpha K(X)$  (see [6], [7], [17], [21], [4]). One interesting phenomenon is the difference between a torsion class and a non-torsion one: for torsion  $\alpha$ , we have a natural notion of twisted vector bundle or twisted sheaf; for a non-torsion  $\alpha$ , there is no geometric notion of vector bundle and one has to use infinite-dimensional analysis.

The case of an orbifold or even a more general singular space is much more interesting. This naturally relates to equivariant theories if we specialize to the case of  $\mathcal{X} = [M/G]$  where  $M$  is a smooth manifold and  $G$  is a compact Lie group acting almost freely on  $M$ . One can consider a cohomology class  $\alpha \in H^3(BG, \mathbb{Z})$  and its corresponding twisted K-theory, where  $BG$  is the classical classifying space for  $G$ . This was the set-up of [3] for orbifold twisted K-theory  ${}^\alpha K_{orb}(\mathcal{X})$  using discrete torsion. Twisted K-theory has been generalized to K-theory twisted by gerbes (see [19]), and also using the framework of groupoids (see [18]). In the general case, one can think that the twisting is a cohomology class<sup>3</sup>  $\alpha \in H^3(B\mathcal{X}, \mathbb{Z})$  where  $B\mathcal{X}$  is now the classifying space of the orbifold  $\mathcal{X}$ .

One advantage of working with orbifolds is the nontrivial cohomological counterpart called *Chen-Ruan* cohomology of orbifolds,  $H_{CR}^*(\mathcal{X}, \mathbb{C})$ . One can use discrete torsion [27] (more generally torsion gerbes [25]) to obtain a twisted Chen-Ruan cohomology  $H_{CR}^*(\mathcal{X}, \mathcal{L}_\alpha)$ . Moreover, the twisted Chen-Ruan cohomology has an important internal product making  $H_{CR}^*(\mathcal{X}, \mathcal{L}_\alpha)$  a ring. On the other hand, the tensor product produces a map

$${}^\alpha K_{orb}(\mathcal{X}) \otimes {}^\beta K_{orb}(\mathcal{X}) \rightarrow {}^{\alpha+\beta} K_{orb}(\mathcal{X}).$$

Note that it shifts the twisting to  $\alpha + \beta$ ; one natural question is if there is an internal “stringy” product for  ${}^\alpha K_{orb}(\mathcal{X})$ ? Freed, Hopkins and Teleman [14] have proved the beautiful result that the twisted equivariant K-theory  ${}^\alpha K_G^{\dim G}(G)$  for the adjoint action is isomorphic to the

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<sup>3</sup>To be totally precise, we will actually be twisting with cocycles.

Verlinde algebra of representations of the central extension of the loop algebra  $\mathcal{L}G$  for a semi-simple Lie group  $G$ . This algebra carries a very important ring structure via the Verlinde product, whose structure constants encode the information for so-called conformal blocks. Using the group structure of  $G$ , one can also construct a ring structure (via the Pontryagin product) for  ${}^\alpha K_G^{\dim G}(G)$ ; these rings turn out to be isomorphic.

Due to the importance of the Verlinde product in representation theory, the existence of a stringy product on the twisted K-theory for more general spaces becomes an important question. This is the problem we will address in this article and its sequel.

Our main observation is that there is indeed a stringy product for the twisted K-theory of orbifolds. Moreover, the key information determining such a stringy product does not lie in  $H^3(B\mathcal{X}, \mathbb{Z})$  as one conventionally believes; instead, it lies in  $H^4(B\mathcal{X}, \mathbb{Z})$ . Given a class  $\phi \in H^4(B\mathcal{X}, \mathbb{Z})$ , it induces a class  $\theta(\phi) \in H^3(B \wedge \mathcal{X}, \mathbb{Z})$  where  $\wedge \mathcal{X}$  is the inertia stack of  $\mathcal{X}$  and thus we can define a twisted K-theory  $\theta(\phi)K(\wedge \mathcal{X})$ . The inertia stack  $\wedge \mathcal{X}$  can be viewed as the moduli space of constant loops on  $\mathcal{X}$ . Furthermore, there is a key multiplicative formula for  $\theta(\phi)$  characterized by the effect of  $\phi$  on the moduli space  $\mathcal{M}$  of constant morphisms from a Riemann surface. This map, which can be thought of as the inverse of the classical transgression map, appears in [12] for finite group cohomology. Based on this we derive a simple extension for orbifold groupoids and explicitly prove its multiplicative property (a more geometric version of this formula appears in [20]).

Our second ingredient is more subtle: experience from Chen-Ruan cohomology tells us that a naive definition does not give an associative product. The reason lies in the fact that the fixed-point sets  $X_g, X_h$  for  $g \neq h$  in general do not intersect each other transversely. It is known that in Chen-Ruan cohomology theory one can correct the naive definition by introducing a certain obstruction bundle. Combining these two ingredients, we obtain an associative product which can be viewed as a K-theoretic counterpart of the Chen-Ruan product for orbifold cohomology.

**Theorem 1.1** *Let  $\mathcal{X}$  denote a compact, almost complex orbifold, and let  $\tau$  be a  $U(1)$ -valued 2-cocycle for the inertia orbifold  $\wedge\mathcal{X}$  which is in the image of the inverse transgression. Then there is an associative product on  ${}^\tau K_{orb}(\wedge\mathcal{X})$  which generalizes both the Pontryagin and the orbifold cohomology product.*

Our construction is in fact motivated by the so-called Pontryagin product on  $K_G(G)$ , for  $G$  a finite group, which is what our construction amounts to, for  $\mathcal{X} = \wedge[*]/G$  in the untwisted case. As an application, we use our construction to clarify the *twisted Pontryagin product*; it may not always exist, and when it exists, it may not be unique either. We provide an explicit calculation of the inverse transgression map for the cohomology of finite groups, showing that in fact it can be computed using the natural multiplication map  $\mathbb{Z} \times Z_G(h) \rightarrow G$ , where  $Z_G(h)$  denotes the centralizer of  $h \in G$ . Using this we exhibit a group,  $G = (\mathbb{Z}/2)^3$  and an integral cohomology class  $\phi \in H^4(G, \mathbb{Z})$  such that under the inverse transgression it maps non-trivially for every properly twisted sector, yielding an interesting product structure on  ${}^{\theta(\phi)}K_G(G)$ .

One of the original motivations for the introduction of the twisted theory in orbifolds was the hope of describing the cohomology of desingularizations of an orbifold. Joyce constructed five classes of topologically different desingularization of  $T^6/\mathbb{Z}_4$  [16], arising from a representation  $\mathbb{Z}/4 \subset SU(3)$ . It is known that Joyce's desingularizations are not captured by discrete torsion. For a while, there was the expectation that they may be captured by 1-gerbes. The computation in [2] shows that the high hopes for 1-gerbes is probably misplaced; however, we notice that  $H^4(B(T^6/\mathbb{Z}_4), \mathbb{Z})$  seems to contain precisely the information related to desingularization. We hope to return to this question later.

We would like to make a comment about notation: throughout this paper we will be using the language of orbifold groupoids, hence given an orbifold  $\mathcal{X}$  we will be thinking of it in terms of a Morita equivalence class of orbifold groupoids, represented by  $\mathcal{G}$ ; in this context  ${}^\tau K_{orb}(\mathcal{X})$  is interpreted as  ${}^\tau K(\mathcal{G})$ , using the notion of twisted  $K$ -theory of groupoids, which we will summarize in Section 3.

The results in this article were first announced by the second author

at the Florida Winter School on Mathematics and Physics in December, 2004. Here, we present our construction for the orbifold case. The construction for general stacks will appear elsewhere. During the course of this work, we received an article by Jarvis-Kaufmann-Kimura [15] which also deals with a stringy product in  $K$ -theory; indeed the restriction of our twisted  $K$ -theory  ${}^\tau K(\wedge \mathcal{X})$  to the non-twisted sector gives their small orbifold  $K$ -theory  $K(| \wedge \mathcal{X}|)$ . The authors would like to thank MSRI and PIMS for their hospitality during the preparation of this manuscript, and the third author would like to thank the MPI-Bonn for its generous support.

## 2 Preliminaries on Orbifolds and Groupoids

In this section, we summarize some basic facts about orbifolds, using the point of view of groupoids. Our main reference is the book [1], but [24] is also a useful introduction. Recall that an orbifold structure can be viewed as an orbifold Morita equivalence class of orbifold groupoids; we shall present all of our constructions in this framework.

Suppose that  $\mathcal{G} = \{s, t : G_1 \rightarrow G_0\}$  is an orbifold groupoid, namely, a proper, étale Lie groupoid, we will use  $|\mathcal{G}|$  to denote its orbit space, i.e., the quotient space of  $G_0$  under the equivalence relation:  $x \sim y$  iff there is an arrow  $g : x \mapsto y$ . Conversely, we call  $\mathcal{G}$  an orbifold presentation of  $|\mathcal{G}|$ .

Recall that a groupoid homomorphism  $\phi : \mathcal{H} \rightarrow \mathcal{G}$  between (Lie) groupoids  $\mathcal{H}$  and  $\mathcal{G}$  consists of two (smooth) maps,  $\phi_0 : H_0 \rightarrow G_0$  and  $\phi_1 : H_1 \rightarrow G_1$ , that together commute with all the structure maps for the two groupoids  $\mathcal{H}$  and  $\mathcal{G}$ . Obviously, a groupoid homomorphism  $\phi : \mathcal{H} \rightarrow \mathcal{G}$  induces a continuous map  $|\phi| : |\mathcal{H}| \rightarrow |\mathcal{G}|$ .

**Definition 2.1** Let  $\phi, \psi : \mathcal{H} \rightarrow \mathcal{G}$  be two homomorphisms. A natural transformation  $\alpha$  from  $\phi$  to  $\psi$  is a smooth map  $\alpha : H_0 \rightarrow G_1$ , giving for each  $x \in H_0$  an arrow  $\alpha(x) : \phi(x) \rightarrow \psi(x)$  in  $G_1$ , natural in  $x$  in the sense that for any  $h : x \rightarrow x'$  in  $H_1$ , the identity  $\psi(h)\alpha(x) = \alpha(x')\phi(h)$  holds.

**Definition 2.2** Let  $\phi : \mathcal{H} \rightarrow \mathcal{G}$  and  $\psi : \mathcal{K} \rightarrow \mathcal{G}$  be homomorphisms of Lie groupoids. The groupoid fibered product  $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$  is the Lie groupoid whose objects are triples  $(y; g; z)$  where  $y \in H_0$ ,  $z \in K_0$  and  $g : \phi(y) \rightarrow \psi(z)$  in  $G_1$ . Arrows  $(y; g; z) \rightarrow (y'; g'; z')$  in  $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$  are pairs  $(h; k)$  of arrows,  $h : y \rightarrow y'$  in  $H_1$  and  $k : z \rightarrow z'$  in  $K_1$  with the property that  $g'\phi(h) = \psi(k)g$ . Composition in  $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$  is defined in the natural way.

Next we recall the notion of equivalence of groupoids.

**Definition 2.3** A homomorphism  $\phi : \mathcal{H} \rightarrow \mathcal{G}$  between Lie groupoids is called an equivalence if the map  $t\pi_1 : G_1 \times_{s \times \phi} H_0 \rightarrow G_0$  is a surjective submersion and the square

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi} & G_1 \\ (s, t) \downarrow & & \downarrow (s, t) \\ H_0 \times H_0 & \xrightarrow{\phi \times \phi} & G_0 \times G_0 \end{array}$$

is a fibered product of manifolds.

**Definition 2.4** Two orbifold groupoids  $\mathcal{G}$  and  $\mathcal{G}'$  are said to be orbifold Morita equivalent if there is a third orbifold groupoid  $\mathcal{H}$  and two equivalences  $\phi : \mathcal{H} \rightarrow \mathcal{G}$  and  $\phi' : \mathcal{H} \rightarrow \mathcal{G}'$ . An orbifold homomorphism from  $\mathcal{H}$  to  $\mathcal{G}$  is a triple  $(\mathcal{K}, \epsilon, \phi)$ , where  $\mathcal{K}$  is another orbifold groupoid,  $\epsilon : \mathcal{K} \rightarrow \mathcal{H}$  is an equivalence and  $\phi : \mathcal{K} \rightarrow \mathcal{G}$  is a groupoid homomorphism. The equivalence relation for orbifold homomorphisms is generated by natural transformations of  $\phi$  and equivalences of  $\mathcal{K}$ .

**Definition 2.5** The category of orbifolds is the category whose objects are the orbifold Morita equivalence classes of orbifold groupoids and the morphisms are equivalence classes of orbifold homomorphisms.

**Remark** In this paper, we will use the term homomorphism for a *groupoid homomorphism*, an orbifold homomorphism will be clearly identified when it arises.

There are several important constructions which play a fundamental role in stringy orbifold theory. Given  $r > 0$  an integer, we can consider

the  $r$ -tuples of composable arrows in  $\mathcal{G}$ , i.e.

$$G_r = \{(g_1, \dots, g_r) \in G_1^r \mid t(g_i) = s(g_{i+1}), i = 1, \dots, r\}.$$

These fit together to form a simplicial space, whose geometric realization is the classifying space  $B\mathcal{G}$  of the groupoid  $\mathcal{G}$ . In our discussion of homological invariants of groupoids, we will be considering cochains arising from this complex. Recall that the inertia groupoid  $\wedge\mathcal{G}$  is a groupoid canonically associated with  $\mathcal{G}$  which is defined as follows:

**Definition 2.6** For any groupoid  $\mathcal{G}$ , we can associate an inertia groupoid  $\wedge\mathcal{G}$  as

$$(\wedge\mathcal{G})_0 = \{g \in G_1 \mid s(g) = t(g)\}, (\wedge\mathcal{G})_1 = \{(a, v) \in G_2 \mid a \in (\wedge\mathcal{G})_0\}$$

where

$$s(a, v) = a, t(a, v) = v^{-1}av.$$

More generally, we can define the groupoid of  $k$ -sectors  $\mathcal{G}^k$  as

$$\begin{aligned} (\mathcal{G}^k)_0 &= \{(a_1, a_2, \dots, a_k) \in G_1^k \mid s(a_1) = t(a_1) = \dots = s(a_k) = t(a_k)\} \\ (\mathcal{G}^k)_1 &= \{(a_1, a_2, \dots, a_k, u) \in G_1^{k+1} \mid s(a_1) = t(a_1) = \dots \\ &= s(a_k) = t(a_k) = s(u)\} \end{aligned}$$

with

$$s(a_1, \dots, a_k, u) = (a_1, \dots, a_k), t(a_1, \dots, a_k, u) = (u^{-1}a_1u, \dots, u^{-1}a_ku).$$

The construction of the inertia groupoid and  $\mathcal{G}^k$  in general is completely functorial. Namely, a homomorphism of groupoids induces a homomorphism between  $k$ -sectors and an equivalence of orbifold groupoids induces an equivalence between them. In case of orbifold groupoids, the inertia groupoid can be identified as the space of constant loops on  $\mathcal{G}$ ; more generally,  $\mathcal{G}^{k-1}$  can be identified as the space of constant morphisms from an orbifold sphere with  $k$ -orbifold points to  $\mathcal{G}$ . We will come back to these descriptions later.

Another important notion is that of quasi-suborbifold; before defining it we first point out that for a groupoid  $\mathcal{G}$ , and an open subset  $V \subset G_0$ ,  $\{s, t : s^{-1}(V) \cap t^{-1}(V) \rightarrow V\}$  is a groupoid, which we will denote by  $\mathcal{G}|_V$ .

**Definition 2.7** A homomorphism of orbifold groupoids  $\phi : \mathcal{G} \rightarrow \mathcal{H}$  is a quasi-embedding if

- $\phi : G_0 \rightarrow H_0$  is an immersion.
- For any  $y \in \text{im}(\phi) \subset H_0$ , with isotropy group  $H_y$ ,  $\phi^{-1}(y)$  is in an orbit of  $\mathcal{G}$ , and for any  $x \in \phi^{-1}(y)$ ,  $\phi : G_x \rightarrow H_y$  is injective.
- For any  $y \in \text{im}(\phi)$ , and any  $x \in \phi^{-1}(y)$ , there are neighborhoods  $U_y$  of  $y$  and  $V_x$  of  $x$  such that  $\mathcal{H}|_{U_y} = U_y \rtimes H_y$ ,  $\mathcal{G}|_{V_x} = V_x \rtimes G_x$  and  $\mathcal{G}|_{\phi^{-1}(U_y)} \cong (H_y \times_{\phi(G_x)} V_x) \rtimes H_y$ .
- $|\phi| : |\mathcal{G}| \rightarrow |\mathcal{H}|$  is proper.

**Definition 2.8**  $\mathcal{G}$  together with  $\phi$  is called a *quasi-suborbifold* of  $\mathcal{H}$ .

The following are important examples of quasi-suborbifolds.

**Example 2.9** Suppose that  $\mathcal{G} = X \rtimes G$  is a global quotient groupoid (i.e. a quotient by a finite group). We often use the stacky notation  $[X/G]$  to denote the groupoid. An important object is the inertia groupoid  $\wedge \mathcal{G} = (\sqcup_g X_g) \rtimes G$  where  $X_g$  is the fixed point set of  $g$  and  $G$  acts on  $\sqcup_g X_g$  as  $h : X_g \rightarrow X_{hgh^{-1}}$  by  $h(x) = hx$ . By our definition,  $\phi : \wedge \mathcal{G} \rightarrow \mathcal{G}$  induced by the inclusion map  $X_g \rightarrow X$  is a quasi-embedding.

**Example 2.10** Let  $\mathcal{G}$  be the global quotient groupoid defined as in the previous example. We would like to define an appropriate notion of the diagonal  $\Delta$  for  $\mathcal{G} \times \mathcal{G}$ . We define it as  $\Delta = (\sqcup_g \Delta_g) \rtimes (G \times G)$  where  $\Delta_g = \{(x, gx), x \in X\}$ . Our definition of quasi-suborbifold includes this example.

More generally, we define the diagonal  $\Delta(\mathcal{G})$  as the groupoid fibered product  $\mathcal{G} \times_{\mathcal{G}} \mathcal{G}$ . One can check that  $\Delta(\mathcal{G}) = \mathcal{G} \times_{\mathcal{G}} \mathcal{G}$  is locally of the desired form and hence a quasi-suborbifold of  $\mathcal{G} \times \mathcal{G}$ . Notice that the map  $x \rightarrow (x, 1_x, x)$  allows us to identify  $\mathcal{G}$  as a component of  $\Delta(\mathcal{G})$ .



**Example 2.11** For  $l \leq k$ , there are natural evaluation morphisms  $e_{i_1, \dots, i_l} : \mathcal{G}^k \rightarrow \mathcal{G}^l$  given by

$$e_{i_1, \dots, i_l}(a_1, \dots, a_k) = (a_{i_1}, \dots, a_{i_l}).$$

Furthermore, we have  $e : \mathcal{G}^k \rightarrow \mathcal{G}$  given by

$$e(a_1, \dots, a_k) = s(a_1) = t(a_1) = \dots = s(a_k) = t(a_k).$$

The latter one corresponds to taking the image of constant morphism. We leave as an exercise for the reader to check that  $e$  and the  $e_{i_1, \dots, i_l}$  are quasi-embeddings and that  $\mathcal{G}^k$  is a quasi-suborbifold of  $\mathcal{G}^l$ .

One of the main tools is the notion of a normal bundle. If  $i : \mathcal{G} \rightarrow \mathcal{H}$  is a quasi-embedding,  $i^*T\mathcal{H}$  is a groupoid vector bundle over  $\mathcal{G}$  such that  $T\mathcal{G}$  is a subbundle. Then we can define the normal bundle  $N_{\mathcal{G}|\mathcal{H}} = i^*T\mathcal{H}/T\mathcal{G}$ .  $N_{\mathcal{G}|\mathcal{H}}$  behaves as the normal bundle does for smooth manifolds. Next we introduce the notion of intersection for quasi-suborbifolds.

**Definition 2.12** Let  $f : \mathcal{G}_1 \rightarrow \mathcal{H}$  and  $g : \mathcal{G}_2 \rightarrow \mathcal{H}$  denote quasi-suborbifolds. We define their intersection  $\mathcal{G}_1 \cap \mathcal{G}_2$  as the restriction of the pullback  $\mathcal{G}_1 \times_{\mathcal{H}} \mathcal{G}_2$  to the component  $\mathcal{H}$  in  $\mathcal{H} \times_{\mathcal{H}} \mathcal{H}$ .

Note that under this definition it makes sense to intersect a quasi-suborbifold with itself. Under certain conditions these intersections can have nice properties, analogous to the situation for manifolds. We will be interested in the notion of a *clean intersection*.

**Definition 2.13** Suppose that  $f : \mathcal{G}_1 \rightarrow \mathcal{H}, g : \mathcal{G}_2 \rightarrow \mathcal{H}$  are smooth quasi-suborbifolds, we say that  $\mathcal{G}_1$  intersects  $\mathcal{G}_2$  *cleanly* if the intersection orbifold  $\mathcal{G}_1 \cap \mathcal{G}_2$  is a smooth quasi-suborbifold of  $\mathcal{H}$  (where as before  $\mathcal{H}$  is viewed as a component of  $\Delta(\mathcal{H})$ ) such that for every  $x \in (\mathcal{G}_1)_0 \cap (\mathcal{G}_2)_0$ ,  $T_{(x, 1_x, x)}(\mathcal{G}_1 \cap \mathcal{G}_2) = T_x \mathcal{G}_1 \cap T_x \mathcal{G}_2$ .

**Example 2.14** As we mentioned before, the evaluation map  $e : \wedge \mathcal{G} \rightarrow \mathcal{G}$  is a quasi-suborbifold. Then,  $e$  with itself forms a clean intersection.

Indeed the question is local, and locally it corresponds to the intersection of fixed point sets  $V^g \cap V^h$ . This is clearly a clean intersection. More generally,  $e_{i_1, \dots, i_l} : \mathcal{G}^k \rightarrow \mathcal{G}^l$  is a quasi-embedding. Then, two different quasi-embeddings  $e_{i_1, \dots, i_l}, e_{j_1, \dots, j_l} : \mathcal{G}^k \rightarrow \mathcal{G}^l$  form a clean intersection. We leave it as an exercise for our readers.

As in manifold theory, there is also the notion of transversality for quasi-suborbifolds.

**Definition 2.15** Suppose that  $f : \mathcal{G}_1 \rightarrow \mathcal{H}$ ,  $g : \mathcal{G}_2 \rightarrow \mathcal{H}$  are smooth homomorphisms. We say that  $f \times g$  is transverse to the diagonal  $\Delta$  if locally  $f \times g$  is transverse to every component of the diagonal  $\Delta$  on the object level. We say that  $f, g$  are transverse to each other if  $f \times g$  is transverse to the diagonal  $\Delta$ .

**Example 2.16** Suppose that  $f : \mathcal{G}_1 \rightarrow \mathcal{H}$ ,  $g : \mathcal{G}_2 \rightarrow \mathcal{H}$  are quasi-embeddings which are transverse to each other. Then the intersection  $\mathcal{G}_1 \cap \mathcal{G}_2$  is a quasi-suborbifold of  $\mathcal{H}$ .

Note that a clean intersection need not be transverse, this failure of transversality plays a role in the definition of orbifold cohomology and K-theory.

### 3 Gerbes and Twisted K-Theory

We now consider the cohomology and  $K$ -theory of orbifold groupoids.

**Definition 3.1** Let  $\mathcal{G}$  denote a Lie groupoid, then we define the continuous  $U(1)$ -valued  $k$ -cochains on  $\mathcal{G}$  as

$$C^k(\mathcal{G}, U(1)) = \{\phi : G_k \rightarrow U(1) \mid \phi \text{ is continuous}\}.$$

The differential on this abelian group (using additive notation) is defined via

$$\begin{aligned} \delta\phi(g_1, \dots, g_{k+1}) &= \phi(g_2, \dots, g_{k+1}) \\ &+ \sum_{i=1}^k (-1)^i \phi(g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}) + (-1)^{k+1} \phi(g_1, \dots, g_k). \end{aligned}$$

By a result due to Moerdijk [23], if  $\mathcal{G}$  is an étale groupoid then the cohomology of this chain complex is the Čech cohomology of  $B\mathcal{G}$  with coefficients in the sheaf  $C(U(1))$  of  $U(1)$ -valued continuous functions over the classifying space  $B\mathcal{G}$ . By the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow C(\mathbb{R}) \rightarrow C(U(1)) \rightarrow 0,$$

we obtain a long exact sequence in cohomology,

$$H^k(B\mathcal{G}, C(\mathbb{R})) \rightarrow H^k(B\mathcal{G}, C(U(1))) \rightarrow H^{k+1}(B\mathcal{G}, \mathbb{Z}) \rightarrow H^{k+1}(B\mathcal{G}, C(\mathbb{R})).$$

Since  $C(\mathbb{R})$  is a fine sheaf, the connecting homomorphism is an isomorphism, and so for  $k > 0$ ,

$$H^k(B\mathcal{G}, C(U(1))) \cong H^{k+1}(B\mathcal{G}, \mathbb{Z}).$$

We recall

**Definition 3.2** An  $n$ -gerbe on  $\mathcal{G}$  is a pair  $(\mathcal{H}, \theta)$  consisting of

- a refinement  $\mathcal{G} \xleftarrow{\epsilon} \mathcal{H}$  (i.e.,  $\epsilon$  is an equivalence)
- an  $(n+1)$ -cocycle  $\phi : H_{n+1} \rightarrow U(1)$ .

Next we define equivalence of gerbes.

**Definition 3.3** Given two  $n$ -gerbes  $(\mathcal{H}, \theta)$  and  $(\mathcal{H}', \theta')$  on  $\mathcal{G}$  we have the following:

- $(\mathcal{H}, \theta)$  is equivalent to  $(\mathcal{H}', \theta')$  if there is a common refinement  $\mathcal{H}' \leftarrow \mathcal{H}'' \rightarrow \mathcal{H}$  such that the induced  $(n+1)$ -cocycles on  $\mathcal{H}''$  are the same.
- $(\mathcal{H}, \theta)$  is isomorphic to  $(\mathcal{H}', \theta')$  if there is a common refinement  $\mathcal{H}''$  such that the induced  $(n+1)$ -cocycles on  $\mathcal{H}''$  differ by a coboundary.

Let  $\mathcal{K} \xrightarrow{\epsilon} \mathcal{G}$  be an equivalence. If  $(\mathcal{H}, \theta)$  is a  $n$ -gerbe over  $\mathcal{K}$ , by definition, it is a  $n$ -gerbe over  $\mathcal{G}$ . For an  $n$ -gerbe  $(\mathcal{H}, \theta)$  over  $\mathcal{G}$ ,  $\mathcal{H}' = \mathcal{H} \times_{\mathcal{G}} \mathcal{K}$  is an orbifold which is equivalent to both  $\mathcal{H}$  and  $\mathcal{K}$  by projections  $p_1 : \mathcal{H}' \rightarrow \mathcal{H}$  and  $p_2 : \mathcal{H}' \rightarrow \mathcal{K}$ . So we have the pull-back  $n$ -gerbe  $(\mathcal{H}', p_1^* \theta)$  over  $\mathcal{K}$ . It is easy to see that under these operations, equivalent (isomorphic) gerbes go to equivalent (isomorphic) ones, therefore gerbes behave well under orbifold Morita equivalence.

**Definition 3.4** An  $n$ -gerbe on an orbifold is an equivalence class of pairs  $(\mathcal{G}, \theta)$ , where  $\mathcal{G}$  is a presentation of the orbifold,  $\theta$  is a  $n+1$ -cocycle on  $\mathcal{G}$ , and the equivalence relation is gerbe isomorphism.

From the definition, it is clear that an  $n$ -gerbe defines a Čech  $(n+1)$ -cocycle for the sheaf  $C(U(1))$  of continuous  $U(1)$ -valued functions on the classifying space  $B\mathcal{G}$  ( $B\mathcal{H}$  and  $B\mathcal{G}$  are weakly homotopy equivalent). Hence it will define a cohomology class in  $H^{n+1}(B\mathcal{G}, C(U(1))) \cong H^{n+2}(B\mathcal{G}, \mathbb{Z})$ . The image of  $\theta$  under the connecting homomorphism in  $H^{n+2}(B\mathcal{G}, \mathbb{Z})$  is called its characteristic class or Dixmier-Douady class.

We can associate twisted  $K$ -theory to a 1-gerbe. For simplicity we assume that the 2-cocycle  $\theta$  is defined on  $\mathcal{G}$ , i.e., we are dealing with  $(\mathcal{G}, \theta)$ , and the twisted  $K$ -theory  ${}^\theta K(\mathcal{G})$  will be defined. We follow the treatment of [18] to describe this. Let  $\mathbb{H}$  be a separable Hilbert space; it is well-known that the characteristic class of a principal  $PU(\mathbb{H})$ -bundle over  $\mathcal{G}$  also lies in  $H^3(B\mathcal{G}, \mathbb{Z})$ . Hence, given a 1-gerbe, we should be able to associate a  $PU(\mathbb{H})$  bundle with the same characteristic class; in fact we can associate a canonical principal  $PU(\mathbb{H})$ -bundle. We outline its construction.

For the orbifold groupoid  $\mathcal{G} = \{s, t : G_1 \rightarrow G_0\}$ , let  $R = G_1 \times U(1)$  be the topologically trivial central extension, and

$$(g_1, r_1)(g_2, r_2) = (g_1 g_2, \theta(g_1, g_2) r_1 r_2),$$

which makes  $\{\tilde{s}, \tilde{t} : R \rightarrow G_0\}$  a Lie groupoid, where  $\tilde{s}(g, r) = s(g)$ ,  $\tilde{t}(g, r) = t(g)$ .

Now let  $G^x = t^{-1}(x)$ ; there is a system of measure (Haar system)  $\lambda = (\lambda^x)_{x \in G_0}$ , where  $\lambda^x$  is a measure with support  $G^x$  such that for all  $f \in C_c(G_1)$ ,  $x \rightarrow \int_{g \in R^x} f(g) \lambda^x(dg)$  is continuous. By  $\mathcal{L}_x^2$ , we denote the space  $L^2(G^x)$  consisting of functions defined on  $G^x$  which are  $L^2$  with respect to the Haar measure. Let  $E_x = \mathcal{L}_x^2 \otimes \mathbb{H}$ ,  $E = \sqcup_x E_x$ . Then  $E$  is a countably generated continuous field of infinite dimensional Hilbert spaces over the finite dimensional space  $G_0$ , and therefore is a locally trivial Hilbert bundle according to the Dixmier-Douady theorem [13].

The Lie groupoid  $R$  acts naturally on  $E$ :  $U(1)$  acts on  $\mathbb{H}$  by complex multiplication. Therefore,  $E$  is naturally a Hilbert bundle over  $\{\tilde{s}, \tilde{t} : R \rightarrow G_0\}$ . Notice  $E$  is not a Hilbert bundle over  $\mathcal{G}$ . However,  $P(E)$  is a projective bundle over  $\mathcal{G}$  with precisely the same characteristic class of  $\theta$ . Let  $\mathcal{B}$  be the principal bundle of orthonormal frames of  $E$ ; it is a  $U(\mathbb{H})$ -principal bundle. By our previous argument,  $P\mathcal{B}$  is a principal  $PU(\mathbb{H})$ -bundle over  $\mathcal{G}$ .

Let  $\theta$  be a 1-gerbe and  $P_\theta$  be the associated  $PU(\mathbb{H})$ -bundle constructed above. Let  $Fred^0(\mathbb{H})$  be the space of Fredholm operators endowed with the  $*$ -strong topology and  $Fred^1(\mathbb{H})$  be the space of self-adjoint elements in  $Fred^0(\mathbb{H})$ . Let  $\mathcal{K}(\mathbb{H})$  be the space of compact operators endowed with the

norm-topology. Now consider the associated bundles

$$Fred_\theta^i(\mathbb{H}) := P_\theta \times_{PU(\mathbb{H})} Fred^i(\mathbb{H}) \rightarrow G_0,$$

$$\mathcal{K}_\theta(\mathbb{H}) := P_\theta \times_{PU(\mathbb{H})} \mathcal{K}(\mathbb{H}) \rightarrow G_0.$$

By  $\mathbf{F}_\theta^i$ , we denote the space of norm-bounded,  $G_1$ -invariant, continuous sections  $x \rightarrow T_x$  of the bundle  $Fred_\theta^i(\mathbb{H}) \rightarrow G_0$  such that there exists a norm-bounded,  $G_1$ -invariant, continuous section  $x \rightarrow S_x$  of  $\mathcal{K}_\theta(\mathbb{H}) \rightarrow G_0$  with the property that  $1 - T_x S_x$  and  $1 - S_x T_x$  are continuous sections of  $\mathcal{K}_\theta(\mathbb{H})$  vanishing at infinity of  $|\mathcal{G}|$ .

**Definition 3.5** For any section  $T$  of  $\mathbf{F}_\theta^i$ . We define the support  $supp(T)$  as the set of point  $x \in |\mathcal{G}|$  such that  $T_{x'}$  is not invertible for any  $x'$  in the preimage of  $x$ .

Then we have

**Definition 3.6** Let  $\mathcal{G}$  be an orbifold groupoid and  $(\mathcal{G}, \theta)$  be a 1-gerbe. We define its  $\theta$ -twisted K-theory as

$${}^\theta K^i(\mathcal{G}) = \{[T] \mid T \in \mathbf{F}_\theta^i\},$$

where  $[T]$  denotes the homotopy class of  $T$  where  $T$  is compactly supported.

Note that since the space of invertible operators is contractible, any  $T$  with compact support is homotopic to a section which is the identity outside a compact subset. Suppose that  $i : U \rightarrow G_0$  is an open subset, using the property above, we have a natural extension

$$i_* : {}^{i^*\theta} K^i(\mathcal{G}|_U) \rightarrow {}^\theta K^i(\mathcal{G}).$$

**Remark** Suppose that we have a cocycle  $\alpha = \beta + \delta\rho$ . Then there is a canonical isomorphism between central extensions of groupoids

$$\psi_\rho : R_\alpha \rightarrow R_\beta$$

given by  $\psi_\rho(g, r) = (g, \rho(g)r)$ . Hence it induces an isomorphism

$$P_\alpha(E) \rightarrow P_\beta(E)$$

and also a canonical isomorphism

$$\psi_\rho : {}^\alpha K^i(\mathcal{G}) \rightarrow {}^\beta K^i(\mathcal{G}).$$

Suppose that in fact  $\rho$  is a cocycle, i.e.,  $\delta\rho = 0$ . Then  $\alpha = \alpha + \delta\rho$  and hence we have an automorphism  $\psi_\rho : {}^\alpha K^i(\mathcal{G}) \rightarrow {}^\alpha K^i(\mathcal{G})$ . Furthermore, if  $\rho = \delta\gamma$  is a coboundary, then  $\psi_\rho$  is the identity. Hence  $H^1(B\mathcal{G}, U(1))$  acts as automorphisms of twisted K-theory. It is easy to check in many examples that they are nontrivial automorphisms. In the literature, twisted K-theory is often referred to as being twisted by a Čech cohomology class or characteristic class of a 1-gerbe. This is a rather ambiguous statement, as cohomologous 1-gerbes induce isomorphic twisted K-theory, but this is not canonical. This observation is particularly important when we define a product structure on twisted K-theory.

To summarize: for an orbifold groupoid  $\mathcal{G}$ , and a 1-gerbe  $(\mathcal{G}, \theta)$ , twisted K-theory  ${}^\theta K(\mathcal{G})$  is well-defined up to isomorphism (see [18]).

**Definition 3.7** For an orbifold and a 1-gerbe on it, taking a presentation  $(\mathcal{G}, \theta)$  of the gerbe, the twisted K-theory of the orbifold is defined to be  ${}^\theta K(\mathcal{G})$ .

There is a natural addition operator for  ${}^\alpha K^*(\mathcal{G})$  induced by the Hilbert space addition  $\mathbb{H} \cong \mathbb{H} \oplus \mathbb{H}$ . On the other hand, the multiplication operation induced by Hilbert space tensor product  $\mathbb{H} \cong \mathbb{H} \otimes \mathbb{H}$  shifts the twisting

$${}^\alpha K^*(\mathcal{G}) \otimes {}^\beta K^*(\mathcal{G}) \rightarrow {}^{\alpha+\beta} K^*(\mathcal{G}).$$

**Remark** Strictly speaking, there is an issue about canonicity because of the way we identify  $\mathbb{H} \oplus \mathbb{H}$  and  $\mathbb{H} \otimes \mathbb{H}$  with  $\mathbb{H}$ . Since  $U(\mathbb{H})$  is contractible, any identification will give the same homotopy classes, therefore the same K-theory element.

We also want to point out that the product of an element in twisted K-theory with a vector bundle is to be understood in the following sense. Let  $P$  be a family of Fredholm operators on  $\mathbb{H}$  parameterized by a space  $M$  and  $E$  a complex vector bundle over  $M$  of finite rank. Then  $P \cdot E$  is a family of Fredholm operators on  $\mathbb{H} \otimes E$  parameterized by  $M$ , which at every point  $x \in M$  has  $P \cdot E(x) = P(x) \otimes Id_E$ . Hence it is easy to see that if  $P \in {}^\theta K^i(\mathcal{G})$  and  $E$  is a  $\mathcal{G}$ -bundle, then  $P \cdot E \in {}^\theta K^i(\mathcal{G})$  (just like the above remark, the way to identify  $\mathbb{H} \otimes E$  with  $\mathbb{H}$  does not change the element). So even though  $E$  may not be an element of  $K^0(\mathcal{G})$ , the product with  $E$  makes sense.

To define our stringy product, we will need a version of the push-forward map in the context of twisted K-theory. For smooth manifolds, such a push-forward map has already been worked out by Carey-Wang [9]. In the case of orbifold groupoids, we follow their treatment, the extra effort will be needed to deal with the groupoid structure.

Let  $\mathcal{G}, \mathcal{H}$  be almost complex groupoids,  $f : \mathcal{G} \rightarrow \mathcal{H}$  be a homomorphism which preserves the almost complex structures, and  $\alpha$  be a 1-gerbe on  $\mathcal{H}$ . We will define the push-forward map  $f_* : f^* \alpha K^*(\mathcal{G}) \rightarrow \alpha K^*(\mathcal{H})$ .

Let  $\mathcal{G} = \{s, t : G_1 \rightarrow G_0\}$  be a groupoid and  $E$  a rank  $n$  complex vector bundle over  $\mathcal{G}$ , i.e.,  $\pi : E \rightarrow G_0$  is a complex vector bundle with compatible  $\mathcal{G}$ -action. We first establish the Thom homomorphism  $\Phi : \alpha K^*(\mathcal{G}) \rightarrow \pi^* \alpha K^*(\mathcal{G} \rtimes E)$ , where  $\mathcal{G} \rtimes E$  is the transformation groupoid with object set  $E$  and arrow set  $G_1 \times_{G_0} E$ .

Fix any invariant hermitian metric on  $E$  [28], then for any  $g \in G_0$ ,  $e \in E_g = \pi^{-1}(g)$ , we use  $e^*$  to denote the dual of  $e$  with respect to the fixed hermitian metric.

The complex  $\mathcal{G}$ -bundle  $\pi : E \rightarrow G_0$  defines a complex of  $\mathcal{G}$ -bundles over  $E$ ,

$$\lambda_E = (\Lambda^{even} \pi^* E, \Lambda^{odd} \pi^* E, \phi),$$

where  $\phi_{(g,e)} = e \wedge -e^*$ . Notice that in ordinary  $K$ -theory, this is the Thom element.

For any element  $x \in \alpha K^*(\mathcal{G})$ , it is represented by a  $\mathcal{G}$ -equivariant section  $x : G_0 \rightarrow Fred_\alpha^*(\mathbb{H})$  with  $supp(x)$  compact.

**Proposition 3.8** *For any 1-gerbe  $\alpha$ , there is a  $K^0(\mathcal{G})$ -module homomorphism*

$$\Phi : \alpha K^*(\mathcal{G}) \rightarrow \pi^* \alpha K^*(\mathcal{G} \rtimes E)$$

*which is the standard Thom isomorphism in the case of equivariant K-theory.*

*Proof.* For any element  $x \in \alpha K^i(\mathcal{G})$ , it is given by a  $\mathcal{G}$ -equivariant section  $x : G_0 \rightarrow Fred_\alpha^i(\mathbb{H})$ . Therefore, we have a section  $\pi^*(x) : E \rightarrow \pi^* Fred_\alpha^i(\mathbb{H})$ . Now  $supp(\pi^*(x)) = \pi^{-1}(supp(x))$ .

For any point  $(g, e) \in E$ , let us consider the operator:

$$D : \mathbb{H} \otimes \Lambda^{even} E_g \oplus \mathbb{H} \otimes \Lambda^{odd} E_g \rightarrow \mathbb{H} \otimes \Lambda^{even} E_g \oplus \mathbb{H} \otimes \Lambda^{odd} E_g$$

defined by

$$D_{(g,e)} = \begin{pmatrix} (\pi^* x)(g) \otimes 1 & -1 \otimes \phi_{(g,e)}^* \\ 1 \otimes \phi_{(g,e)} & (\pi^* x)(g)^* \otimes 1 \end{pmatrix}$$

where  $*$  means the adjoint operator. This is the so-called “graded tensor product” of Fredholm operators. It is easy to check that  $D_{(g,e)}$  is Fredholm and if  $e \neq 0$ , then it is invertible.

Globalizing this construction, we have a family  $D$  of Fredholm operators parameterized by  $E$ . In fact, it is a fiberwise Fredholm operator on Hilbert

bundles  $\Lambda^\bullet \pi^* E \otimes \mathbb{H}$  over  $E$ , as we remarked, the identification of fiber with  $\mathbb{H}$  does not matter.

Notice that

- $\text{supp}(D) = \pi^{-1} \text{supp}(x) \cap i(G_0)$ , where  $i : G_0 \rightarrow E$  is the zero section.
- $D$  is a section of  $\pi^* \text{Fred}_\alpha^i(\mathbb{H}) \cong \text{Fred}_{\pi^* \alpha}^i(\mathbb{H})$ .
- $D$  is  $\mathcal{G} \times E$ -equivariant.

Therefore  $D$  represents an element of  $\pi^{*\alpha} K^i(\mathcal{G} \times E)$ .

Now we define

$$\begin{aligned} \Phi : {}^\alpha K^*(\mathcal{G}) &\rightarrow \pi^{*\alpha} K^*(\mathcal{G} \times E) \\ x &\mapsto D \end{aligned}$$

Up to homotopy, as in ordinary  $K$ -theory, this definition does not depend on any choice, so it is well-defined. Because it is the graded tensor product with  $\lambda_E$ ,  $\Phi$  is a  $K(\mathcal{G})$ -module homomorphism. Furthermore, by the definition, we see that it is a generalization of the Thom isomorphism in equivariant  $K$ -theory.  $\square$

We need a slightly more general version of this. Let  $U$  be an open neighborhood of the zero section, from the definition,  $\Phi(x)$  is supported on the zero section, so by restriction, we have the following Thom homomorphism.

**Proposition 3.9** *For any 1-gerbe  $\alpha$ , there is a  $K(\mathcal{G})$ -module homomorphism*

$$\Phi : {}^\alpha K^*(\mathcal{G}) \rightarrow \pi^{*\alpha} K^*(\mathcal{G} \times E|_U).$$

To handle the general situation, let us first recall a lemma for Lie groupoids [22].

**Lemma 3.10** *Let  $p : F \rightarrow G_0$  be a smooth surjective submersion, then the groupoid  $F \times_p \mathcal{G}$  is equivalent to  $\mathcal{G}$ , where  $(F \times_p \mathcal{G})_1 = (F \times F)_{p \times p} \times_{s \times t} G_1$  and  $(F \times_p \mathcal{G})_0 = F$ , and the new source map is  $s : (F \times F)_{p \times p} \times_{s \times t} G_1 \rightarrow F, ((x, y), g) \mapsto x$ , the new target map is  $t : (F \times F)_{p \times p} \times_{s \times t} G_1 \rightarrow F, ((x, y), g) \mapsto y$ .*

For a Lie groupoid homomorphism  $f : \mathcal{G} \rightarrow \mathcal{H}$ , if we apply the lemma above to the space  $F = G_0 \times H_0$ , and take  $\mathcal{K}$  to be  $F \times_p \mathcal{H}$ , we can prove the next lemma, where all the maps are the natural ones.



**Lemma 3.11** *Let  $f : \mathcal{G} \rightarrow \mathcal{H}$  be a homomorphism of Lie groupoids, then there exists a Lie groupoid  $\mathcal{K}$  and homomorphisms  $g : \mathcal{G} \rightarrow \mathcal{K}$ ,  $h : \mathcal{K} \rightarrow \mathcal{H}$ , where  $g_0 : G_0 \rightarrow K_0$  is an embedding and  $h$  is an equivalence, such that  $f$  is the composition of  $g$  and  $h$ . In other words, any homomorphism is an embedding on the object level up to Morita equivalence.*

Now we can prove the existence of a push-forward map in twisted K-theory.

**Theorem 3.12** *If  $f : \mathcal{G} \rightarrow \mathcal{H}$  is a homomorphism between almost complex groupoids which preserves the almost complex structures, such that  $|f| : |\mathcal{G}| \rightarrow |\mathcal{H}|$  is proper, and  $f_1(\mathcal{G}.x) = \mathcal{H}.f_0(x)$  for any  $x \in G_0$ , then there is a push-forward map*

$$f_* : f^* \alpha K^*(\mathcal{G}) \rightarrow \alpha K^*(\mathcal{H}).$$

*Proof.* Given our last lemma, we may assume that  $f_0 : G_0 \rightarrow H_0$  is a proper embedding. By our assumption, the normal bundle  $E$  of  $G_0$  in  $H_0$  is a complex  $\mathcal{G}$ -bundle, and we can identify an open neighborhood  $U$  of the zero section in the normal bundle with a neighborhood of  $f_0(G_0)$  in  $H_0$ , i.e. we have an embedding  $j : U \rightarrow H_0$  as an open subset. It defines a homomorphism:  $j_* : j^* \alpha K^*(\mathcal{G} \times E|_U) \rightarrow \alpha K^*(\mathcal{H}|_{j(U)})$ , because the action of  $\mathcal{G}$  on the normal bundle is induced from the  $\mathcal{H}$  action, in this case any  $\mathcal{G}$ -equivariant section is  $\mathcal{H}$ -equivariant.

It is clear that  $f\pi$  is homotopic to  $j$ . Therefore,  $\pi^* f^* \alpha = j^* \alpha + \delta\rho$  for some  $\rho$ . The choice of  $\rho$  is not unique; for example, we can add a 1-cocycle. This corresponds exactly to the non-canonicity of the dependence of twisted K-theory on the cohomology class of a 1-gerbe. However,  $f\pi = j$  on the zero section; therefore, we can choose  $\rho$  such that  $\rho = 0$  on the zero section. Since  $U$  deformation retracts to the zero section, it fixes  $\rho$  uniquely.

Now we have homomorphisms:

$$f^* \alpha K^*(\mathcal{G}) \xrightarrow{\Phi} \pi^* f^* \alpha K^*(\mathcal{G} \times E|_U) \xrightarrow{P\rho} \alpha K^*(\mathcal{G} \times E|_U) \xrightarrow{j_*} \alpha K^*(\mathcal{H}|_{j(U)}) \rightarrow \alpha K^*(\mathcal{H})$$

where the last homomorphism is extension for an open saturated subgroupoid. The composition is the push-forward map  $f_*$ .  $\square$

Given our explicit definition, it is easy to check the following properties of the push-forward map.

**Proposition 3.13** *Let  $f : \mathcal{G} \rightarrow \mathcal{H}$  as before, then there exists an element  $c = c(\mathcal{G}, \mathcal{H})$  such that for any  $a \in f^* \alpha K^*(\mathcal{G})$  and  $b \in \beta K^*(\mathcal{H})$ , we have*

$$f^* f_*(a) = a \cdot c$$

$$f_*(a \cdot f^*(b)) = f_*(a) \cdot b.$$

In particular for quasi-suborbifolds, we have following result.

**Corollary 3.14** *If  $i : \mathcal{G}_1 \rightarrow \mathcal{G}$  is a quasi-suborbifold, then there is a push-forward map*

$$i_* : i^* \alpha K^*(\mathcal{G}_1) \rightarrow \alpha K^*(\mathcal{G})$$

*satisfying the above properties.*

For later purposes we would like to introduce

**Definition 3.15** *If  $E \rightarrow \mathcal{G}$  is a complex orbifold bundle, then its  $K$ -theoretic Euler class  $e_K(E)$  is defined as  $i^* \lambda_E$ , the complex of  $\mathcal{G}$ -vector bundles obtained by pulling back the Thom element  $\lambda_E$  using the zero-section  $i : \mathcal{G} \rightarrow E$ .*

Note that we can define the product  $x \cdot e_K(E)$  as

$$x \cdot e_K(E) = x \cdot (\wedge^{even} E) + (x \cdot \wedge^{odd} E)^*.$$

## 4 The Inverse Transgression for Groupoids

In order to define the stringy product in twisted  $K$ -theory, we will need a cohomological formula to match up the levels which appear in the twistings. The basic construction is the *inverse transgression*, which was defined in [12]. We provide a formulation for groupoids inspired by the case of finite groups. We will also provide some explicit calculations. See [19] for a more geometric view on this.

Recall that

$$(\wedge \mathcal{G})_0 = \{a \in G_1 \mid s(a) = t(a)\},$$

$$(\wedge \mathcal{G})_1 = \{(a, u_1) \in G_1 \times G_1 \mid s(a) = t(a) = s(u_1)\}.$$

It is easy to check that the  $k$ -tuples of composable arrows in  $\wedge \mathcal{G}$  are

$$(\wedge \mathcal{G})_k = \{(a, u_1, \dots, u_k) \in G_{k+1} \mid s(a) = t(a) = s(u_1), t(u_i) = s(u_{i+1})\}.$$

**Definition 4.1** Define  $\theta : C^{k+1}(\mathcal{G}, U(1)) \rightarrow C^k(\wedge \mathcal{G}, U(1))$  by

$$\begin{aligned} \theta(\phi)(a, u_1, \dots, u_k) = \\ (-1)^k \phi(a, u_1, \dots, u_k) + \sum_{i=1}^k (-1)^{i+k} \phi(u_1, \dots, u_i, a_i, u_{i+1}, \dots, u_k), \end{aligned}$$

where  $a_i = (u_1 \cdots u_i)^{-1} a u_1 \cdots u_i$ .

A routine but slightly tedious computation shows that this is in fact a cochain map, i.e.,  $\delta\theta = \theta\delta$ .

We should note that  $\theta$  is a natural map defined for all groupoids. For orbifold groupoids it induces a homomorphism

$$\theta_* : H^k(B\mathcal{G}, U(1)) \rightarrow H^{k-1}(B \wedge \mathcal{G}, U(1)),$$

and hence a homomorphism

$$\theta_* : H^{k+1}(B\mathcal{G}, \mathbb{Z}) \rightarrow H^k(B \wedge \mathcal{G}, \mathbb{Z}).$$

The cochain map  $\theta$  and the induced map in cohomology will be called the inverse transgression.

Recall that the moduli space of constant morphisms  $\overline{\mathcal{M}}_3(\mathcal{G})$  from an orbifold sphere with three orbifold points can be identified with the 2-sector orbifold  $\mathcal{G}^2$ , where

$$(\mathcal{G}^2)_0 = \{(a, b) \in G_2 \mid s(a) = t(a) = s(b) = t(b)\},$$

$$(\mathcal{G}^2)_k = \{(a, b, u_1, \dots, u_k) \in G_{k+2} \mid \\ s(a) = t(a) = s(b) = t(b) = s(u_1), t(u_i) = s(u_{i+1})\}$$

with

$$s(a, b, u_1, \dots, u_k) = (a, b), t(a, b, u_1, \dots, u_k) = (a_k, b_k)$$

where

$$a_i = (u_1 \cdots u_i)^{-1} a u_1 \cdots u_i, \quad b_i = (u_1 \cdots u_i)^{-1} b u_1 \cdots u_i.$$

There are three natural evaluation morphisms

$$e_1 : \mathcal{G}^2 \rightarrow \wedge \mathcal{G} \text{ by } e_1(a, b) = a,$$

$$e_2 : \mathcal{G}^2 \rightarrow \wedge \mathcal{G} \text{ by } e_2(a, b) = b,$$

$$e_{12} : \mathcal{G}^2 \rightarrow \wedge \mathcal{G} \text{ by } e_{12}(a, b) = ab.$$

Furthermore,  $e_1, e_2, e_{12}$  are all quasi-embeddings.

**Definition 4.2** Define  $\mu : C^{k+2}(\mathcal{G}, U(1)) \rightarrow C^k(\mathcal{G}^2, U(1))$  by

$$\mu(\phi)(a, b, u_1, \dots, u_k) = \phi(a, b, u_1, \dots, u_k) \\ + \sum (-1)^{i+j} \phi(u_1, \dots, u_i, a_i, u_{i+1}, \dots, u_j, b_j, u_{j+1}, \dots, u_k)$$

where the summation is taken over the set of pairs  $(i, j)$  satisfying  $0 \leq i \leq j \leq k$  and  $(i, j) \neq (0, 0)$ .

A second key *multiplicative formula* is given by the equation

$$\mu\delta + \delta\mu = e_1^*\theta + e_2^*\theta - e_{12}^*\theta.$$

Note that the function  $\mu$  defines a chain homotopy between  $e_1^*\theta + e_2^*\theta$  and  $e_{12}^*\theta$ . If  $\phi$  is a cocycle, then  $\theta(\phi)$  is a cocycle and the formula above implies that

$$e_1^*\theta(\phi) + e_2^*\theta(\phi) = e_{12}^*\theta(\phi) + \delta\mu(\phi).$$

In particular we see that the difference between the cocycles is given by a *canonical* coboundary, expressed explicitly as a function of  $\phi$ . This will be very important when we make our identifications in twisted K-theory.

We will verify and apply this formula in low degree, which is our main interest here.

**Proposition 4.3** *Let  $\phi$  be an element in  $C^3(\mathcal{G}, U(1))$ . Then*

$$\delta\mu(\phi) + \mu\delta(\phi) = e_1^*\theta(\phi) + e_2^*\theta(\phi) - e_{12}^*\theta(\phi).$$

*Proof.* This can be proved by an explicit calculation.

$$\begin{aligned} \delta\mu(\phi)(a, b, u_1, u_2) &= -\phi(a_1, b_1, u_2) + \phi(a, b, u_1u_2) \\ &\quad - \phi(a, b, u_1) + \phi(a_1, u_2, b_2) - \phi(a, u_1u_2, b_2) + \phi(a, u_1, b_1) \\ &\quad - \phi(u_2, a_2, b_2) + \phi(u_1u_2, a_2, b_2) - \phi(u_1, a_1, b_1), \end{aligned}$$

$$\begin{aligned} \mu\delta(\phi)(a, b, u_1, u_2) &= \phi(b, u_1, u_2) - \phi(u_1, b_1, u_2) + \phi(u_1, u_2, b_2) \\ &\quad + \phi(a_1, b_1, u_2) - \phi(a_1, u_2, b_2) + \phi(u_2, a_2, b_2) - \phi(ab, u_1, u_2) \\ &\quad + \phi(au_1, b_1, u_2) - \phi(au_1, u_2, b_2) - \phi(u_1a_1, b_1, u_2) + \phi(u_1a_1, u_2, b_2) \\ &\quad - \phi(u_1u_2, a_2, b_2) + \phi(a, bu_1, u_2) - \phi(a, u_1b_1, u_2) + \phi(a, u_1u_2, b_2) \\ &\quad + \phi(u_1, a_1b_1, u_2) - \phi(u_1, a_1u_2, b_2) + \phi(u_1, u_2a_2, b_2) - \phi(a, b, u_1u_2) \\ &\quad + \phi(a, u_1, b_1u_2) - \phi(a, u_1, u_2b_2) - \phi(u_1, a_1, b_1u_2) + \phi(u_1, a_1, u_2b_2) \\ &\quad - \phi(u_1, u_2, a_2b_2) + \phi(a, b, u_1) - \phi(a, u_1, b_1) + \phi(a, u_1, u_2) \\ &\quad + \phi(u_1, a_1, b_1) - \phi(u_1, a_1, u_2) + \phi(u_1, u_2, a_2). \end{aligned}$$

We now add these two expressions. Using the identities  $au_1 = u_1a_1$ ,  $bu_1 = u_1b_1$ ,  $a_1u_2 = u_2a_2$ , and  $b_1u_2 = u_2b_2$ , cancelling and collecting terms, yields the expression

$$\begin{aligned} [\mu\delta + \delta\mu](\phi)(a, b, u_1, u_2) &= \phi(a, u_1, u_2) - \phi(u_1, a_1, u_2) + \phi(u_1, u_2, a_2) \\ &\quad + \phi(b, u_1, u_2) - \phi(u_1, b_1, u_2) + \phi(u_1, u_2, b_2) \\ &\quad - \phi(ab, u_1, u_2) + \phi(u_1, a_1b_1, u_2) - \phi(u_1, u_2, a_2b_2). \end{aligned}$$

This expression is exactly  $e_1^*\theta + e_2^*\theta - e_{12}^*\theta$  applied to  $\phi$ , hence the proof is complete.  $\square$

The inverse transgression formula implies that a 2-gerbe  $\phi$  on an orbifold groupoid  $\mathcal{G}$  induces a 1-gerbe  $\theta(\phi)$  on the associated inertia groupoid  $\wedge\mathcal{G}$ . Furthermore, two equivalent (isomorphic) 2-gerbes induce equivalent (isomorphic) 1-gerbes on the inertia groupoid.

Recall that there is an embedding  $e : \mathcal{G} \rightarrow \wedge\mathcal{G}$  by  $e(x) = 1_x$  where  $1_x$  is the identity arrow. The image  $e(\mathcal{G})$  is often referred as non-twisted sector and other components of  $\wedge\mathcal{G}$  are called *twisted sectors*.

**Corollary 4.4** *If  $\phi$  is a cocycle, then  $e^*\theta(\phi)$  is a coboundary.*

*Proof.*  $e^*\theta(\phi)(u, v) = \theta(1, u, v)$ . Using the embedding  $\lambda : \mathcal{G} \rightarrow \mathcal{G}^2$  given by  $x \rightarrow (i_x, i_x)$ , we can pull back  $e_1^*\theta(\phi) + e_2^*\theta(\phi) - e_{12}^*\theta(\phi)$  in cohomology. Note that

$$\lambda^*e_1^*\theta(\phi) = \lambda^*e_2^*\theta(\phi) = \lambda^*e_{12}^*\theta(\phi) = e^*\theta(\phi).$$

Therefore,  $e^*\theta(\phi) = \delta e^*\mu(\phi)$  is a coboundary. This implies that restricted to the untwisted sector, our cocycle  $\theta(\phi)$  gives rise to a trivial cohomology class.  $\square$

## 5 The Inverse Transgression in the Case of a Finite Group

In the case when the original orbifold is  $[*/G]$  where  $G$  is a finite group, the inverse transgression has a classical interpretation in terms of shuffle products. Recall that  $\wedge[*/G]$  can be thought of in terms of  $G$  with the conjugation action; this breaks up into a disjoint union of orbits of the form  $G/Z_G(g)$ , indexed by conjugacy classes. Each of these is in turn equivalent to  $[*/Z_G(g)]$ ; so we have a Morita equivalence  $\wedge[*/G] \cong \sqcup_{(g)} [*/Z_G(g)]$ . Hence we can restrict our attention to these components; in particular we would like to describe each  $\theta_g : C^k(G, U(1)) \rightarrow C^{k-1}(Z_G(g), U(1))$ . Now for a finite group  $G$ , the cochain complex  $C^*(G, U(1))$  is in fact equal to  $Hom_G(B_*(G), U(1))$ , where  $B_*(G)$  is the bar resolution for  $G$  (see [8], page 19).

There is natural homomorphism  $\rho_g : Z_G(g) \times \mathbb{Z} \rightarrow G$  given by  $\rho_g(x, t^i) = xg^i$ , where  $t$  is a generator for  $\mathbb{Z}$ ; the fact that  $Z_G(g)$  centralizes  $g$  is crucial here. This homomorphism induces a map in integral homology

$$H_*(Z_G(g), \mathbb{Z}) \otimes H_*(\mathbb{Z}, \mathbb{Z}) \cong H_*(Z_G(g) \times \mathbb{Z}, \mathbb{Z}) \rightarrow H_*(G, \mathbb{Z}).$$

Classically it is known that multiplication is induced by the shuffle product on the chain groups (see [8], page 117–118); i.e. there is a chain map  $B_*(Z_G(g)) \otimes B_*(\mathbb{Z}) \rightarrow B_*(G)$  which will induce  $\rho_{g^*}$  in homology. Let  $t$  denote a generator of the cyclic group  $\mathbb{Z}$ . The shuffle product we are interested in is  $B_k(Z_G(g)) \otimes B_1(\mathbb{Z}) \rightarrow B_{k+1}(G)$  given by

$$[g_1|g_2|\dots|g_k] \star [t^i] = \sum_{\sigma} \sigma[g_1|g_2|\dots|g_k|g_{k+1}]$$

where  $g_{k+1} = g^i$ ,  $\sigma$  ranges over all  $(k, 1)$ -shuffles and

$$\sigma[g_1|g_2|\dots|g_{k+1}] = (-1)^{\text{sign}(\sigma)} [g_{\sigma(1)}|g_{\sigma(2)}|\dots|g_{\sigma(k+1)}].$$

A  $(k, 1)$ -shuffle is an element  $\sigma \in S_{k+1}$  such that  $\sigma(i) < \sigma(j)$  for  $1 \leq i < j \leq k$ . These are precisely the cycles:

$$1, (k \ k+1), (k-1 \ k \ k+1), (k-2 \ k-1 \ k \ k+1), \dots, (1 \ 2 \ 3 \ \dots \ k \ k+1).$$

Note that there are  $k+1$  of them. This can be dualized, using  $U(1)$  coefficients, but for cohomology purposes it's easier to use integral coefficients. Given a cocycle  $\phi \in C^{k+1}(G, \mathbb{Z})$ , we see that  $\theta_g(\phi) \in C^k(Z_G(g), \mathbb{Z})$  can be defined as

$$\theta_g(\phi)([g_1|g_2|\dots|g_k]) = \phi([g_1|g_2|\dots|g_k] \star [g])$$

where  $g_1, g_2, \dots, g_k \in Z_G(g)$ .

As a consequence of this we see that  $\theta_g^* : H^{k+1}(G, \mathbb{Z}) \rightarrow H^k(Z_G(g), \mathbb{Z})$  is induced by the multiplication map

$$\rho_g^* : H^{k+1}(G, \mathbb{Z}) \rightarrow H^k(Z_G(g), \mathbb{Z}) \otimes H^1(\mathbb{Z}, \mathbb{Z}).$$

To be precise, if  $\nu$  is the natural generator for  $H^1(\mathbb{Z}, \mathbb{Z})$ , then

$$\rho_g^*(u) = \text{res}_{Z_G(g)}^G(u) \otimes 1 + \theta_g^*(u) \otimes \nu.$$

This discussion clarifies the geometric arguments in [12], and will also allow us to do some computations in cohomology.

**Example 5.1** Finite group cohomology is difficult to compute, especially over the integers. The simple examples such as cyclic and quaternion groups are not so interesting in this context, as their odd dimensional cohomology (with trivial  $\mathbb{Z}$  coefficients) is zero. The first interesting example is  $G = (\mathbb{Z}/2)^2$ . In this case  $H^*(G, \mathbb{F}_2)$  is a polynomial algebra on two degree one generators  $x, y$ . In degree four there is a natural basis given by  $x^4, y^4, x^3y, x^2y^2, xy^3$ . For an elementary abelian 2-group, the mod 2 reduction map for  $k > 0$  is a

monomorphism  $H^k(G, \mathbb{Z}) \rightarrow H^k(G, \mathbb{F}_2)$ , and so we can understand it as the kernel of the Steenrod operation  $Sq^1 : H^k(G, \mathbb{F}_2) \rightarrow H^{k+1}(G, \mathbb{F}_2)$ . Hence we see that  $H^4(G, \mathbb{Z})$  can be identified with the subspace generated by  $x^4, y^4$  and  $x^2y^2$ . These are all squares, hence when we apply  $\theta_g^* : H^4(G, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$  for any  $g \in G$ , the result will always be zero.

Next we consider  $G = (\mathbb{Z}/2)^3$ ; in this case  $H^*(G, \mathbb{F}_2)$  is a polynomial algebra on three degree one generators  $x, y, z$ . In this case we have an element  $\alpha = Sq^1(xy z) = x^2yz + xy^2z + xyz^2$  which represents a non-square element in  $H^4(G, \mathbb{Z})$ . By analyzing the multiplication map in cohomology we obtain the following.

**Lemma 5.2** *Let  $g = x^a y^b z^c$  be an element in  $G = (\mathbb{Z}/2)^3$ , where we are writing it in terms of the standard basis (identified with its dual by abuse of notation). Then*

$$\theta_g^*(\alpha) = a(y^2z + z^2y) + b(x^2z + xz^2) + c(x^2y + xy^2)$$

and so it is non-zero on every component except the one corresponding to the trivial element in  $G$ .

Now for an abelian group, the multiplicative formula implies that for all  $g, h \in G$ ,  $\theta_g^* + \theta_h^* = \theta_{gh}^*$  in cohomology, or up to coboundaries. In particular this implies that the correspondence  $g \mapsto \theta_g(\alpha)$  defines a homomorphism  $G \rightarrow H^3(G, \mathbb{Z})$  of elementary abelian groups, in this case an isomorphism.

## 6 The Twisted Pontryagin Product for Finite Groups

Let  $G$  denote a finite group, and consider the orbifold defined by its action on a point. Then the inertia groupoid  $\wedge \mathcal{G}$  can be identified with the groupoid determined by the conjugation action of  $G$  on itself. In this case the untwisted orbifold K-theory is simply  $K_G(G)$ , which is additively isomorphic to  $\bigoplus_{(g)} R(Z_G(g))$ , where as before  $Z_G(g)$  denotes the centralizer of  $g$  in  $G$ , and the sum is taken over conjugacy classes. This group can be endowed with a certain product, known as the *Pontryagin product*, defined as follows. An equivariant vector bundle over  $G$  (with the conjugation action) can be thought of as a collection of finite dimensional vector spaces  $V_g$  with a  $G$ -module structure on  $\bigoplus_{g \in G} V_g$  such that  $gV_h = V_{ghg^{-1}}$ . The product of two of these bundles is now defined as:

$$(V \star W)_g = \bigoplus_{\{g_1, g_2 \in G, g_1 g_2 = g\}} V_{g_1} \otimes W_{g_2}.$$

This formula has been referred to as the holomorphic orbifold model in the physics literature [11].

This product admits an alternate description, which will admit a geometric generalization. In this case we can identify  $\mathcal{G}^2$  with the orbifold defined by considering  $G \times G$  with the conjugation action on both coordinates. Our maps  $e_1$ ,  $e_2$  and  $e_{12}$  correspond to  $(g, h) \mapsto g$ ,  $(g, h) \mapsto h$ ,  $(g, h) \mapsto gh$  respectively, which are  $G$ -equivariant with respect to the conjugation action. Then, if  $\alpha, \beta$  are elements in  $K_G(G)$ , the Pontryagin product can also be defined as

$$\alpha \star \beta = e_{12*}(e_1^*(\alpha) \cdot e_2^*(\beta)).$$

We propose to extend this definition to twisted K-theory, with certain conditions on the twisting cocycle. Note that given a 2-cocycle  $\tau = \theta(\phi)$  in the image of the inverse transgression, then by our multiplicative formula we have  $e_1^*\tau + e_2^*\tau = e_{12}^*\tau + \delta\mu(\phi)$ .

**Definition 6.1** Let  $\tau$  be a  $U(1)$  valued 2-cocycle for the orbifold defined by the conjugation action of a finite group  $G$  on itself which is in the image of the inverse transgression. The Pontryagin product on  ${}^\tau K_G(G)$  is defined by the following formula: if  $\alpha, \beta \in {}^\tau K_G(G)$ , then

$$\alpha \star \beta = e_{12*}(e_1^*\alpha \cdot e_2^*\beta).$$

Note that if  $\tau = \theta(\phi)$  then by our multiplicative formula we have

$$e_1^*\tau + e_2^*\tau = e_{12}^*\tau + \delta\mu(\phi)$$

and so the product  $e_1^*(\alpha) \cdot e_2^*(\beta)$  lies in

$$e_1^*\tau + e_2^*\tau K_G(G) = e_{12}^*\tau + \delta\mu(\phi) K_G(G) \cong e_{12}^*\tau K_G(G).$$

Now applying  $e_{12*}$ , this is mapped to  ${}^\tau K_G(G)$ ; and so we have a product on our twisted K-theory; it is elementary to verify that this defines an associative product.

Our approach will define a twisted Pontryagin product for any cocycle in the image of the inverse transgression. This cocycle could very well be a coboundary; but that does not necessarily imply that the corresponding product on  $K_G(G)$  is the standard Pontryagin product. It is also clear that we may choose twistings which give rise to a twisted K-theory without any product.<sup>4</sup>

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<sup>4</sup>For much more on Pontryagin products please consult [14] and its sequels.



**Example 6.2** If  $G$  is an abelian group, then what we are doing is using the identification  $\theta(\phi)_g + \theta(\phi)_h = \theta(\phi)_{gh}$  to define a product on the abelian group

$$\mathcal{X}(G) = \sum_{g \in G} \theta(\phi)_g R(G)$$

via the pairing

$$\theta(\phi)_g R(G) \otimes \theta(\phi)_h R(G) \rightarrow \theta(\phi)_{gh} R(G).$$

In the case described in 5.1 for  $G = (\mathbb{Z}/2)^3$  and a cocycle  $\phi$  representing the cohomology class  $xy^2z + xyz^2 + x^2yz$ ,  $\theta(\alpha)$  establishes a group homomorphism  $G \rightarrow H^3(G, \mathbb{Z})$ , with image the subgroup generated by  $xy^2 + x^2y$ ,  $xz^2 + x^2z$  and  $yz^2 + y^2z$ . In this case we see that for  $g \neq 1$ ,  $\theta(\phi)_g R(G)$  has rank equal to two, and so  $\mathcal{X}(G)$  is of rank equal to twenty-two, with the twisted Pontryagin product described above. This can be made explicit.

The case of the Pontryagin product should be considered as motivation for the case of orbifold groupoids. As long as we twist with a cocycle in the image of the inverse transgression, the levels will match up as required. Hence the main difficulty is geometric—as we shall see in the next section, there is an obstruction bundle which plays an important role.

## 7 Twisted K–theory of Orbifolds

During the course of our investigation of possible stringy products on the twisted K-theory of orbifolds, we came to realize that one needs to use the very same information required to construct the Chen-Ruan cohomology of orbifolds ([10]). We first briefly recall the situation for orbifold cohomology, and then proceed to develop the tools necessary to deal with twisted K–theory. For a very interesting but different approach we refer the reader to [15].

First we recall that  $H_{CR}^*(\mathcal{G}, \mathbb{C})$  is additively the same as  $H^*(\wedge \mathcal{G}, \mathbb{C})$ ; what is interesting is the ring structure. Recall that there are three evaluation maps  $e_1, e_2, e_{12} : \mathcal{G}^2 \rightarrow \wedge \mathcal{G}$ . A naive definition of the stringy product for orbifold cohomology (which would generalize the Pontryagin product) would be  $\alpha \star \beta = (e_{12})_*(e_1^* \alpha \cup e_2^* \beta)$ . However, one soon discovers that this product is not associative due to the fact that  $e_1, e_{12}$  are not transverse in general. In fact, the correction term is precisely the obstruction bundle to the transversality of  $e_1$  and  $e_{12}$ . A natural idea was to modify the definition of the product via

$$\alpha \star \beta = (e_{12})_*(e_1^* \alpha \cup e_2^* \beta \cup e(E_{\mathcal{G}^2}))$$

where we need to construct a bundle  $E_{\mathcal{G}^2}$  in a fashion that is consistent with the obstruction to transversality of  $e_1, e_{12}$ , and  $e(E_{\mathcal{G}^2})$  denotes its Euler class. A key observation is that the *obstruction bundle* in the construction of the Chen-Ruan product provides such a choice. We will adapt this same idea to  $K$ -theory.

Throughout this section, we assume that  $\mathcal{G}$  is a compact, almost complex orbifold groupoid. Then  $\mathcal{G}^k$  naturally inherits an almost complex structure such that the evaluation map  $e_{i_1, \dots, i_l} : \mathcal{G}^k \rightarrow \mathcal{G}^l$  is an almost complex quasi-embedding.  $\mathcal{G}^2$  can be identified with the space of constant morphisms from an orbifold sphere with three marked point to  $\mathcal{G}$ ; we now make this identification precise. Consider an orbifold Riemann sphere with three orbifold points  $(\mathbb{S}^2, (x_1, x_2, x_3), (m_1, m_2, m_3))$ ; in this context we simply denote it by  $\mathbb{S}^2$ . Suppose that  $f$  is a constant morphism from  $\mathbb{S}^2$  to  $\mathcal{G}$ . Here, the term constant means that the induced map  $|f| : \mathbb{S}^2 \rightarrow |\mathcal{G}|$  is constant. Let  $y = \text{im}(|f|)$  and  $U_y/G_y$  be an orbifold chart at  $y$ . By the results in [1],  $f$  is classified by the conjugacy class of a homomorphism  $\pi_f : \pi_1^{orb}(\mathbb{S}^2) \rightarrow G_y$ . Recall that

$$\pi_1^{orb}(\mathbb{S}^2) = \{\lambda_1, \lambda_2, \lambda_3; \lambda_i^{k_i} = 1, \lambda_1 \lambda_2 \lambda_3 = 1\},$$

where  $\lambda_i$  is represented by a loop around the marked point  $x_i$ .  $\pi_f$  is uniquely determined by a pair of elements  $(g_1, g_2)$  with  $g_i \in G_y$  where  $g_i = \pi_f(\lambda_i)$ ; on the other hand,  $(g_1, g_2) \in \mathcal{G}_0^2$ . It is clear that the same method can be used to identify the moduli space of constant morphisms from an orbifold sphere with  $k + 1$  marked points to  $\mathcal{G}$  with the groupoid of  $k$ -multisectors  $\mathcal{G}^k$ .

For any  $f \in \mathcal{G}^2$  viewed as a constant morphism, we can form an elliptic complex

$$\bar{\partial}_f : \Omega^0(f^*T\mathcal{G}) \rightarrow \Omega^{0,1}(f^*T\mathcal{G}),$$

where  $f^*T\mathcal{G}$  is a complex vector bundle by our assumption. This defines an orbibundle  $E$  with  $E_{\mathcal{G}^2}|_f = \text{Coker } \bar{\partial}_f$ . We now examine  $E_{\mathcal{G}^2}$  in more detail. Let  $g_1, g_2 \in \mathcal{G}_0^2$ ; by the definition,  $g_1, g_2 \in G_x$  for  $x = s(g_i) = t(g_i)$ . Let  $N$  be the subgroup of  $G_x$  generated by  $g_1, g_2$ . By Lemma 4.5 in [1],  $N$  depends only on the component of  $\mathcal{G}^2$ . Let  $e : \mathcal{G}^2 \rightarrow \mathcal{G}$  be an evaluation map. Clearly  $N$  acts on  $e^*T\mathcal{G}$  while fixing  $T\mathcal{G}^2$ .

There is an obvious surjective homomorphism  $\pi : \pi_1^{orb}(\mathbb{S}^2) \rightarrow N$  and  $\text{Ker } \pi$  is therefore a subgroup of finite index. Suppose that  $\tilde{\Sigma}$  is the orbifold universal cover of  $\mathbb{S}^2$ . By [1] (see Chapter II),  $\tilde{\Sigma}$  is smooth. Let  $\Sigma = \tilde{\Sigma}/\text{Ker } \pi$ ; then  $\Sigma$  is compact and we have a quotient map  $\Sigma \rightarrow \mathbb{S}^2 = \Sigma/N$ . Since  $N$  contains the relation  $g_i^{m_i} = 1$ ,  $\Sigma$  is smooth.

It is clear that  $f$  lifts to an ordinary constant map  $\tilde{f} : \Sigma \rightarrow U_y$ ; hence  $\tilde{f}^*T\mathcal{G} = T_y\mathcal{G}$  is a trivial bundle over  $\Sigma$ . Then we can lift the elliptic complex

to  $\Sigma$

$$\bar{\partial}_\Sigma : \Omega^0(\tilde{f}^*T\mathcal{G}) \rightarrow \Omega^{0,1}(\tilde{f}^*T\mathcal{G}).$$

The original elliptic complex is just the  $N$ -invariant part of the current one. However,  $\text{Ker}(\bar{\partial}_\Sigma) = T_y\mathcal{G}$  and  $\text{Coker}(\bar{\partial}_\Sigma) = H^{0,1}(\Sigma) \otimes T_y\mathcal{G}$ . Now we vary  $y$  in a component  $\mathcal{G}_{(\gamma)}^2$  to obtain  $e_{(\gamma)}^*T\mathcal{G}$  for the evaluation map  $e_{(\gamma)} : \mathcal{G}_{(\gamma)}^2 \rightarrow \mathcal{G}$  and  $H^{0,1}(\Sigma) \otimes e_{(\gamma)}^*T\mathcal{G}$ .  $N$  acts on both. It is clear that  $(e_{(\gamma)}^*T\mathcal{G})^N = T\mathcal{G}_{(\gamma)}$  as we claim. The obstruction bundle  $E_{(\gamma)}$  we want is the invariant part of  $H^{0,1}(\Sigma) \otimes_{\mathbb{C}} e_{(\gamma)}^*T\mathcal{G}$ , i.e.,  $E_{(\gamma)} = (H^{0,1}(\Sigma) \otimes_{\mathbb{C}} e_{(\gamma)}^*T\mathcal{G})^N$ . We remark that  $E_{(\gamma)}$  can have different dimensions at the different components of  $\mathcal{G}^2$ . We can obviously use the same method to construct a bundle  $E_{\mathcal{G}^k}$  over  $\mathcal{G}^k$  whose fiber is the cokernel of  $\bar{\partial}_f$  for  $f \in \mathcal{G}^k$ .

Consider the evaluation maps  $e_1, e_2, e_{12} : \mathcal{G}^2 \rightarrow \wedge\mathcal{G}$ . Suppose that the local chart of  $\mathcal{G}$  is  $U/G$ . Then, the local chart of  $\wedge\mathcal{G}$  is  $(\sqcup_{g \in G} U^g)/G$ . The local chart of  $\mathcal{G}^2$  is  $(\sqcup_{g_1, g_2 \in G} U^{g_1} \cap U^{g_2})/G$ . The  $e_1, e_2, e_{12}$  are quasi-embeddings and hence  $\mathcal{G}^2$  can be thought of as a quasi-suborbifold of  $\wedge\mathcal{G}$  via three different quasi-embeddings, denoted by  $e_{1*}\mathcal{G}^2, e_{2*}\mathcal{G}^2, e_{12*}\mathcal{G}^2$ . It is clear that  $\mathcal{G}^3 = \mathcal{G}^2 \times_{e_{12}, e_1} \mathcal{G}^2$  and so  $\mathcal{G}^3$  can be viewed as the intersection of the quasi-suborbifolds  $e_{12*}\mathcal{G}^2$  and  $e_{1*}\mathcal{G}^2$ :

$$\begin{array}{ccc} \mathcal{G}^3 & \xrightarrow{\pi_3} & \mathcal{G}^2 \\ \pi_1 \downarrow & & \downarrow e_1 \\ \mathcal{G}^2 & \xrightarrow{e_{12}} & \wedge\mathcal{G} \end{array}$$

The problem is that  $e_{12*}\mathcal{G}^2, e_{1*}\mathcal{G}^2$  are not intersecting transversely in general. Let

$$\nu = (e_{12}\pi_1)^*T \wedge \mathcal{G} / \pi_1^*T\mathcal{G}^2 + \pi_2^*T\mathcal{G}^2$$

where  $\pi_i : \mathcal{G}^3 \rightarrow \mathcal{G}^2$  are the natural projection maps. This is the so-called excess bundle of the intersection. A crucial ingredient in the proof of associativity for the Chen-Ruan product is the bundle formula:

**Theorem 7.1**

$$E_{\mathcal{G}^3} \cong \pi_1^*E_{\mathcal{G}^2} \oplus \pi_2^*E_{\mathcal{G}^2} \oplus \nu.$$

The proof is given in [10] by gluing arguments. We shall call this the *obstruction bundle equation*.

These bundles will play a key role in our definition of the product. It ties in to the transversality question mentioned before via the following lemma, which is an orbifold analogue of the clean intersection formula first described by Quillen [26]:

**Lemma 7.2** *Suppose that  $i_1 : \mathcal{H}_1 \rightarrow \mathcal{G}$  and  $i_2 : \mathcal{H}_2 \rightarrow \mathcal{G}$  are quasi-suborbifolds of  $\mathcal{G}$  forming a clean intersection  $\mathcal{H}_3$ . Then, if  $u \in {}^\alpha K(\mathcal{H}_1)$  we have*

$$i_2^* i_{1*} u = \pi_{2*} [\pi_1^* u \cdot e_K(\nu)]$$

where  $\pi_1 : \mathcal{H}_3 \rightarrow \mathcal{H}_1$  and  $\pi_2 : \mathcal{H}_3 \rightarrow \mathcal{H}_2$  are the natural projections,  $\nu$  is the excess bundle of the intersection, and  $e_K(\nu) \in K(\mathcal{H}_3)$  denotes its Euler class.

Note that this lemma will allow us to connect facts about the geometry of the quasi-suborbifolds  $\mathcal{G}^2$  and  $\mathcal{G}^3$  in  $\wedge \mathcal{G}$  with products, and the obstruction bundle  $E$  plays a key role here. In fact the only information we need about  $E$  is the obstruction bundle equation mentioned above.

**Definition 7.3** *Suppose that  $\mathcal{G}$  is an almost complex orbifold. Let  $\phi$  denote a  $U(1)$ -valued 3-cocycle for  $\mathcal{G}$ , and  $\theta(\phi)$  its inverse transgression, i.e. a  $U(1)$ -valued 2-cocycle for  $\wedge \mathcal{G}$ . For  $\alpha, \beta \in {}^{\theta(\phi)} K(\wedge \mathcal{G})$ , we define*

$$\alpha \star \beta = e_{12*} (e_1^* \alpha \cdot e_2^* \beta \cdot e_K(E_{\mathcal{G}^2})).$$

Our goal is to show that this defines an associative product. However, we want to do this using very general properties of our construction, so that it will be a natural extension of the Chen–Ruan product. We will abbreviate  $E_2 = E_{\mathcal{G}^2}$ ,  $E_3 = E_{\mathcal{G}^3}$ .<sup>5</sup>

As with the twisted Pontryagin product, we must explain in what sense this defines a product. Given that  $e_1, e_2, e_{12} : \mathcal{G}^2 \rightarrow \wedge \mathcal{G}$ , then  $e_i^*(\alpha) \in e_{i*}({}^{\theta(\phi)} K(\mathcal{G}^2))$  for  $i = 1, 2$ . Hence

$$e_1^*(\alpha) \cdot e_2^*(\beta) \cdot e_K(E_{\mathcal{G}^2}) \in e_{1*}^{\theta(\phi)} e_{2*}^{\theta(\phi)} K(\mathcal{G}^2).$$

Now here we must be careful. We use the fact that the twisting cocycle is in fact equal to  $e_{12}^* \theta(\phi) + \delta \mu(\phi)$ . We have a *canonical* isomorphism

$$e_{12*}^{\theta(\phi) + \delta \mu(\phi)} K(\mathcal{G}^2) \cong e_{12*}^{\theta(\phi)} K(\mathcal{G}^2).$$

Next we apply the push forward  $e_{12*}$  to our expression, which now lands in  ${}^{\theta(\phi)} K(\wedge \mathcal{G})$ . Note that the product is clearly commutative; it is associativity that requires a proof.

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<sup>5</sup>We have chosen to work only with even dimensional  $K$ -theory, but this can be readily extended to odd dimensions; the signs work out appropriately after a tedious computation.

**Theorem 7.4** *This product is associative:*

$$(\alpha \star \beta) \star \gamma = \alpha \star (\beta \star \gamma).$$

*Proof.* We pull back the quasi-embeddings  $e_1$ ,  $e_2$  and  $e_{12}$  to  $\mathcal{G}^3$  as follows: let  $\tilde{e}_1 = e_1\pi_1$ ;  $\tilde{e}_2 = e_2\pi_1$ ;  $\tilde{e}_3 = e_2\pi_2$  and  $\tilde{e}_{123} = e_{12}\pi_2$ . We will also make use of an operator defined for triples, namely let  $I_3 : \mathcal{G}^3 \rightarrow \mathcal{G}^3$  be defined by

$$(g_1, g_2, g_3) \mapsto (g_2, g_3, g_3^{-1}g_2^{-1}g_1g_2g_3).$$

Note that  $\tilde{e}_i I_3$  is equal to  $\tilde{e}_{i+1}$  up to conjugation (modulo three); hence they induce the same map in K-theory. Note that  $e_K(E_3)$  is invariant under  $I_3^*$  by construction.

We are now ready to start computing:

$$\begin{aligned} (\alpha \star \beta) \star \gamma &= e_{12*}(e_1^*(\alpha \star \beta) \cdot e_2^*\gamma \cdot e_K(E_2)) \\ &= e_{12*}(e_1^*(e_{12*}(e_1^*\alpha \cdot e_2^*\beta \cdot e_K(E_2))) \cdot e_2^*\gamma \cdot e_K(E_2)). \end{aligned}$$

We can apply the clean intersection formula to this expression, whence we obtain

$$\begin{aligned} (\alpha \star \beta) \star \gamma &= e_{12*}(\pi_{2*}(\pi_1^*(e_1^*\alpha \cdot e_2^*\beta \cdot e_K(E_2)) \cdot e(\nu)) \cdot e_2^*\gamma \cdot e_K(E_2)) \\ &= e_{12*}(\pi_{2*}(\tilde{e}_1^*\alpha \cdot \tilde{e}_2^*\beta \cdot \pi_1^*e_K(E_2) \cdot e_K(\nu)) \cdot e_2^*\gamma \cdot e_K(E_2)) \\ &= e_{12*}\pi_{2*}(\tilde{e}_1^*\alpha \cdot \tilde{e}_2^*\beta \cdot \pi_1^*e_K(E_2) \cdot e_K(\nu) \cdot \pi_2^*e_2^*\gamma \cdot \pi_2^*e_K(E_2)). \end{aligned}$$

Note here that we are using the following property of the pushforward:  $\pi_{2*}(x \cdot \pi_2^*y) = \pi_{2*}(x) \cdot y$ . Thus we have

$$\begin{aligned} (\alpha \star \beta) \star \gamma &= \tilde{e}_{123*}(\tilde{e}_1^*\alpha \cdot \tilde{e}_2^*\beta \cdot \tilde{e}_3^*\gamma \cdot \pi_1^*e_K(E_2) \cdot \pi_2^*e(E_2) \cdot e_K(\nu)) \\ &= \tilde{e}_{123*}(\tilde{e}_1^*\alpha \cdot \tilde{e}_2^*\beta \cdot \tilde{e}_3^*\gamma \cdot e_K(E_3)). \end{aligned}$$

Now we consider

$$\begin{aligned} \alpha \star (\beta \star \gamma) &= (\beta \star \gamma) \star \alpha = \tilde{e}_{123*}(\tilde{e}_1^*\beta \cdot \tilde{e}_2^*\gamma \cdot \tilde{e}_3^*\alpha \cdot e_K(E_3)) \\ &= \tilde{e}_{123*}I_{3*}I_3^*(\tilde{e}_1^*\beta \cdot \tilde{e}_2^*\gamma \cdot \tilde{e}_3^*\alpha \cdot e_K(E_3)) \\ &= \tilde{e}_{123*}(I_3^*\tilde{e}_1^*\beta \cdot I_3^*\tilde{e}_2^*\gamma \cdot I_3^*\tilde{e}_3^*\alpha \cdot I_3^*e_K(E_3)) \\ &= \tilde{e}_{123*}(\tilde{e}_2^*\beta \cdot \tilde{e}_3^*\gamma \cdot \tilde{e}_1^*\alpha \cdot e_K(E_3)) \\ &= \tilde{e}_{123*}(\tilde{e}_1^*\alpha \cdot \tilde{e}_2^*\beta \cdot \tilde{e}_3^*\gamma \cdot e_K(E_3)) \\ &= (\alpha \star \beta) \star \gamma. \end{aligned}$$

Hence our proof of associativity is complete.  $\square$

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