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SPIN MODELS, ASSOCIATION SCHEMES AND $\Delta-$Y TRANSFORMATIONS

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Abstract

In this paper we extended a result given by Francois Jaeger to compute the partition function of a spin model defined on planar graphs (see [10]) to the computation on classes of non-planar graphs. Moreover, we present some results about the classification of spin models in terms of association schemes.

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1 Introduction

In the world of knot and link invariants, we are interested in spin models and their classification in term of association schemes. In [13] V. Jones introduced a construction of a link invariant based on the statistical mechanical concept of spin model. Jones studied only the symmetric case; Kawagoe, Munemasa and Watatani[22] generalized it by removing the symmetry condition.

A spin model is defined on a directed graph G by assigning to each edge e a square matrix $w(e)$ (with complex entries) whose rows and columns are indexed by a given finite set X. Let $c : V(G) \to X$ be an arbitrary coloring of the vertices of G with elements of X. Then with

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each edge e from v to v' is associated the $(c(v), c(v'))$ entry of $w(e)$. The product over all edges of these numbers is called the weight of the coloring c, and the sum of weights over all possible colorings is called the partition function.

The main idea of Jones is to represent every link by a plane graph with signed edges. Jones defines on this signed graph a spin model for which the matrix associated with any edge is choosen according to signs among two matrices. Then he gives a set of equations which, when satisfied by the two matrices, guarantee that the partition function (after an adequate normalization) is a link invariant.

F. Jaeger studied the relationship of spin models and association schemes (Bose-Mesner algebras) in paper [9]. Those results where the first that showed this relation. association schemes, a structure from Algebraic Combinatorics, are important in several areas of Combinatorics, for example, distance-regular graphs, codes, design theory, etc.

The question about the relationship between spin models and association schemes was finally settled by K. Nomura [19], for the symmetric case: he gave a simple algebraic relation. The second invariance equation on a spin model generates a BM-algebra ($N(W)$) and the third invariance equation tells us that the weight matrix of the spin model belongs to the BM-algebra of Nomura. The non-symmetric case is treated by Jaeger, Matsumoto, and Nomura in [12]. In particular every non-symmetric spin model generates a dual pair of BM-algebras. We are interested in knowing when there exists a spin model for a dual pair of BM-algebras. Afterwards F. Jaeger, by using a topological point of view, showed another relation between spin models and Bose-Mesner algebras [11]. Jaeger defines the partition function Z of a spin model on a plane tangle diagram. Z converts the vertical and horizontal products of tangles into the ordinary and Hadamard products of matrices, and the rotation through angle $\frac{\pi}{2}$ into a duality map.

On the other hand, in [10] Jaeger computed the partition function by using only local transformations on graphs. For this one assumes that all matrices assigned to the edges of a graph belong to a given BM-algebra. This is always possible by using the BM-algebra of Nomura. If a graph contains loops, pendant edges, edges in series or in parallel, one can easily compute the partition function on a reduced graph for which the assignment of matrices to edges has been modified in an appropriate way consistent with the reductions. In particular if a graph is series-parallel, the partition function can be computed by an iterative process. Moreover, Jaeger extended the concept of series-parallel evaluation to all plane graphs by considering also the $\Delta-$Y transformations. The evaluation process relies on Epifanov's Theorem on Δ-Y reducibility of planar graphs and the fact that all matrices assigned to the edges belong to a BM-algebra (*exactly triple regular*). Moreover, we give a simple extension to important classes of nonplanar graphs.

2 BM-Algebras

In this work we let $X = \{1, \ldots, n\}$ and $\mu(X)$ will denote the set of square matrices of order n with complex entries. For any A, B in $\mu(X)$, AB denotes the usual product of matrices, I is the identity matrix of the usual product, $A \circ B$ denotes the Hadamard product of matrices, J is the identity matrix with the Hadamard product, and A^T the traspose of A. A *d-class association scheme* on X is a finite family of {0,1}-matrices of order n, $\{A_i | i = 0, \ldots, d\}$ such that the following properties hold:

B0) $A_i \circ A_j = \delta_{ij} A_i$

B1) $\sum_{i=0}^{d} A_i = J$

B2) $A_0 = I$

B3) For each $i \in \{0, \ldots, d\}$ there exists $\sigma(i) \in \{0, \ldots, d\}$ such that $A_i^T = A_{\sigma(i)}$

B4) $A_i A_j = A_j A_i = \sum_{k=0}^{d} p_{ij}^k A_k$.

The numbers p_{ij}^k are called the intersection parameters and they must satisfy $p_{ij}^k = p_{ji}^k$. The numbers $n_i = p_{i\sigma(i)}^0$, for $i = 0, \ldots, d$ are usually called the valencies of the association scheme.

From (B1) we see that the matrices A_i are linearly independent and by (B2)-(B4) we see that they generate a commutative (d +1)-dimensional algebra $\mathbb{A}$. This algebra is called the Bose-Mesner algebra of the association scheme (BM-algebra).

The matrices $A_0, A_1, \ldots, A_d$ are called the canonical basis of the BM-algebra. Since the matrices A_i commute, they can be diagonalized simultaneously, that is, there exists a unitary matrix U such that for each $A \in \mathbb{A}$, U^*AU is a diagonal matrix.
We have $\mathbb{C}^n = V_0 \bigoplus V_1 \bigoplus \cdots \bigoplus V_d$ where each V_i is a common eigenspace

of $A_0, A_1, \ldots, A_d$. Let E_i be the orthogonal projection $\mathbb{C}^n \to V_i$ expressed in matrix form with respect to the canonical basis. Then these matrices satisfy:

A0) $E_iE_j = \delta_{ij}E_i$

A1) $\sum_{i=0}^{d} E_i = I$

A2) $E_0 = \frac{1}{n}J$

A3) For each $i \in \{0, \ldots, d\}$ there exists $\sigma(i) \in \{0, \ldots, d\}$ such that $E_i^T = \bar{E}_i = E_{\sigma(i)}$

A4) $E_i \circ E_j = \frac{1}{n}\sum_{k=0}^{d} q_{ij}^k E_k$.

The numbers q_{ij}^k are called the Krein parameters and the integer numbers $m_i = dimV_i = rank\ E_i$ are called the multiplicities of the association scheme.
The matrices $E_0, E_1, \ldots, E_d$ are a basis of orthogonal idempotents for the BM-algebra $\mathbb{A}$.
Let $\mathcal{P}$ and $\frac{1}{n}\mathcal{Q}$ be the matrices relating our two bases for $\mathbb{A}$, then:

$$A_j = \sum_{i=0}^{d} P_j(i)E_i$$

$$E_j = \frac{1}{n}\sum_{i=0}^{d} Q_j(i)A_i$$

$\mathcal{P}$ and $\mathcal{Q}$ are called the first eigenmatrix and the second eigenmatrix respectively. It is easy to see that

$$A_iE_j = P_i(j)E_j$$

and

$$A_i \circ E_j = \frac{1}{n}Q_j(i)A_i$$

The first equation tells us that each column vector of E_j is an eigenvector of A_i with eigenvalue $P_i(j)$, and the second equation tells us that E_j is constant in each entry where A_i is different from zero, moreover, we can consider informally that each column vector of A_i is an "eigenvector" of E_j with the Hadamard product and $\frac{1}{n}Q_j(i)$ is the respective "eigenvalue".

Now we introduce several notions of isomorphisms for BM-algebras. Let $\mathbb{A}, \mathbb{B}$ be two BM-algebras and $\psi : \mathbb{A} \to \mathbb{B}$ a linear isomorphism.

Definition 2.1 *ψ is a BM-isomorphism if and only if $\psi(AB) = \psi(A)\psi(B)$ and $\psi(A \circ B) = \psi(A) \circ \psi(B)$, for all $A, B \in \mathbb{A}$.*

A classical example is $\psi(A) = P^{-1}AP$, where P is a permutation matrix, but not all BM-isomorphism can be obtained in this form. In fact this type of BM-isomorphism is called a combinatorial isomorphism. There exist examples of BM-isomorphisms that are not combinatorial isomorphisms.

Definition 2.2 *ψ is a duality if and only if $\psi(AB) = \psi(A) \circ \psi(B)$ and $\psi(A \circ B) = \frac{1}{n}\psi(A)\psi(B)$, for all $A, B \in \mathbb{A}$.*

Remark 2.3 *It is easy to prove that $\frac{1}{n}\psi^{-1}$ is a duality from $\mathbb{B}$ to $\mathbb{A}$.*

Usually we call $(\mathbb{A}, \mathbb{B})$ a dual pair of BM-algebras if there exists a *duality* from $\mathbb{A}$ to $\mathbb{B}$.

2.1 Some results on dual pairs of BM-algebras

In this part we present some properties that every dual pair of BM-algebras must have. We let $\bullet$ denote the composition map.

Lemma 2.4 *The next statements always hold.*

i) The composition between a duality and a BM-isomorphism is a duality,

ii) The composition between two dualities (under certain normalization) is a BM-isomorphism.

Proof: Part (i) is clear. For (ii) take $(\mathbb{A}, \mathbb{B})$ and $(\mathbb{B}, \mathbb{D})$ two dual pairs of BM-algebras with dualities ψ_1 and ψ_2 respectively. It is easy to check that $\frac{1}{n}\psi_2 \bullet \psi_1$ is a BM-isomorphism from $\mathbb{A}$ to $\mathbb{D}$. ■

Proposition 2.5 *Up to composition with a BM-isomorphism, duality between BM-algebras is unique.*

Proof: The next diagram is commutative.

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\psi_1"] \arrow[dr, "\psi_2"'] & \mathbb{B} \arrow[d, "\psi"] \\ & \mathbb{B} \end{tikzcd}$$

Where $\psi = \psi_2 \bullet \psi_1^{-1}$ and by lemma 2.4$[ii]$, ψ is a BM-isomorphism. ■

Proposition 2.6 *Let* $(\mathbb{A}, \mathbb{B})$ *be a dual pair of BM-algebras with duality* ψ*. Then the following statements are true.*

i) The intersection numbers of $\mathbb{A}$ *are equal to the Krein parameters of* $\mathbb{B}$ *and viceversa.*

ii) The first eigenmatrix of $\mathbb{A}$ *(under certain rearrangement) is equal to the second eigenmatrix of* $\mathbb{B}$ *and viceversa.*

iii) Any duality commutes with the trasposition map.

Proof: For any matrix M, $tr(M)$ will denote the trace of M and $sum(M)$ will denote the sum of all entries of M.
Let $\{A_0, A_1, \ldots, A_d\}$ be the canonical basis for $\mathbb{A}$. It is easy to prove that $\{\psi(A_0), \psi(A_1), \ldots, \psi(A_d)\}$ is the basis of orthogonal idempotents for $\mathbb{B}$ and by the definition of duality we have (i). Similarly, if $\{E_0, E_1, \ldots, E_d\}$ is the basis of orthogonal idempotents for $\mathbb{B}$ it is easy to check that $\{\psi(E_0), \psi(E_1), \ldots, \psi(E_d)\}$ is the canonical basis of $\mathbb{B}$. Now by the relation $\psi(A_iE_j) = \psi(P_i(j)E_j) = P_i(j)\psi(E_j) = \psi(A_i) \circ \psi(E_j)$ we have (ii). To prove (iii) recall that the multiplicities of the association scheme satisfy $q^0_{ij} = m_i\delta_{i\sigma(j)}$ and $sum(E_i) = n\delta_{i,0}$ (see [2],[3]), in fact $q^0_{ij} \neq 0$ if and only if $\sigma(i) = j$. Since $tr(E_i \circ E_j) = sum(E_iE_j^T)$ and suppose that $\psi(A_i)^T = E_{T(i)}$ we have, $nq^0_{iT(i)} = sum(\psi(A_i) \circ \psi(A_i)^T) = tr(\psi(A_i)\psi(A_i)) = tr(\psi(A_i)) = tr(E_{\psi(i)}) = m_{\psi(i)} \neq 0$. ∎

The splitting field K of an association scheme is

$$\mathbb{Q}(P_i(j)(0 \leq i, j \leq d)) = \mathbb{Q}(Q_i(j)(0 \leq i, j \leq d)).$$

Where $\mathbb{Q}$ denotes the set of rational numbers.

Theorem 2.7 *[21] If the Krein parameters are all rational, then the splitting field* K *is contained in a cyclotomic number field.*

Corollary 2.8 *The eigenvalues of a dual pair belong to a cyclotomic field. In fact they are in its integer ring.*

Proof: Since the Krein parameters are integers, by proposition 2.6[i], we can apply the above theorem. ∎

Let $\psi : \mathbb{A} \to \mathbb{A}$ be a *duality*, we shall say that ψ is a *strong duality* if $\psi^2 = n\tau_{\mathbb{A}}$ where $\tau_{\mathbb{A}}$ is the trasposition map on $\mathbb{A}$. And in this case we shall say that $\mathbb{A}$ is a *self-dual* BM-algebra.

3 Spin models

Let W be a square matrix with non-zero complex entries, we introduce $\underline{W}$ (the floor matrix of W) defined by $\underline{W}(i,j) = \frac{1}{W(i,j)}$.
A spin model is a pair $S = (X, W)$ where W is a $n \times n$ matrix with non-zero complex entries, such that:

(**I**) $W \circ I = aI, \;\; JW = WJ = Da^{-1}J, \;\; J\underline{W} = \underline{W}J = DaJ$

(**II**) $W\underline{W}^T = nI$

(**III**) For every $i, j, k \in X$,
$\sum_{x\in X} W(x,i)W(x,j)\underline{W}(x,k) = \sqrt{n}W(i,j)\underline{W}(k,j)\underline{W}(i,k)$

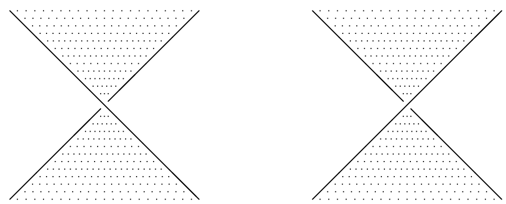
In [13] Jones used the concept of spin models to construct invariants of links and knots, but he treats only the symmetric case. Kawagoe, Munemasa and Watatani established the general case in [22]. The main idea is to represent any connected diagram $\vec{L}$ of an oriented link as a signed planar graph $G(\vec{L})$ as follows. Color the regions in black and white so that the infinite region is colored with white and adjacent regions receive different colors. Then $G(\vec{L})$ has one vertex in each black region and one edge for each crossing. Each crossing has a sign $+$ or $-$ which is defined by the next figure. If $e \in E(G(\vec{L}))$, we denote the sign of e by $s(e)$, the initial vertex of e by $i(e)$, and the terminal vertex by $t(e)$. If (X, W) is a Spin Model, the partition function is defined by

$$Z(\vec{L}) = \sum_{\sigma} \Pi_e w_{s(e)}(\sigma(i(e)), \sigma(t(e)))$$

Where

$$w_{s(e)}(i,j) = \begin{cases} W(i,j) & \text{if } s(e) = + \\ \underline{W}^T(i,j) & \text{if } s(e) = - \end{cases}$$

The product is taken over all edges and the sum is taken over all mappings σ from the set of vertices to X.



Finally, the following result can be found in [22], for the symmetric case see [13].

Proposition 3.1 *For a complex number* $\mathbf{a}$ *and* $\vec{L}$ *a link diagram, the number*

$$Z(\vec{L}) = \mathbf{a}^{-T(\vec{L})} n^{-\frac{|V(G(\vec{L}))|}{2}} Z(V(G(\vec{L})), w_s)$$

is a link invariant.

Where $T(\vec{L})$ is the Tait number of the link diagram.

When the matrix is symmetric, $Z(G(\vec{L}), w_s)$ *does not depend on the orientation of* $G(\vec{L})$*. In this case we have the definition of spin models given in [13], and we shall call this model a symmetric spin model.*

We shall say that a square matrix W with non-zero complex entries is a type *II* matrix if it satifies one of the following conditions, each of which is equivalent to condition (*II*) above.

i) $\sum_{i=1}^{n} \frac{W(j,i)}{W(k,i)} = n\delta(j,k) \quad \forall j,k \in X.$

ii) $\sum_{i=1}^{n} \frac{W(i,j)}{W(i,k)} = n\delta(j,k) \quad \forall j,k \in X.$

In [19] Nomura introduced a BM-algebra for a spin model, the construction of this algebra required only the second invariance equations and he treated only the symmetric case. Jaeger, Matsumoto and Nomura in [12] generalized this result. We present part of this work.

Let W a type *II* matrix, we introduce for each $(i,j) \in X \times X$ two column n-dimensional vectors Y_{ij} and Y'_{ij} where the k-entry is equal to:

a) $Y_{ij}(k) = \frac{W(k,i)}{W(k,j)}$

b) $Y'_{ij}(k) = \frac{W(i,k)}{W(j,k)}$

Moreover, the star-triangle equation can be written as

$$(ST) \qquad WY_{ij} = \frac{\sqrt{n}}{W(j,i)} Y_{ij}$$

Let

$$N(W) = \{A \in M_{n\times n}(\mathbb{C}) | Y_{ij} \ \ is \ \ an \ \ eigenvector \ \ of \ \ A \ \ \forall i,j\}$$

and

$$N'(W) = \{A \in M_{n\times n}(\mathbb{C}) | Y'_{ij} \ \ is \ \ an \ \ eigenvector \ \ of \ \ A \ \ \forall i,j\}$$

Theorem 3.2 *[12]*
$(N(W), N'(W))$ is a dual pair of BM-algebras. If W is a symmetric matrix, we have $N(W) = N'(W)$ and $N(W)$ is a self-dual BM-algebra.

The star-triangle equation (ST) tell us that $W \in N(W)$.

Proposition 3.3 *[12] The following properties are equivalent:*

(i) $W \in N(W)$

(ii) *W satisfies the star-triangle equation for some $D \in \mathbb{C}^*$*

Inverse Problem: We are interested in the inverse problem; suppose we have a dual pair of BM-algebras $(\mathbb{A}, \mathbb{B})$ when is it the case that there exists a type *II* matrix W such that $N(W) = \mathbb{A}$ and $N'(W) = \mathbb{B}$.

3.1 Some results about the inverse problem and examples

Let $\mathbf{H}$ be a cyclic group of order m generated by g.
The group association scheme on $X = \mathbf{H}$ is generated by the matrix A_1, for $x, y \in \mathbf{H}$.

$$A_1(x,y) = \begin{cases} 1 & \text{if } xy^{-1} = g \\ 0 & \text{otherwise} \end{cases}$$

This association scheme is usually denoted by $\mathcal{X}(\mathbf{H})$. The matrices of the association scheme are $A_i = A_1^i$, for $i = 0, \ldots, n-1$ Let $\mathcal{X}(\mathbf{H}) =$

$\langle A_0, A_1, \ldots, A_{n-1}\rangle = \langle A_1\rangle$ be the BM-algebra of the group association scheme.

The idempotents matrices $E_0, E_1, \ldots, E_{n-1}$ of $\mathbb{A}$ are given by

$$A_i = \sum_{j=0}^{n-1} \zeta^{ij} E_j$$

where ζ is a primitive n-th root of unity. The first eigenmatrix is $P = (\zeta^{ij}),\ \ 0 \le i \le n-1,\ \ 0 \le j \le n-1.$

$$\psi : \mathbb{A} \to \mathbb{A},\ \ \psi(A_i) = nE_i$$

defines a strong duality, $\mathcal{X}(\mathbf{H})$ is a self-dual BM-algebra.
For any abelian group $\mathbf{H}$ of finite order, we have that $\mathbf{H} = \mathbf{H_1} \times \mathbf{H_2} \times \cdots \times \mathbf{H_m}$ is a direct product of cyclic groups. Then $\mathcal{X}(\mathbf{H}) = \mathcal{X}(\mathbf{H_1}) \otimes \mathcal{X}(\mathbf{H_2}) \otimes \cdots \otimes \mathcal{X}(\mathbf{H_m})$ where $\otimes$ is the Kroenecker product of matrices. If $P_1, P_2, \ldots, P_m$ are the first eigenmatrices of $\mathcal{X}(\mathbf{H_1}), \mathcal{X}(\mathbf{H_2}), \ldots, \mathcal{X}(\mathbf{H_m})$ respectively, then $P = P_1 \otimes P_2 \otimes \cdots \otimes P_m$ is the first eigenmatrix of $\mathcal{X}(\mathbf{H})$.

Theorem 3.4 *Let P the first eigenmatrix of the BM-algebra $\mathbb{A}$. P is a type II matrix if and only if $\mathbb{A}$ is the BM-algebra of some abelian group. Moreover $N(P) = \mathbb{A}$.*

Proof: Suppose P is a type II matrix, then from the orthogonal relation (see [2] or [3]) of the first eigenmatrix we can see that $n_i = 1$ for any i, hence all elements of the canonical basis are permutation matrices and are closed under the usual product. The canonical basis is an abelian group. The other implication it is very simple. For the last part we can assume that $\mathbf{H}$ is a cyclic group, it is easy to see that $N(P) = \mathbb{A}$. For the general case, if $\mathbf{H}$ is an abelian group then $\mathbf{H} = \mathbf{H_1} \times \mathbf{H_2} \times \cdots \times \mathbf{H_m}$ is a direct product of cyclic groups, $P = P_1 \otimes P_2 \otimes \cdots \otimes P_m$, the first eigenmatrix of $\mathbb{A}$, is a type II matrix and $N(P) = N(P_1) \otimes N(P_2) \otimes \cdots \otimes N(P_m) = \mathbb{A}$. ■

In this part we present some examples of type II matrices and their BM-algebras.

1. For the Hadamard matrix

$$W = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

we have that

$$N(W) = \langle A_0, A_1, A_2, A_3 \rangle$$

where

$$A_0 = I \quad A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

2. For $w = \exp(\frac{2\pi i}{5})$, let

$$W = \begin{pmatrix} 1 & w & w^{-1} & w^{-1} & w \\ w & 1 & w & w^{-1} & w^{-1} \\ w^{-1} & w & 1 & w & w^{-1} \\ w^{-1} & w^{-1} & w & 1 & w \\ w & w^{-1} & w^{-1} & w & 1 \end{pmatrix}$$

we have that

$$N(W) = \langle A_0, A_1, A_2, A_3, A_4 \rangle$$

where

$$A_0 = I \quad A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

All examples above generate BM-algebras with valency one. But this is not true in general as the following example shows.

3. From Proposition 5 [4], let $\varepsilon \in \{1, -1\}$ and $a \in \mathbb{C}^*$

$$W = \begin{pmatrix} a & -\varepsilon a^{-1} & -a & -\varepsilon a^{-1} \\ -\varepsilon a^{-1} & a & -\varepsilon a^{-1} & -a \\ -a & -\varepsilon a^{-1} & a & -\varepsilon a^{-1} \\ -\varepsilon a^{-1} & -a & -\varepsilon a^{-1} & a \end{pmatrix}$$

Take $a = exp(\frac{2\pi i}{5})$, we have

$$N(W) = \langle A_0, A_1, A_2 \rangle$$

Where

$$A_0 = I \;\; A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \;\; A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$n_1 = 2.$

The following examples ilustrate the inverse problem.

4. Let $\mathbb{A} = \langle A_0, A_1, A_2, A_3 \rangle$ and $\mathbb{B} = \langle B_0, B_1, B_2, B_3 \rangle$ where

$$A_0 = I \;\; A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \;\; A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$B_0 = I \;\; B_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \;\; B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$(\mathbb{A}, \mathbb{B})$ is a dual pair of BM-algebras, in fact both are sub-BM-algebras of a BM-algebra generated by a cyclic group of order 6. The duality is generated by the duality of this BM-algebra.
The type II matrix produced by the dual pair of BM-algebras is

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & \zeta^2 & -\zeta & 1 & \zeta^2 & -\zeta \\ 1 & \zeta^2 & -\zeta & -1 & -\zeta^2 & \zeta \\ 1 & -\zeta & \zeta^2 & 1 & -\zeta & \zeta^2 \\ 1 & -\zeta & \zeta^2 & -1 & \zeta & -\zeta^2 \end{pmatrix}$$

Where ζ satify that $\zeta^6 = 1$, $\zeta \neq 1$ or -1.

Unfortunately it is not always true that for any dual pair of BM-algebras there exist a type II matrix for the inverse problem as the next example shows.

5. Let $\mathbb{A} = \langle A_0, A_1, A_2 \rangle$ where

$$A_0 = I \quad A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$\mathbb{A}$ is a self-dual BM-algebra, in fact this BM-algebra is the symmetrization of the BM-algebra generated by a cyclic group of order 5. From [20] the only type II matrices of order 5 are:
The cyclic model

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \\ 1 & \zeta^2 & \zeta^4 & \zeta & \zeta^3 \\ 1 & \zeta^3 & \zeta & \zeta^4 & \zeta^2 \\ 1 & \zeta^4 & \zeta^3 & \zeta^2 & \zeta \end{pmatrix},$$

where $\zeta^5 = 1,\ \zeta \neq 1$, and the Potts model

$$\begin{pmatrix} \alpha & 1 & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 & 1 \\ 1 & 1 & \alpha & 1 & 1 \\ 1 & 1 & 1 & \alpha & 1 \\ 1 & 1 & 1 & 1 & \alpha \end{pmatrix},$$

where $\alpha + \alpha^{-1} + 3 = 0$.
The cyclic model generates a group BM-algebra of order 5 and the Potts model produces the BM-algebra generated by $\{I, J - I\}$.

4 Spin models and Δ−Y transformations

4.1 Series-parallel reductions

In [10] F. Jaeger showed that the computation of the partition function can be performed by using series-parallel reductions of graphs, and later extended this approach to all planar graphs by introducing the star-triangle transformation (Δ−Y transformation), and using a well know theorem of Epifanov which states that all planar graphs are Δ−Y reducible to a vertex. We present a generalization for not necessarily planar graphs.

Let (X, W) be a spin model, $N(W) = \mathbb{A}$ and $\mathbf{G}$ a directed graph with non empty edge-set and provided with an arbitrary ordering of its edges. Let us represent every map w from $E(\mathbf{G})$ to $\mathbb{A}$ by the vector

$$(w(e_1), \ldots, w(e_n)) \in \mathbb{A}^m$$

where $m = |E(\mathbf{G})|$.

The mapping $w \to Z(\mathbf{G}, w)$ defines a m-multilinear form on $\mathbb{A}^m$ which we shall denote by $Z_{\mathbf{G}}$. Let us denote by $\mathbb{A}_{\mathbf{G}}$ the tensor product of vector spaces $\bigotimes_{j=1,\ldots,m} \mathbb{A}_j$, where $\mathbb{A}_j$ corresponds to the j-th edge of $\mathbf{G}$ and is identified with $\mathbb{A}$ for $j = 1, \ldots, m$. We shall identify $Z_{\mathbf{G}}$ with the linear form on $\mathbb{A}_{\mathbf{G}}$ which takes the value $Z(\mathbf{G}, w)$ on $w(e_1) \otimes \cdots \otimes w(e_m)$ for every mapping w from $E(\mathbf{G})$ to $\mathbb{A}$.

Let $C(G, 1)$ (respectively: $D(G, 1)$) be the graph obtained from G by contracting (deleting) the edge e_1. Thus $\mathbb{A}_{C(G,1)}$ and $\mathbb{A}_{D(G,1)}$ are obtained from $\mathbb{A}_G$ by deleting the first factor. For every w in $\mathbb{A}_{C(G,1)} \simeq \mathbb{A}_{D(G,1)}$ we have

$$Z_G(I \otimes w) = Z_{C(G,1)}(w)$$

$$Z_G(J \otimes w) = Z_{D(G,1)}(w)$$

The rules for the computation of Z_G are as follows. Let $R(G,1)$ be the graph obtained from G by reversing the orientation of e_1, then

$$Z_G = Z_{R(G,1)} \bullet (\tau \otimes Id)$$

where $\bullet$ denotes the composition of maps and Id denotes the identity map acting on the appropriate factors.
Let $\theta, \theta^* : \mathbb{A} \to \mathbb{C}$ two linear forms defined by

$$I \circ M = \theta(M)I \qquad JM = MJ = \theta^*(M)J$$

for every matrix $M \in \mathbb{A}$.
Let $\mu, \mu^* : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ defined by

$$\mu(M \otimes N) = MN \qquad \mu^*(M \otimes N) = M \circ N$$

for every $M, N \in \mathbb{A}$. Now we have that:

If G has no edges	*then*	$Z_G = n^{\|V(G)\|}$
If e_1 is a loop	*then*	$Z_G = Z_{D(G,1)} \bullet (\theta \otimes Id)$
If e_1 is a pendant edge	*then*	$Z_G = Z_{C(G,1)} \bullet (\theta^* \otimes Id)$
If e_1, e_2 form a series pair	*then*	$Z_G = Z_{C(G,1)} \bullet (\mu \otimes Id)$
If e_1, e_2 form a parallel pair	*then*	$Z_G = Z_{D(G,1)} \bullet (\mu^* \otimes Id)$

A graph G is series-parallel (see [23]) if and only if it can be reduced to a graph with no edges by repetead applications of one of the following types of transformations which we call *extended series-parallel reductions*:

(i) Deletion of a loop.

(ii) Contraction of a pendant edge.

(iii) Contraction of one of the edges of a series pair.

(iv) Deletion of one of the edges of a parallel pair.

Note: A graph is series-parallel if an only if it has no K_4 minor.

Proposition 4.1 *[10] If G is a connected series-parallel graph. Z_G is a composition $\rho_0 \bullet \rho_1 \cdots \bullet \rho_k$, where ρ_0 is scalar multiplication by n and each of $\rho_1, \ldots, \rho_k$ corresponds to the action of one of the maps $\tau, \theta, \theta^*, \mu, \mu^*$ on some factors of a tensor product of copies of $\mathbb{A}$.*

4.2 Star and triangle projections in association schemes

Let $\mathbb{A}$ be a BM-algebra on X and $\mathcal{S}$ be the complex vector space with basis X. We shall provide $\mathcal{S}\otimes\mathcal{S}\otimes\mathcal{S}$ with a positive definite Hermitian form $\langle\ ,\ \rangle$ such that $\{\alpha\otimes\beta\otimes\gamma|\alpha,\beta,\gamma\in X\}$ is an orthonormal basis. We define the linear maps π (*star projection*) and π^* (*triangle projection*) from $\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A}$ to $\mathcal{S}\otimes\mathcal{S}\otimes\mathcal{S}$ by

$$\pi(A\otimes B\otimes C)=\sum_{\alpha,\beta,\gamma\in X}\Big(\sum_{x\in X}A(x,\alpha)B(x,\beta)C(x,\gamma)\Big)\alpha\otimes\beta\otimes\gamma \quad (S)$$

$$\pi^*(A\otimes B\otimes C)=\sum_{\alpha,\beta,\gamma\in X}A(\beta,\gamma)B(\gamma,\alpha)C(\alpha,\beta)\alpha\otimes\beta\otimes\gamma \qquad (T)$$

Then W satisfies the star-triangle equation(ST) if and only if

$$\pi(W^T\otimes W\otimes\underline{W}^T)=\sqrt{n}\pi^*(\underline{W}^T\otimes\underline{W}\otimes W).$$

Jaeger introduced the concept of *feasible triple* and *dually feasible triple.*

For $i,j,k,u,v,w\in\{0,\dots,d\}$

- (u,v,w) is a *feasible triple* if and only if $p_{uv}^{\sigma(w)}\neq 0$
- (i,j,k) is a *dually feasible triple* if and only if $q_{ij}^{\sigma(k)}\neq 0$

We shall denote by $\mathcal{F}(\mathbb{A})$ the set of feasible triples and $\mathcal{F}^*(\mathbb{A})$ the set of dually feasible triples.
If $(\mathbb{A},\mathbb{B})$ is a dual pair of BM-algebras we have that $\mathcal{F}(\mathbb{A})=\mathcal{F}^*(\mathbb{B})$ and $\mathcal{F}^*(\mathbb{A})=\mathcal{F}(\mathbb{B})$ which follows from Proposition 2.6[i]. In particular, if $\mathbb{A}$ is self-dual, $\mathcal{F}(\mathbb{A})=\mathcal{F}^*(\mathbb{A})$.
Finally, from [10], for $Y_{ijk}=\pi(E_I\otimes E_j\otimes E_k)$ and $\Delta_{uvw}=\pi^*(A_u\otimes A_v\otimes A_w)$, we have that $\{Y_{ijk}|(i,j,k)\in\mathcal{F}^*(\mathbb{A})\}$ is an orthogonal basis of $Im(\pi)$ and $\{\Delta_{uvw}|(u,v,w)\in\mathcal{F}(\mathbb{A})\}$ is an orthogonal basis of $Im(\pi^*)$.

We shall say that a BM-algebra $\mathbb{A}$ is *triply regular* if and only if there exists a linear map k: $\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A}\to\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A}$ such that the following holds

$$\pi=\pi^*\bullet\mathrm{k}.$$

Similarly, we shall say that a BM-algebra $\mathbb{A}$ is *dually triply regular* if and only if there exists a linear map $\mathrm{k}^*:\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A}\to\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A}$ such that the following holds

$$\pi^*=\pi\bullet\mathrm{k}^*.$$

We now define an *exactly triply regular* BM-algebra as a BM-algebra which is both triply regular and dually triply regular.

The next propositions give a characterization of *triply regular* BM-algebras. Given a BM-algebra $\mathbb{A}$, we define two linear maps k, k* from $\mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A}$ to itself by

$$\mathrm{k}(E_i \otimes E_j \otimes E_k) = \sum_{(u,v,w)\in\mathcal{F}(\mathbb{A}} \frac{\langle Y_{ijk}, \Delta_{uvw}\rangle}{\langle \Delta_{uvw}, \Delta_{uvw}\rangle} A_u \otimes A_v \otimes A_w$$

$$\mathrm{k}^*(A_u \otimes A_v \otimes A_w) = \sum_{(i,j,k)\in\mathcal{F}^*(\mathbb{A}} \frac{\langle \Delta_{uvw}, Y_{ijk}\rangle}{\langle Y_{ijk}, Y_{ijk}\rangle} E_i \otimes E_j \otimes E_k$$

$Im(f)$ denote the image of the function f.

Proposition 4.2 *[10] The following properties are equivalent*

(i) The BM-algebra $\mathbb{A}$ is triply regular

(ii) $Im(\pi) \subseteq Im(\pi^*)$

(iii) The linear map k satisfies $\pi = \pi^* \bullet k$.

Proposition 4.3 *[10] The following properties are equivalent*

(i) The BM-algebra $\mathbb{A}$ is dually triply regular

(ii) $Im(\pi^*) \subseteq Im(\pi)$

(iii) The linear map k^* satisfies $\pi^* = \pi \bullet k^*$.

Proposition 4.4 *[10]*

(i) The BM-algebra $\mathbb{A}$ is exactly triply regular if and only if $Im(\pi) = Im(\pi^*)$

(ii) The triply regular BM-algebra $\mathbb{A}$ is exactly triply regular if and only if $|F(\mathbb{A})| = |F^*(\mathbb{A})|$

(iii) Every self-dual triply regular BM-algebra is exactly triply regular.

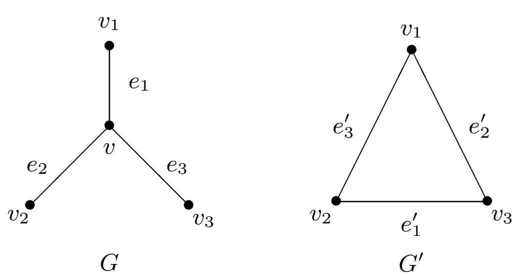
In this context, we are interested in studying the following problem.

Let W a type *II* matrix. Under what conditions $N(W)$ is an *exactly triply regular* BM-algebra.

4.3 $\Delta - \mathrm{Y}$ transformations

In [10] Jaeger extended the computation of the partition function to all planar graphs by using $\Delta-$Y transformations. Here we give a simple extension to important clases of nonplanar graphs.

The $\Delta-$Y transformation is a local transformation on graphs. If v is a degree three vertex (a wye) adjacent to three vertices v_1, v_2 and v_3 by edges e_1, e_2 and e_3 respectively, vertex v and edges e_1, e_2 and e_3 can be deleted and replaced by edges $e'_1 = (v_2, v_3)$, $e'_2 = (v_1, v_3)$ and $e'_3 = (v_1, v_3)$ (see figure). We shall say that G' is obtained from G by a Y$-\Delta$ transformation. If $e'_1 = (v_2, v_3)$, $e'_2 = (v_1, v_3)$ and $e'_3 = (v_1, v_3)$ are edges of G' (a delta or triangle), they can be deleted and replaced by adding a new vertex v adjacent to v_1, v_2 and v_3. We shall say that G is obtained from G' by a $\Delta-$Y transformation. We shall say that G is $\Delta-$Y reducible when G can be reduced to the trivial graph with one vertex by series-parallel reduction, $\Delta-$Y transformations and Y$-\Delta$ transformations.



$\Delta-$Y Transformation

Theorem 4.5 (Epifanov's Theorem) *Every connected plane graph is $\Delta - Y$ reducible.*

We now considerer a (directed) plane graph G and the associated form Z_G.
Let us assume that G is obtained from G' by a $\Delta-$Y transformation. Then[10]

$$Z_G = Z_{G'} \bullet (\mathrm{k} \otimes Id)$$

where k acts on the first three factors of $\mathbb{A}_G = \mathbb{A}_{G'}$.

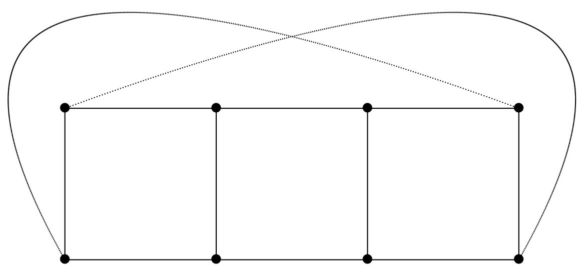
And

$$Z_{G'} = Z_G \bullet (\mathrm{k}^* \otimes Id)$$

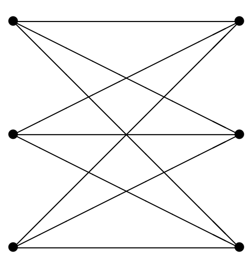
The next theorem gives a way to calculate the partition function on planar graphs.

Proposition 4.6 *[10] Let $\mathbb{A}$ be an exactly triply regular BM-algebra. If G is a connected plane graph, the linear form Z_G on $\mathbb{A}_G$ is a composition $\rho_0 \bullet \rho_1 \cdots \bullet \rho_k$, where ρ_0 is scalar multiplication by n and each of $\rho_1, \ldots, \rho_k$ corresponds to the action of one of the maps $\tau, \theta, \theta^*, \mu, \mu^*, k, k^*$ on some factors of a tensor product of copies of $\mathbb{A}$.*

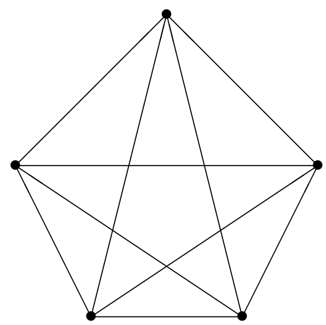
Proposition 4.7 *The graphs $K_{3,3}$, K_5 and V_8 are $\Delta - Y$ reducible.*



V_8



K_{33}



K_5

Let $G = (V, E)$ be a connected graph, and $A \subseteq V$ be a minimal articulation set, that is, the deletion of A produces a disconnected graph, but no proper subset of A has this property. Choose subsets T_1 and T_2 of V, such that (T_1, A, T_2) is a partition of V, and no edge joins a vertex in T_1 to a vertex in T_2. Add a set F of new edges joining each pair of noadjacent vertices in A. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be subgraphs so that $V_i = T_i \cup A$ $(i = 1, 2)$, $E_1 \cup E_2 = E \cup F$ and $G_1 \cap G_2 = (A, E_1 \cap E_2)$ is a complete graph. Then if $|A| = k$ $(1 \leq k \leq 3)$, G is called a k-sum of G_1 and G_2 (see [6]).

A variation on $\Delta-$Y reducibility is to forbid reduction on some distinguished vertices. Specifically, let $T \subset V(G)$ be a set of terminals. A terminal cannot be deleted in a degree-one or series reduction, nor can it be deleted in a Y$-\Delta$ transformation. If a graph with terminals can be reduced to eliminate all non-terminal vertices, then we say it is (Terminal) $\Delta-$Y reducible (For more details see [5], [6], [8]).

Theorem 4.8 *[6] Every 2-connected plane graph with two terminals is $\Delta - Y$ reducible to a single edge.*

Theorem 4.9 (Wagner's Theorem I) *[6] Every connected graph without K_5 minors can be obtained by means of k-sums (k=1,2,3) starting from planar graphs and copies of V_8.*

Theorem 4.10 (Wagner's Theorem II) *[6] Every connected graph without $K_{3,3}$ minors can be obtained by means of k-sums (k =1,2,3) starting from planar graphs and copies of K_5.*

Theorem 4.11 (Gitler's Theorem) *[5] A 2-connected plane graph with three terminals is $\Delta - Y$ reducible to a Δ (or Y) where the vertices are the original three terminals.*

Theorem 4.12 *[5] A graph with no K_5 minor is $\Delta - Y$ reducible.*

Proof: The proof is by an inductive procedure. If the present graph is planar apply Epifanov's theorem. If the graph is V_8 then it is reducible by proposition 4.7. Otherwise G is $k-$sum of G_1 and G_2, where G_2 is a planar graph or V_8. We have several cases depending on k.
If $k = 3$, when G_2 is a planar graph. Consider the vertices of A as terminals of G_2 and apply Theorem 4.11.
If $k = 2$, we have two cases depending on whether G_2 is planar or V_8. When G_2 is a planar graph then by theorem 4.8 applied when considering the vertices of A as terminals in the conclusion. Otherwise G_2 is V_8 and the result follows by proposition 4.7.
If $k = 1$, the conclusion follows by proposition 4.7 when G_2 is V_8 and by theorem 4.5, if G_2 is a planar graph.
We have covered all cases, thus obtaining the result. ■

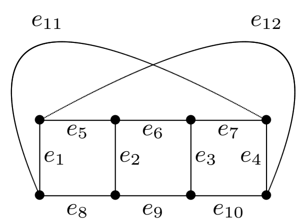
Theorem 4.13 *[5] A graph with no $K_{3,3}$ minor is $\Delta - Y$ reducible.*

Proof: By proposition 4.8 we have that K_5 is reducible and terminal reducible for the cases of one and two terminals. Theorem 4.8 covers the reducibility of the planar two terminal case. By an inductive argument similar to the proof of theorem 4.12 (when $k = 1$ and 2), the result follows. ■

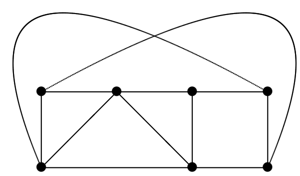
We have a generalization of proposition 4.6.

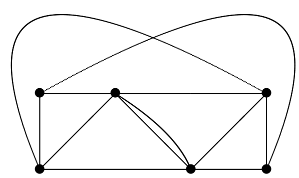
Proposition 4.14 *Let $\mathbb{A}$ be an exactly triply regular BM-algebra. If G is a connected graph without K_5 minors or without $K_{3,3}$ minors, the linear form Z_G on $\mathbb{A}_G$ is a composition $\rho_0 \bullet \rho_1 \bullet \cdots \bullet \rho_k$, where ρ_0 is scalar multiplication by n and each of $\rho_1, \ldots, \rho_k$ corresponds to the action of one of the maps $\tau, \theta, \theta^*, \mu, \mu^*, k, k^*$ on some factors of a tensor product of copies of $\mathbb{A}$.*

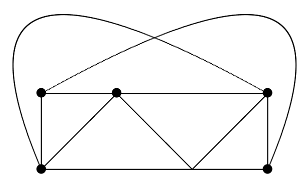
Proof: First, we show a reduction of V_8.

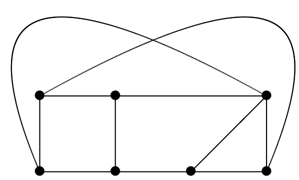


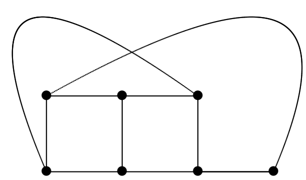
V_8

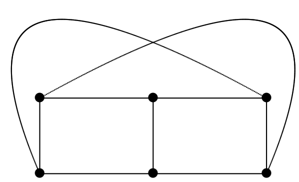


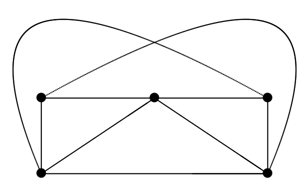


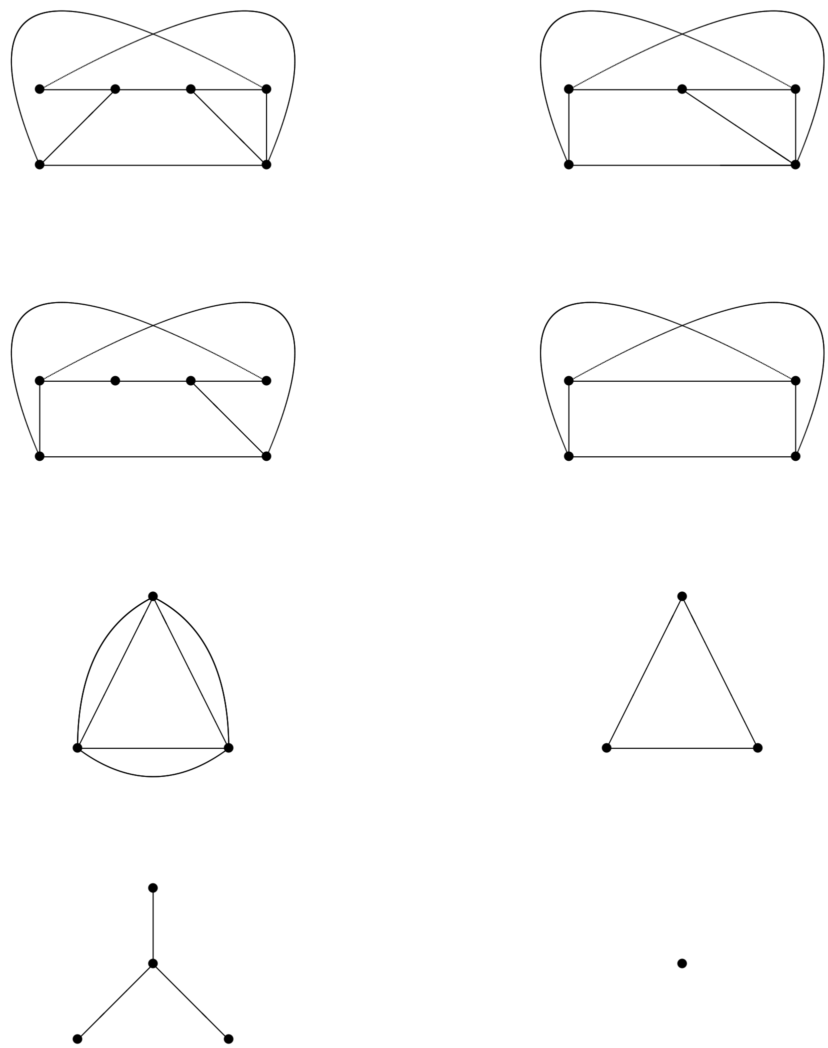












Thus

$$Z(V_8) = n\theta^* \bullet k \bullet \mu^* \bullet k \bullet \mu \bullet k \bullet \mu \bullet k \bullet k^* \bullet \mu \bullet k \bullet \mu^* \bullet k^*.$$

Similarly we have:

$$Z(K_{33}) = n\theta^* \bullet \theta \bullet k^*,$$

and

$$Z(K_5) = n\theta^* \bullet k \bullet \mu^* \bullet k \bullet \mu \bullet k \bullet \mu \bullet k.$$

Where the maps act on the appropriate factors.

If G is a graph with no K_5 minor it is the k-sum ($k = 1, 2, 3$) starting from planar graphs and copies of V_8. We apply theorem 4.6 for each component, similarly if G is a graph with no $K_{3,3}$ minor. ■

In general, we have:

Theorem 4.15 *Let $\mathbb{A}$ be an exactly triply regular BM-algebra. If G is a $\Delta - Y$ reducible connected graph, the linear form Z_G on $\mathbb{A}_G$ is a composition $\rho_0 \bullet \rho_1 \bullet \cdots \bullet \rho_k$, where ρ_0 is scalar multiplication by n and each of $\rho_1, \ldots, \rho_k$ corresponds to the action of one of the maps $\tau, \theta, \theta^*, \mu, \mu^*, k, k^*$ on some factors of a tensor product of copies of $\mathbb{A}$.*

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