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Approximation on arcs and dendrites going to infinity in $\mathbb{C}^n$ (Extended version) *

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Abstract

The Stone-Weierstrass approximation theorem is extended to certain unbounded sets in $\mathbb{C}^n$. In particular, on arcs which are of locally finite length and are going to infinity, each continuous function can be approximated by entire functions.

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1 Introduction

This work is the original version of the paper: *Approximation on arcs and dendrites going to infinity in* $\mathbb{C}^n$ [14]. This version could not be published in its extended form because of size limitations. However, we wish to publish it because it contains a sketch of the proof of Alexander-Stolzenberg's theorem, which we announced in [14], and several lemmas on tangential approximation by polynomial and meromorphic functions which could not be included on [14]. For example, we include a not-so-well-known result of Arakelian in Proposition 3.1.

A famous theorem of Torsten Carleman [7] asserts that for each continuous function f on the real line $\mathbb{R}$ and for each positive continuous function ϵ on $\mathbb{R}$, there exists an entire function g on $\mathbb{C}$ such that

$$|f(x) - g(x)| < \epsilon(x), \text{ for all } x \in \mathbb{R}.$$

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Carleman's theorem was extended to $\mathbb{C}^n$ by Herbert Alexander [3] who replaced the line $\mathbb{R}$ by a piecewise smooth arc going to infinity in $\mathbb{C}^n$ and by Stephen Scheinberg [20] who replaced the real line $\mathbb{R}$ by the real part $\mathbb{R}^n$ of $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. In the present work, we approximate on closed subsets of area zero in $\mathbb{C}^n$ and extend Alexander's theorem to closed connected subsets $\Gamma \subset \mathbb{C}^n$ which are of locally finite *length* and contain no closed curves.

It would be a quite difficult task to give a complete description of all the results before the words of Carleman, Alexander and Scheinberg. Nevertheless, we have added a special section (§4) at the end of this paper, trying to compile the historic results which have drove us to the theorems we are proving in this paper.

Let X be a subset of $\mathbb{C}^n$. X is a continuum if it is a compact connected set. The *length* and area of X are the Hausdorff 1-measure and 2-measure of X respectively. The set X is said to be of finite *length* at a point $x \in X$ if this point has a neighbourhood in X of finite *length*, and X is said to be of locally finite *length* if X is of finite *length* at each of its points. Notice that if X is a set of locally finite *length*, then each compact subset of X has finite *length* (though X itself need not be of finite *length*). We denote the polynomial hull of a compact set X by $\widehat{X}$. The algebra of continuous functions defined on X is denoted by $\mathcal{C}(X)$. Finally, the definition and some properties of the first Čech cohomology group with integer coefficients $\check{H}^1(X)$ are presented in [10] and [23].

2 The Alexander-Stolzenberg theorem

John Wermer laid the foundations of approximation on curves in $\mathbb{C}^n$ and prepared the way for a fundamental result of Gabriel Stolzenberg [21] concerning hulls and smooth curves (for history see [22]). In [2], Alexander comments that Stolzenberg's theorem can be improved to consider *continua of finite length* instead of *smooth curves*. We shall refer to the following version as the Alexander-Stolzenberg Theorem.

Theorem 2.1 (Alexander-Stolzenberg) *Let X and Y be two compact subsets of $\mathbb{C}^n$, with X polynomially convex and $Y \setminus X$ of zero area. Then,*

A *Every continuous function on $X \cup Y$ which is uniformly approximable on X by polynomials is uniformly approximable on $X \cup Y$ by rational functions.*

Suppose, moreover, there exists a continuum $\Upsilon \subset \mathbb{C}^n$ such that $\Upsilon \setminus X$ has locally finite length and $Y \subset (X \cup \Upsilon)$. Then:

B $\widehat{X \cup Y} \setminus (X \cup Y)$ *is (if non-empty) a pure one-dimensional analytic subset of* $\mathbb{C}^n \setminus (X \cup Y)$.

C *If the map $\check{H}^1(X \cup Y) \to \check{H}^1(X)$ induced by $X \subset X \cup Y$ is injective, then $X \cup Y$ is polynomially convex.*

Notice that in this Alexander-Stolzenberg Theorem, locally finite *length* is required only for parts **B** and **C**. Moreover, the set $\Upsilon \setminus X$ may be of infinity *length*; we only need it to have locally finite *length*. On the other hand, the main ideas in the proof of points **A** and **C** are essentially contained in [22, pp. 187-188]). However, we need to introduce several changes due to the new hypotheses.

2.1 Proof of part A of Theorem 2.1

We shall prove part **A** by considering two cases, depending on whether Y is itself of zero area or not. We only need to cite Stolzenberg's ideas when Y has area zero (we obviously replace K by Y in the original paper).

"By the theory of antisymmetric sets (see[15]) it suffices to prove that if $p \in Y \setminus X$ then for each $q \neq p$ in $Y \cup X$ there is a real-valued f, with $f(q) \neq f(p)$, which is uniformly approximable by rational functions on $Y \cup X$."

"Since X is polynomially convex there is a polynomial g such that $g(p) = 1$ and $\Re(g) \leq 0$ on $X \cup \{q\}$. Let c be a real-valued continuous function on $g(Y \cup X)$ which is identically 0 for $\Re(\zeta) \leq \frac{1}{2}$ and with $c(1) = 1$. The following argument of Wermer shows that c is a uniform limit of rational functions on $g(Y \cup X)$."

"Namely, it suffices to prove that any measure μ on $g(Y \cup X)$ which annihilates all uniform limits of rational functions also annihilates c. This will be done if we can show that any such μ is supported on $\{\Re(\zeta) \leq \frac{1}{2}\}$. But Y has area zero and g is a polynomial, so $g(Y)$ has area zero and, hence, $\int (z-\zeta)^{-1} d\mu(z) = 0$ for almost all ζ with $\Re(\zeta) > \frac{1}{2}$. Therefore, by Fubini's Theorem, for almost all open disks $\Delta \subset \{\Re(\zeta) > \frac{1}{2}\}$, if ∂=the boundary of Δ then

$$0 = \frac{-1}{2\pi i} \int_{\partial} d\zeta \int \frac{d\mu(z)}{z-\zeta} =$$

$$= \int \frac{d\mu(z)}{2\pi i} \int_{\partial} \frac{d\zeta}{\zeta - z} = \int \chi_{\Delta}(z) d\mu(z),$$

where χ_Δ is the characteristic function of Δ. It follows that $\mu = 0$ on $\{\Re(\zeta) > \frac{1}{2}\}$."

"Hence c is a uniform limit of rational functions on $g(Y \cup X)$ and, hence, $f = c \circ g$ is a continuous real-valued function on $Y \cup X$, with $f(q) \neq f(p)$, which is uniform limit of rational functions."

This settles part A when Y has area zero.

Now suppose we merely know that $Y \setminus X$ has zero area. Let f be a continuous function on $X \cup Y$ which is uniformly approximable on X by polynomials and let $\epsilon > 0$. There exists a polynomial p such that $|f - p| < \epsilon/2$ on X. Since X is polynomially convex, it has a fundamental system of neighbourhoods which are polynomial polyhedra [29, Lemma 7.4]. From the continuity of $f - p$, it follows that $|f - p| < \epsilon/2$ on some polynomial polyhedron $\widetilde{X}$ containing X in its interior. Extend $p|_{\widetilde{X}}$ to a continuous function $\tilde{p}$ on $Y \setminus \widetilde{X}$ so that $|f - \tilde{p}| < \epsilon/2$ on $\widetilde{X} \cup Y$. It is easy to see that the closure K of $Y \setminus \widetilde{X}$ has area zero because $K \subset Y \setminus X$, and so $\widetilde{X} \cup Y$ can be written as the union of $\widetilde{X}$ with a compact set of area zero K; it follows from the first part of this proof that there is a rational function h such that $|\tilde{p} - h| < \epsilon/2$ on $\widetilde{X} \cup Y$. By the triangle inequality, $|f - h| < \epsilon$ on $X \cup Y$ which concludes the proof of **A**.

2.2 Deduction of part C from part B in Theorem 2.1

Here we also need to replace Lemma 1 of [22, p. 188]) by the following proposition.

Proposition 2.2 *Let X and Y be two compact subsets of $\mathbb{C}^n$, with X rationally convex and $Y \setminus X$ of zero area. Then, $X \cup Y$ is rationally convex. If, moreover, X is polynomially convex, then given a point p in the complement of $X \cup Y$, there is a polynomial f such that $f(p) = 0$, $0 \notin f(X \cup Y)$ and $\Re f(z) < -1$ for $z \in X$.*

Proof: The set X has a fundamental system of neighbourhoods which are rational polyhedra [21, p. 283] or [23]. Given a point p in the complement of $X \cup Y$, choose a compact rational polyhedron $\widetilde{X}$ which contains X in its interior, but $p \notin \widetilde{X}$. Along with $\widetilde{X}$, the closure K of $Y \setminus \widetilde{X}$ is also rationally convex because it has zero area (notice that $K \subset Y \setminus X$ and see [10, p. 71] recalling that projections preserve the zero area condition), so there are two polynomials g and h such that $0 \notin g(K)$, $0 \notin h(\widetilde{X})$ and $g(p) = h(p) = 0$.

The rational function (h/g) is smooth on K, and so $(h/g)(K)$ has zero area. Thus, we can find a complex number $\lambda \notin (h/g)(K)$ whose absolute value $|\lambda|$ is so small that the polynomial $f = h - \lambda g$ has no zeros on $\widetilde{X} \cup K$. Since $X \cup Y \subset \widetilde{X} \cup K$ and $f(p) = 0$, it follows that $X \cup Y$ is rationally convex.

If, in addition, X is polynomially convex, one has just to choose $\widetilde{X}$ to be a compact polynomial polyhedron (see [21] or [29, Lemma 7.4]) and the polynomial h to satisfy $\Re(h) < -1$ on $\widetilde{X}$; and so, for sufficiently small λ, $\Re(f) < -1$ on X.

Now we can conclude the proof of point **C** by citing Stolzenberg's ideas.

"Consider any $p \notin Y \cup X$ and choose an f as in Proposition 2.2. Then f is a continuous invertible function on $Y \cup X$ with a continuous logarithm on X. But, for any T, $\check{H}^1(T)$ is isomorphic to the group of all continuous invertible complex-valued functions of T modulo those with continuous logarithms. Therefore, since $\check{H}^1(Y \cup X) \to \check{H}^1(X)$ is injective, there is a continuous branch of $\log(f)$ on all of $Y \cup X$. However, by part **B**, $\widehat{X \cup Y} \setminus (X \cup Y)$ is (if non-empty) a one-dimensional analytic subset of $\mathbb{C}^n \setminus (Y \cup X)$; so by the argument principle (see, for instance [21, p. 271]) f has no zeroes on $\widehat{X \cup Y} \setminus (X \cup Y)$. Hence, any such p is not in $\widehat{Y \cup X}$, so $Y \cup X$ is polynomially convex."

2.3 Proof of part B of Theorem 2.1

This proof is implicitly contained in Alexander's paper [2], but we need to make several remarks.

Set $\Gamma = X \cup Y$ and suppose there is a point $p \in \widehat{\Gamma} \setminus \Gamma$. From Proposition 2.2, there is a polynomial f such that $f(p) = 0$, $0 \notin f(\Gamma)$ and $\Re(f) < -1$ on X. Fix the compact set $L = f(\Gamma)$, the closed half-plane $H = \{\Re(z) \geq -1/2\}$ and the open set Ω to be the connected component of $\mathbb{C} \setminus L$ which contains the origin. Alexander's arguments [2] can be slightly modified to show that $\widehat{\Gamma} \cap f^{-1}(\Omega)$ is a one-dimensional analytic subset of $f^{-1}(\Omega)$. Notice that $p \in f^{-1}(\Omega)$. Alexander uses the hypothesis that the set L has finite *length* in the whole plane $\mathbb{C}$. However, his argument works even if we restrict the set L to have finite *length* just in the half-plane H. Indeed, the intersection $L \cap H$ is the polynomial image of the compact set $\Gamma \cap f^{-1}(H)$ of finite *length*; recall that $\Gamma \cap f^{-1}(H) = (Y \setminus X) \cap f^{-1}(H)$ has finite *length* because it is compact and contained in the set $\Upsilon \setminus X$ of locally finite *length*. Now we shall rewrite the preparatory lemmas of [2] with their respective modifications.

Lemma 2.3 (Lemma 1 of [2]) *Let $\mathcal{X}$ be a second-countable topological space, $\mathcal{Y}$ a set, $f : \mathcal{X} \to \mathcal{Y}$ a function, σ a non-zero positive measure on $\mathcal{Y}$ such that if V is open in $\mathcal{X}$, then $f(V)$ is σ-measurable. Then for σ-almost all $y \in \mathcal{Y}$, the image under f of each neighbourhood in $\mathcal{X}$ of each point of $f^{-1}(y)$ has positive σ-measure.*

Lemma 2.4 (Lemma 2 of [2]) *Let D be a closed Jordan domain in $\mathbb{C}$ with boundary of finite length, K a compact subset of ∂D of positive length, Q a polynomially convex set in $\mathbb{C}^n$, f a polynomial in $\mathbb{C}^n$, s a positive integer. Assume that $Q = (f^{-1}(\partial D) \cap Q)^\wedge$ and that $f|_Q$ is at most s-to-1 over points of K (i.e., if $\lambda \in K$ then $f^{-1}(\lambda) \cap Q$ has at most s points). Then $f^{-1}(D^o) \cap Q$ is a (possibly empty) pure 1-dimensional analytic subset of $f^{-1}(D^o)$. Here D^o stands for the interior of D.*

The hypotheses of the previous two lemmas need not be changed, so we refer to their original proofs in Alexander's paper [2, p. 66]. In the following lemmas, the notation $\#(E)$ stands for the number $(\le \infty)$ of elements of the set E.

Lemma 2.5 (Lemma 3 of [2]) *Let Γ be a compact set in $\mathbb{C}^n$ and f a polynomial in $\mathbb{C}^n$ such that $\Gamma \cap f^{-1}(H)$ has finite length. For $x \in \mathbb{R}$, set $N(x) = \#\{p \in \Gamma; \Re f(p) = x\}$. Then $\int_{-1/2}^{\infty} N(x)dx < \infty$.*

For the proof that N is a Lebesgue measurable function, see [18, p. 216].

Proof: By replacing Γ by its homeomorphic image in $\mathbb{C}^{n+1}$ under the mapping $z \mapsto (f(z), z)$, a Lipschitz mapping preserving the finiteness of *length*, we may assume that $f(z) = z_1$, the first coordinate projection.

Let $\epsilon_m \downarrow 0$. Then for each m there exists a finite collection $\mathcal{C}_m$ of closed balls in $\mathbb{C}^n$ each of diameter less than ϵ_m such that $\mathcal{C}_m$ covers $\Gamma \cap f^{-1}(H)$ and if α_m denotes the sum of the diameters of the members of $\mathcal{C}_m$, then $\alpha_m \uparrow \text{length}(\Gamma \cap f^{-1}(H))$. Let $N_m(x) = \#\{B; B \in \mathcal{C}_m \text{ and } x \in \Re z_1(B)\}$. Then clearly $\int_{-\infty}^{\infty} N_m(x)dx = \alpha_m$. Also $\underline{\lim} N_m(x) \ge N(x)$ whenever $x \ge -1/2$; in fact, if $N(x) \ge k$, and $p_1, p_2, \ldots p_k$ are distinct points in $\Gamma \cap (\Re z_1)^{-1}(x)$, then $N_m(x) \geq k$ as soon as $\epsilon_m < \min\{\|p_i - p_j\|; i \neq j\}$. Thus, by Fatou's lemma

$$\int_{-1/2}^{\infty} N(x)dx \leq \underline{\lim} \int_{-1/2}^{\infty} N_m(x)dx \leq$$

$$\leq \underline{\lim}\alpha_m = \text{length}(\Gamma \cap f^{-1}(H)) < \infty.$$

Lemma 2.6 (Lemma 4 of [2]) *Let $I = [0,1]$ be the closed unit interval of the real line and $F \in \mathcal{C}(I)$ be such that $\Re F$ is of bounded variation. Define for $x \in \mathbb{R}$, $N(x) = \#\{t \in I; \Re F(t) = x\}$. Then $\int_{-1/2}^{\infty} N(x)dx < \infty$.*

Proof: Let $\Gamma \subset \mathbb{C}^1$ be the set $\{(\Re F(t), t); t \in I\}$ and take $f(z) = z$ in Lemma 2.5.

Definition. Let L be a closed subset of $\mathbb{C}$. Let Ω_1 and Ω_2 be components of $\mathbb{C} \setminus L$. We shall say that the pair (Ω_1, Ω_2) is *amply adjacent* provided the following holds: there exist real numbers $b > a > -1/2$ and $c_2 > c_1$, and a compact subset $K_1 \subset [a,b]$ of positive *length* such that $[a,b] \times \{c_j\} \subset \Omega_j$ for $j = 1,2$ and $K = (K_1 \times [c_1, c_2]) \cap L$ is a subset of $\partial\Omega_1 \cap \partial\Omega_2$ such that the projection π_1 maps K homeomorphically (and so 1-to-1) onto K_1 (we are identifying $\mathbb{C}$ and $\mathbb{R} \times \mathbb{R}$, so $\pi_1(x,y) = x$).

Lemma 2.7 (Lemma 5 of [2]) *Let $L \subset \mathbb{C}$ be compact and such that $\int_{-1/2}^{\infty} N(x)dx < \infty$ where $N(x) = \#\{q \in L; \Re(q) = x\}$. Then, for every component Ω of $\mathbb{C} \setminus L$ which meets the half-plane H, there exists a finite sequence Ω_0, Ω_1, $\ldots \Omega_m$ of components of $\mathbb{C} \setminus L$ with Ω_0 equal to the unbounded component, $\Omega_m = \Omega$ and (Ω_{j-1}, Ω_j) amply adjacent through rectangles $R_j = [a,b] \times [c_{j-1}, c_j]$ contained in H for $j = 1, 2, \ldots, m$.*

The proof of this lemma is exactly the same as the original one presented by Alexander in his paper [2, p. 69]; he chooses a line segment $[a,b] \times c \subset \Omega$ and uses the fact that $\int_b^a N(x) < \infty$. Thus, we shall have exactly the same result by choosing a horizontal line segment $[a,b] \times c \subset \Omega \cap H$ and following the original proof word for word.

Lemma 2.8 (Lemma 6 of [2]) *Let Γ be a compact subset of $\mathbb{C}^n$ and f a polynomial in $\mathbb{C}^n$. Set $L = f(\Gamma) \subset \mathbb{C}$. Suppose that $\int_{-1/2}^{\infty} N(x)dx < \infty$ for $N(x) = \#\{p \in \Gamma; \Re f(p) = x\}$, and that $L \cap H$ is contained in a continuum L_1 whose intersection $L_1 \cap H$ is of finite length. Let (Ω_1, Ω_2) be a pair of components of $\mathbb{C} \setminus (L \cup L_1)$ which are amply adjacent through a rectangle $R = [a, b] \times [c_1, c_2]$ with $b > a > -1/2$. Suppose $\widehat{\Gamma} \cap f^{-1}(\Omega_i)$ is a (possibly empty) pure 1-dimensional analytic subset of $f^{-1}(\Omega_i)$ for $i = 1$. Then, the same is true for $i = 2$.*

Again, the proof of this lemma follows word for word the original one presented by Alexander in [2, p. 70], we only need to add the new trivial condition $b > a > -1/2$. Alexander proves that $\widehat{\Gamma} \cap f^{-1}(D^o)$ is a pure 1-dimensional analytic subset of $f^{-1}(D^o)$, where D^o is an open set contained in $R \cap \Omega_2$. He deduces then that $\widehat{\Gamma} \cap f^{-1}(\Omega_2)$ is also a pure 1-dimensional analytic set in $f^{-1}(\Omega_2)$ by using Lemma 11 of [22]. This lemma is quite amazing because the component Ω_2 may not be completely contained in H.

The following lemma needs no changes in its hypotheses, so we refer its proof to the original paper [2, p. 71].

Lemma 2.9 (Lemma 7 of [2]) *Let $\Gamma \subset S$ be two compact sets in $\mathbb{C}^n$ and suppose that $\widehat{S} \setminus S$ is a pure 1-dimensional analytic subset of $\mathbb{C}^n \setminus S$. Then so is $\widehat{\Gamma} \setminus S$ (if non-empty).*

We conclude the proof of part **B** of Theorem 2.1 following Alexander's arguments. If the equality $X \cup \Upsilon = \Gamma = X \cup Y$ holds, we let $L = f(\Gamma)$ and Ω be the connected component of $\mathbb{C} \setminus L$ which contains the origin $0 = f(p)$. Apply Lemmas 2.5 and 2.7 to get a sequence Ω_0, $\Omega_1, \ldots \Omega_m = \Omega$. Notice that $\Upsilon \cap f^{-1}(H) = Y \cap f^{-1}(H)$ has finite *length* because it is compact and contained in the set $\Upsilon \setminus X$ of locally finite *length*; so we can take the continuum $L_1 = f(\Upsilon)$ in Lemma 2.8 because $L_1 \cap H = L \cap H$ has finite *length.* We conclude inductively that $\widehat{\Gamma} \cap f^{-1}(\Omega)$ is either empty or a pure 1-dimensional analytic subset of $f^{-1}(\Omega)$, for $L = L_1 \cup L$ and $\widehat{\Gamma} \cap f^{-1}(\Omega_0) = \emptyset$. Hence $\widehat{X \cup Y} \setminus (X \cup Y)$ is analytic (empty or pure 1-dimensional) at an arbitrary point $p \in \widehat{X \cup Y} \setminus (X \cup Y)$.

Now suppose that $X \cup Y$ is strictly contained in $X \cup \Upsilon$. Let $p \in \widehat{\Gamma} \setminus \Gamma$ as above. Modify Υ to obtain Υ_0 such that $p \notin \Upsilon_0$ but Υ_0 is a continuum with $Y \subset X \cup \Upsilon_0$ and $\Upsilon_0 \setminus X$ of finite *length* (say by radial projection to the boundary inside a ball containing p in its interior, centered off Υ, and disjoint from Γ). By the previous paragraph, $\widehat{X \cup \Upsilon_0} \setminus (X \cup \Upsilon_0)$ is a pure

1-dimensional analytic subset of $\mathbb{C}^n \setminus (X \cup \Upsilon_0)$, and so is $\widehat{X \cup Y} \setminus (X \cup \Upsilon_0)$ because of Lemma 2.9. Therefore, the set $\widehat{X \cup Y} \setminus (X \cup Y)$ is analytic (pure 1-dimensional) at p.

An arc Υ is the homeomorphic image of a closed interval of the real line. A direct consequence of the Alexander-Stolzenberg theorem is that every compact arc Υ which is of locally finite *length* everywhere except perhaps at finitely many of its points is polynomially convex and the approximation condition $\mathcal{C}(\Upsilon) = P(\Upsilon)$ holds; notice that Υ may be of infinity *length*.

It is natural to ask whether the connectivity can be dropped in these considerations. In fact, Alexander [4] gave an example of a compact disconnected set Y of finite *length* in $\mathbb{C}^2$ for which $\widehat{Y} \setminus Y$ is *not* a pure one-dimensional analytic subset of $\mathbb{C}^2 \setminus Y$. Thus, the connectivity cannot be dropped in the Alexander-Stolzenberg theorem. Moreover, the following example shows that we cannot finesse Theorem 2.1 by enclosing Y in a continuum of finite *length*, although it is known that one can always construct a compact arc Υ which meets every component of Y (so $Y \cup \Upsilon$ is connected) and $\Upsilon \setminus Y$ is of locally finite *length*.

Example 2.10 *There exists a discrete bounded set in* $\mathbb{C} \setminus \{0\}$ *such that no continuum containing this sequence has finite length.*

Consider the set E consisting of the complex numbers $w_{j,k} = k/j^2 + \sqrt{-1}/j$, for $j = 1, 2, \dots$ and $k = 0, 1, \dots, j$. It is easy to see that E is contained in the disjoint union of the closed balls $\overline{B}_{j,k}$ with respective centers $w_{j,k}$ and radii $\frac{1}{2(j+1)^2}$. Hence, each continuum which contains E has to meet the center and the boundary of each ball $\overline{B}_{j,k}$, so its *length* has to be greater than $\sum_{j>1} \frac{j+1}{2(j+1)^2} = \infty$.

3 Approximation on unbounded sets

Now we shall analyse approximation on closed subsets of $\mathbb{C}^n$ rather than on compact sets. Let Γ be a closed subset of $\mathbb{C}^n$ and $\mathcal{F}$ be a subclass of $\mathcal{C}(\Gamma)$. We say that a function $f : \Gamma \to \mathbb{C}$ can be uniformly (resp. tangentially) approximated by functions in $\mathcal{F}$ if for each positive constant $\epsilon > 0$ (resp. positive continuous function $\epsilon : \Gamma \to \mathbb{R}$) there is $g \in \mathcal{F}$ such that $|f - g| < \epsilon$ on Γ. We are mainly interested in two subclasses $\mathcal{F}$, that which is the restriction to Γ of the class $\mathcal{O}(\mathbb{C}^n)$ of entire functions, and that which is the restriction to Γ of the class of meromorphic functions on $\mathbb{C}^n$ whose singularities do not meet Γ.

Recall that any meromorphic function on $\mathbb{C}^n$ whose singularities do not meet Γ can be expressed as a quotient p/q of two entire functions p and q with $q(z) \neq 0$ for all $z \in \Gamma$, for the second Cousin problem can be solved in $\mathbb{C}^n$. If Γ is compact, then of course, uniform and tangential approximation are equivalent; and we may even replace the classes of entire and meromorphic functions on $\mathbb{C}^n$ by the classes of polynomials and rational functions respectively.

We say that Γ is a set of uniform (resp. tangential) approximation by functions in the class $\mathcal{F}$ if each $f \in \mathcal{C}(\Gamma)$ can be uniformly (resp. tangentially) approximated by functions in $\mathcal{F}$. Of course, as we have defined them, such sets Γ cannot have any interior. In the literature, one also finds a more generous notion of sets of uniform or tangential approximation, which allows some sets having interior.

Before going any further, we should point out that, sets of uniform approximation and sets of tangential approximation by holomorphic functions are in fact the same. This was proved by Norair Arakelian in his doctoral dissertation [5] in $\mathbb{C}$. His proof works verbatim in $\mathbb{C}^n$. Since this fact is not well known and the proof is short we include it.

Proposition 3.1 (Arakelian) *Let Γ be a closed subset of $\mathbb{C}^n$ and let $\mathcal{F}$ be either the class of functions holomorphic on Γ or the class of entire functions. Then, Γ is a set of uniform approximation by functions in the class $\mathcal{F}$ if and only if it is a set of tangential approximation by functions in the same class.*

Proof: Suppose Γ is a set of uniform approximation, $f \in \mathcal{C}(\Gamma)$ and $\epsilon : \Gamma \to \mathbb{R}$ is a positive continuous function. Set $\psi = \ln \epsilon$. There exists a function $g_1 \in \mathcal{F}$ such that $|\psi - g_1| < 1$ on Γ. Setting $h = \exp(g_1 - 1)$, consider the functions $f/h \in C(\Gamma)$. There exists a function $g_2 \in \mathcal{F}$ such that $|f/h - g_2| < 1$ on Γ. Then, $|f - hg_2| < |h| = \exp(\Re(g_1) - 1) < \exp \psi = \epsilon$. This completes the proof. $\square$

The following is a non-compact version of the Stone-Weierstrass Theorem.

Proposition 3.2 *A closed set $\Gamma \subset \mathbb{C}^n$ is a set of tangential approximation by entire functions if and only if one can approximate (in the tangential sense) the real part projections $\Re(z_m)$ for $m = 1, \ldots, n$.*

Proof: The necessity is trivial. Moreover, if one can approximate the real part $\Re(z_m)$, one can approximate the imaginary part $\Im(z_m)$ as well, since $\Im(z_m) = i(\Re(z_m) - z_m)$. Let I be the natural diffeomorphism of $\mathbb{C}^n$ onto the real part $\mathbb{R}^{2n}$ of $\mathbb{C}^{2n}$. That is: $I_1(z) = \Re(z_1)$, $I_2(z) = \Im(z_1)$, $I_3(z) = \Re(z_2)$, $I_4(z) = \Im(z_2)$, etc., for $z \in \mathbb{C}^n$. Given two continuous function $f, \epsilon \in \mathcal{C}(\Gamma)$ with ϵ real positive, we may extend both of them continuously to all of $\mathbb{C}^n$ while keeping ϵ positive. By the theorem of Scheinberg (see introduction), there is an entire function $F \in \mathcal{O}(\mathbb{C}^{2n})$ such that $|f(z) - F \circ I(z)| < \epsilon(z)/2$ for $z \in \mathbb{C}^n$.

Since F is uniformly continuous on compact subsets of $\mathbb{C}^{2n}$, and the diffeomorphism I is proper, there is a positive continuous function δ on $\mathbb{C}^n$ such that $|F \circ I(z) - F(w)| < \epsilon(z)/2$, for each $z \in \mathbb{C}^n$ and each $w \in \mathbb{C}^{2n}$ for which $|I(z) - w| < \delta(z)$.

By hypotheses, we can approximate each component of I on Γ by entire functions and so there exists an entire mapping $h : \mathbb{C}^n \to \mathbb{C}^{2n}$ with $|I - h| < \delta$ on Γ. Thus, $|F \circ I - F \circ h| < \epsilon/2$ on Γ. By the triangle inequality, $|f - F \circ h| < \epsilon$ on Γ. The function $F \circ h$ is entire because h and F are holomorphic. $\square$

An interesting consequence of this result is that neither projection $\Re(z)$ nor $\Im(z)$, in the complex plane $z \in \mathbb{C}$, can be tangentially approximated on the classical examples where the tangential approximation fails to hold, although uniform approximation may sometimes be possible.

Example 3.3 *Let*

$$\Gamma = \bigcup_{j=0}^{\infty} \Gamma_j,$$

where $\Gamma_0 = [0, +\infty) \times \{0\}$*, and for* $j = 1, 2, \cdots$*,*

$$\Gamma_j = \left([0, j] \times \{\frac{1}{2j}, \frac{1}{2j+1}\} \right) \cup \left(\{j\} \times [\frac{1}{2j}, \frac{1}{2j+1}] \right).$$

Then, both functions $\Re(z)$ *and* $\Im(z)$ *can be approximated uniformly, but not tangentially, by entire functions on* Γ*.*

Proof: In his doctoral thesis, Arakelian [5] gave a complete characterization for sets of uniform approximation, from which it follows that Γ is not a set of uniform approximation and *a fortiori* not a set of tangential approximation. Thus, by Proposition 3.2, the functions $\Re(z)$ and

$\Im(z)$ cannot be approximated tangentially. We show that they can be approximated uniformly.

Fix $\epsilon > 0$ and set $Z_\epsilon = \{z \in \mathbb{C} : |\Im(z)| \leq \epsilon\}$ and $W_\epsilon = \Gamma \setminus Z_\epsilon$. We may assume Z_ϵ and W_ϵ disjoint (by choosing an appropriate smaller ϵ if necessary). Now, define the function

$$f(x) = \begin{cases} \epsilon & \text{for } x \in Z_\epsilon, \\ \Im(x) & \text{for } x \in W_\epsilon. \end{cases}$$

Invoking again Arakelian's work (see [5], [13, p.245] or [9]), we deduce the existence of an entire function g such that $|f - g| < \epsilon$ on $Z_\epsilon \cup W_\epsilon$. Hence, $|\Im - g| < 2\epsilon$ on Γ. So $\Im(z)$ and $\Re(z) = z - i\Im(z)$ can both be approximated uniformly on Γ by entire functions. $\square$

It is interesting to compare Propositions 3.1 and 3.2 in the light of the previous example.

We should also notice that Proposition 3.1 also holds if we consider approximation by functions holomorphic in a neighbourhood of Γ instead of approximation by entire functions. That is, we have that each continuous function $f \in \mathcal{C}(\Gamma)$ can be approximated (in the tangential sense) by functions holomorphic in a neighbourhood of Γ if and only if every projection $\Re(z_m)$ can. This result suggests the following:

Proposition 3.4 *Every closed set $\Gamma \subset \mathbb{C}^n$ of area zero is a set of tangential approximation by meromorphic functions in $\mathbb{C}^n$. That is, every continuous function F defined on Γ can be tangentially approximated by meromorphic functions whose singularities do not meet Γ.*

Proof: Let $f, \epsilon \in \mathcal{C}(\Gamma)$ be two continuous functions with ϵ real and positive. We must construct a meromorphic function F such that $|F(z) - f(z)| < \epsilon(z)$ on Γ. Let B_0 be the empty set and $\overline{B}_k$ closed balls of radius k and center in the origin.

Lemma 3.5 *Each continuous function $h \in \mathcal{C}(\overline{B}_k \cup \Gamma)$ which can be uniformly approximated by polynomials in $\overline{B}_k$ can be uniformly approximated on $D = \overline{B}_k \cup (\Gamma \cap \overline{B}_{k+1})$ by rational functions whose singularities do not meet Γ.*

Proof: From Theorem 2.1.A, and for each $\delta > 0$, there exists a rational function $(a/b)(z)$ such that $|(a/b)(z) - h(z)| < \delta$ for $z \in D$ and $0 \notin b(D)$. Notice that $b(\Gamma)$ has zero area, so we may choose a complex number $\lambda \notin b(\Gamma)$ with absolute value so small such that $\lambda \notin b(D)$ and $\left|\frac{a(z)}{b(z)-\lambda} - h(z)\right| < \delta$ for $z \in D$. $\square$

The proof of Proposition 3.4 now follows a classical inductive process. There exists a rational function F_1 whose singularities do not meet Γ and such that $|F_1(z) - f(z)| < (\frac{2}{3} - 2^{-1})\epsilon(z)$ for $z \in \Gamma \cap \overline{B}_1$ by the previous Lemma 3.5. Proceeding by induction, we shall construct a sequence of rational functions F_k which converges uniformly on compact sets to a meromorphic function with the desired properties.

Given a rational function F_k whose singularities do not meet Γ and such that $|F_k(z)-f(z)| < (\frac{2}{3}-2^{-k})\epsilon(z)$ in $\Gamma\cap\overline{B}_k$, let h_k be a continuous function identically equal to zero on $\overline{B}_k$ and such that $|h_k(z) + F_k(z) - f(z)| < (\frac{2}{3} - 2^{-k})\epsilon(z)$ for $z \in \Gamma \cap \overline{B}_{k+1}$ as well. Fix a real number $0 < \lambda_k < 1$ strictly less than $\epsilon(z)$ for every $z \in \Gamma \cap \overline{B}_{k+1}$.

Applying Lemma 3.5, there exists a rational function R_k whose singularities do not meet $\overline{B}_k \cup \Gamma$ and such that $|R_k(z) - h_k(z)| < 2^{-1-k}\lambda_k$ for $z \in \overline{B}_k \cup (\Gamma \cap \overline{B}_{k+1})$. Thus, the singularities of the rational function $F_{k+1}(z) = F_k(z) + R_k(z)$ do not meet Γ and $|F_{k+1}(z) - f(z)| < (\frac{2}{3} - 2^{-1-k})\epsilon(z)$ for $z \in \Gamma \cap \overline{B}_{k+1}$ by the triangle inequality.

Notice that $F_{k+1}(z) - F_k(z)$ is holomorphic and its absolute value is less than 2^{-1-k} inside $\overline{B}_k$, so the sequence F_k converges to a meromorphic function with the desired properties. $\square$

Similar inductive processes were originally employed to prove Carleman's theorem, stated in the introduction, which asserts that the real line $\mathbb{R}$ in $\mathbb{C}$ is a set of tangential approximation by entire functions. Alexander [3] extended Carleman's theorem to piecewise smooth arcs Γ going to infinity in $\mathbb{C}^n$. That is, Γ is the the image of the real axis under a proper continuous embedding (a curve without self-intersections, *going to infinity in both directions*). We should mention that this problem had been considered independently by Bernard Aupetit and Lee Stout (see Aupetit's book [1]). As a consequence of the Alexander-Stolzenberg Theorem, we also have the following further extension of Carleman's theorem, which was conjectured by Aupetit in [1] and announced by Alexander in [3].

Proposition 3.6 *Let Γ be an arc which is of finite length at each one of its points, except perhaps in a discrete subset, and going to infinity in $\mathbb{C}^n$. Besides, let ϵ be a strictly positive continuous function on Γ. Then, for each $f \in \mathcal{C}(\Gamma)$, there exists an entire function g on $\mathbb{C}^n$ such* *that* $|f(z) - g(z)| < \epsilon(z)$ *for all* $z \in \Gamma$*. That is,* Γ *is a set of tangential approximation by entire functions.*

Alexander's proof (see also [1]), for the case that Γ is smooth, relies ingeniously on the topology of arcs and the original Stolzenberg Theorem for smooth curves. It also works when the arc Γ is of locally finite length everywhere except perhaps in a finite subset. One only needs to rewrite Lemma 1 of [3], using the following corollary of Theorem 2.1.

Corollary 3.7 *Let* X *and* Y *be two compact subsets of* $\mathbb{C}^n$ *such that* X *is polynomially convex,* Y *is connected and* $Y \setminus X$ *is of finite length at each one of its points, except perhaps at finitely many of them. If the map* $\check{H}^1(X \cup Y) \to \check{H}^1(X)$ *induced by* $X \subset X \cup Y$ *is injective, then* $X \cup Y$ *is polynomially convex and every continuous function* $f \in \mathcal{C}(X \cup Y)$ *which can be approximated by polynomials in* X *can be approximated by polynomials on the union* $X \cup Y$*.*

Proof: Let Y_0 be the points where $Y \setminus X$ is not of locally of finite *length.* Notice that the inclusion mapping $X \to X \cup Y$ can be decomposed as the composition of the two mappings $X \to X \cup Y_0$ and $X \cup Y_0 \to X \cup Y$. Hence, the induced injective function $\check{H}^1(X \cup Y) \to \check{H}^1(X)$ can also be decomposed as the composition of $\check{H}^1(X \cup Y) \to \check{H}^1(X \cup Y_0)$ and $\check{H}^1(X \cup Y_0) \to \check{H}^1(X)$. It is easy to see that the last two functions are injective as well. Now, suppose $f \in \mathcal{C}(X \cup Y)$ and f can be approximated by polynomials on X. We have that $X \cup Y_0$ is polynomially convex because of the Oka-Weil theorem or Theorem 2.1. Moreover, we also have that $X \cup Y$ is polynomially convex and f can be approximated by polynomials on $X \cup Y$ by Theorem 2.1 again. □

We can also approximate by entire functions on unbounded sets which are more general than arcs, but first, we need to introduce the polynomially convex hull of non-compact sets:

Definition. Given an arbitrary subset Y of $\mathbb{C}^n$, its polynomially convex hull is defined by $\widehat{Y} = \bigcup \Big\{ \widehat{K} : K \subset Y \text{ is compact} \Big\}$.

Proposition 3.8 *Let* Γ *be a closed set in* $\mathbb{C}^n$ *of zero area such that* $\widehat{D \cup \Gamma} \setminus \Gamma$ *is bounded for every compact set* $D \subset \mathbb{C}^n$*. Let* B_1 *be an open ball with center in the origin which contains the closure of* $\widehat{\Gamma} \setminus \Gamma$*. That is, the set* $B_1 \cup \Gamma$ *contains the hull* $\widehat{K}$ *of every compact set* $K \subset \Gamma$*.*

Then, given two continuous functions $f, \epsilon \in \mathcal{C}(\Gamma)$ such that ϵ is real positive and f can be uniformly approximated by polynomials on $\Gamma \cap \overline{B}_1$, there exists an entire function F such that $|F(z) - f(z)| < \epsilon(z)$ for $z \in \Gamma$.

Proof: Let B_0 be the empty set, B_1 as in the hypotheses and B_k open balls with center in the origin such that each B_k contains the closure of $\widehat{\Gamma \cup \overline{B}_{k-1}} \setminus \Gamma$. That is, the set $B_k \cup \Gamma$ contains the hull $\widehat{K}$ of every compact set $K \subset (\Gamma \cup \overline{B}_{k-1})$. Define X_k to be the polynomially convex hull of $\overline{B}_{k+1} \cap (\Gamma \cup \overline{B}_{k-1})$, so $X_k \subset (B_k \cup \Gamma)$. The compact sets X_k and $X_k \cap \overline{B}_k$ are both polynomially convex.

The given hypotheses automatically imply that there exists a polynomial F_1 such that $|F_1(z) - f(z)| < (\frac{2}{3} - 2^{-1})\epsilon(z)$ on $\Gamma \cap \overline{B}_1$. Proceeding by induction, we shall construct a sequence of polynomials F_k which converges uniformly on compact sets to an entire function with the desired properties.

Given a polynomial F_k such that $|F_k(z) - f(z)| < (\frac{2}{3} - 2^{-k})\epsilon(z)$ on $\Gamma \cap \overline{B}_k$, let h_k be a continuous function equal to F_k on $\overline{B}_k$ and such that $|h_k(z) - f(z)| < (\frac{2}{3} - 2^{-k})\epsilon(z)$ for $z \in \Gamma \cap \overline{B}_{k+1}$ as well. Fix a real number $0 < \lambda_k < 1$ strictly less than $\epsilon(z)$ for every $z \in \Gamma \cap \overline{B}_{k+1}$.

Notice that $X_k = (X_k \cap \overline{B}_k) \cup (\Gamma \cap \overline{B}_{k+1})$. Hence, by Theorem 2.1.A, the function h_k can be approximated by rational functions on X_k because $X_k \cap \overline{B}_k$ is polynomially convex and Γ has zero area. Moreover, the functions h_k can be approximated by polynomials by the Oka-Weil theorem. Thus, there exists a polynomial F_{k+1} such that $|F_{k+1}(z) - h_k(z)| < 2^{-1-k}\lambda_k$ for $z \in X_k$, and so $|F_{k+1}(z) - f(z)| < (\frac{2}{3} - 2^{-1-k})\epsilon(z)$ on $\Gamma \cap \overline{B}_{k+1}$.

Finally, the inequality $|F_{k+1}(z) - F_k(z)| < 2^{-1-k}$ holds for $z \in \overline{B}_{k-1}$, so the sequence F_k converges to an entire function with the desired properties. □

On the other hand, if the equality $\widehat{\Gamma} = \Gamma$ holds as well in the last proposition, we can choose the empty set instead of the open ball B_1 (because the proof is an inductive process); and so Γ becomes a set of tangential approximation by entire functions. There are many closed sets Γ which satisfy the hypotheses of the last proposition. For example, we have the following.

Theorem 3.9 *Let Γ be closed connected set of locally finite length in $\mathbb{C}^n$ whose first cohomology group $\check{H}^1(\Gamma)$ vanishes (Γ contains no simple* *closed curves). Then, Γ is a set of tangential approximation by entire functions.*

Proof: The proof strongly uses the topology of Γ. We show that each point of Γ has finite order; that is, has a basis of neighbourhoods in Γ having finite boundaries. Given a point $z \in \Gamma$, let B_r be the open ball in $\mathbb{C}^n$ of radius r and center z. Since Γ is locally of finite *length*, the intersection of Γ with the closed ball $\overline{B_r}$ has finite *length*, so the intersection of Γ with the boundary of B_s must be a finite set for almost all radii $0 < s < r$. Whence, each sub-continuum of Γ is locally connected [17, p. 283]. On the other hand, there are no simple closed curves contained in Γ because $\check{H}^1(\Gamma) = 0$; so each sub-continuum of Γ is a dendrite, that is, a locally connected continuum containing no simple closed curves. In particular, if Γ is compact, then it is a dendrite.

Notice the following lemma.

Lemma 3.10 *Each compact subset $K \subset \Gamma$ is contained in a sub-continuum (dendrite) of Γ.*

Proof: Since Γ is locally connected, the set K is contained in a finite union of sub-continua of Γ. The lemma now follows since Γ is arcwise connected (see Theorem 3.17 of [16]). $\square$

Let D be a compact set in $\mathbb{C}^n$. Notice that $D \cup \Gamma$ may contain simple closed curves Υ with $D \cap \Upsilon \neq \emptyset$ but $\Upsilon \not\subset D$. We shall call such a simple closed curve $\Upsilon \subset (D \cup \Gamma)$ a *loop*. We show there exists a ball which contains all of these loops. Henceforth, let B_r be open balls of radii r and center in the origin, and choose a radius $s > 0$ such that $D \subset B_s$. Recall that $\Gamma \cap \overline{B}_{s+1}$ has finite *length*, so there exists a ball B_t with $s < t < s + 1$ such that Γ meets the boundary of B_t only in a finite number of points $Q = \{q_1, \ldots, q_m\}$. Let $\{\Upsilon_j\}$ be the possible loops which meet the complement of B_t. The set $\bigcup\{\Upsilon_j\} \setminus B_t$ is contained in Γ and can be expressed as the union of compact arcs (not necessarily disjoint) which lie outside of $\overline{B}_t$ except for their two end points which lie in Q. Since Γ cannot contain simple closed curves, two different arcs cannot share the same end points, and there can only be finitely many such arcs. Hence, there exists a ball B_δ which contains all the loops Υ, and $D \subset B_\delta$.

We shall show that $\widehat{D \cup \Gamma} \setminus \Gamma$ is bounded. Without loss of generality, we may suppose that D is a closed ball. Since Γ is connected, the hull $\widehat{D \cup \Gamma}$ is equal to $\bigcup_{r \geq \delta} \widehat{K}_r$, where K_r is the connected component of

$\overline{B}_r \cap (D \cup \Gamma)$ which contains D. We can prove that $\widehat{K}_r = \widehat{K}_\delta \cup K_r$, for every $r \geq \delta$, using Alexander's original argument. The following lemma is a literal translation of Lemma 1.(a) of [3], to our context.

Lemma 3.11 *For every $r \geq \delta$, $\widehat{K}_r = \widehat{K}_\delta \cup \tau_r$ where $\tau_r = \overline{K_r \setminus K_\delta}$.*

Since the notation is quite complicated and different from Alexander's, and we need to invoke Theorem 2.1.B, we shall include the proof of Lemma 3.11, but first we conclude the proof of the theorem.

By Lemma 3.11, the set $\widehat{D \cup \Gamma} \setminus \Gamma$ is bounded because $\widehat{K}_r = \widehat{K}_\delta \cup \tau_r = \widehat{K}_\delta \cup K_r$ and $\widehat{D \cup \Gamma} = \widehat{K}_\delta \cup \Gamma$. Moreover, the equality $\widehat{\Gamma} = \Gamma$ holds as well because each compact subset of Γ is contained in a dendrite of finite *length* and is polynomially convex (see Lemma 3.10 and Alexander's work [2]), so we can deduce from Proposition 3.8 that Γ is a set of tangential approximation. $\Box$

Proof: [Proof of Lemma 3.11] Let $T_r = \widehat{K}_\delta \cup \tau_r$ be the set on the right hand side of the asserted equality. Clearly, we have $T_r \subset \widehat{K}_r \subset \widehat{T}_r$ (the second inclusion is in fact equality). Thus it suffices to show that T_r is polynomially convex. Arguing by contradiction, we suppose otherwise. By Theorem 2.1.B, $\widehat{T}_r \setminus T_r$ is a 1-dimensional analytic subvariety of $\mathbb{C}^n \setminus T_r$.

Let V be a non-empty irreducible analytic component of $\widehat{T}_r \setminus T_r$. We claim that $\overline{V} \setminus K_r$ is an analytic subvariety of $\mathbb{C}^n \setminus K_r$. Since $T_r = \widehat{K}_\delta \cup \tau_r$, it suffices to verify this locally at a point $x \in \overline{V} \cap Q$ where

$$Q = \widehat{K}_\delta \setminus K_\delta.$$

By Theorem 2.1.B, both $\widehat{K}_r$ and Q are analytic near x, where *near* x refers to the intersection of sets with *small enough* neighbourhoods of x, here and below. Furthermore, near x, $\overline{V} \subset \widehat{K}_r$, $V \subset \widehat{K}_r \setminus Q$ and $Q \subset \widehat{K}_r$. Thus, near x, Q is a union of some analytic components of $\widehat{K}_r$. It follows that near x, $\overline{V}$ is just a union of some of the other local analytic components of $\widehat{K}_r$ at x; in fact, near x, $\overline{V} = V \cup \{x\}$. Put

$$W = \overline{V} \setminus K_r.$$

Then W is an irreducible analytic subset of $\mathbb{C}^n \setminus K_r$ and moreover,

$$\overline{W} \setminus W \subset K_\delta \cup \tau_r = K_r.$$

Thus $\overline{W} \subset \widehat{K}_r$ by the maximum principle.

Fix a point $p \in V \subset W$. Since $p \notin T_r$, we have $p \notin \widehat{K}_\delta$ and therefore there exists a polynomial h such that $h(p) = 0$ and $\Re h < 0$ on $\widehat{K}_\delta$. By the open mapping theorem, either $h(W)$ is an open neighbourhood of 0 or $h \equiv 0$ on W. In the latter case, $h \equiv 0$ on $\overline{W}$ and so $\overline{W} \setminus W$ is disjoint from K_δ. This implies that $\overline{W} \setminus W \subset \hat{\tau}_r$ so $W \subset \hat{\tau}_r$. We have a contradiction because τ_r is contained in a dendrite of finite *length* and is polynomially convex (see Lemma 3.10 and Alexander's work [2]), and moreover, a dendrite cannot contain a 1-dimensional analytic set. Hence, the former case holds. Since $h(\tau_r)$ is nowhere dense in the plane (recall that it is of finite *length*), there is a small complex number $\alpha \in h(W)$ such that $\alpha \notin h(\tau_r)$. Now put $g = h - \alpha$. If α is sufficiently small, we conclude that (i) $\Re g < 0$ on $\widehat{K}_\delta$, (ii) $g(q) = 0$ for some $q \in W$ and (iii) $0 \notin g(\tau_r)$.

Now (i) implies that the polynomial g has a continuous logarithm on $\widehat{K}_\delta$ and so, by restriction, on K_δ. We can extend this logarithm of g on K_δ to a continuous logarithm of g on K_r because of (iii), since the ball B_δ was chosen such that every simple closed curve (loop) $\Upsilon \subset K_r$ is contained in B_δ and hence in K_δ. But K_r contains $\overline{W} \setminus W$. Applying the argument principle [21, p. 271] to g on the analytic set W gives a contradiction to (ii). $\square$

We remark that the condition of having zero area is essential in Propositions 3.4 and 3.8, as the following example (inspired by [8]) shows.

Example 3.12 Let $\mathcal{I}$ be the closed unit interval $[0,1]$ of the real line and $K \subset \mathcal{I}$ the compact set $K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. It is easy to see that the $(2+\epsilon)$-dimensional Hausdorff measure of the closed connected set $Y = (\mathcal{I} \times \{0\}) \cup (K \times \mathbb{C})$ in $\mathbb{C}^2$ is equal to zero for every $\epsilon > 0$; moreover, the equality $\widehat{Y} = Y$ holds. However, the following continuous function $f \in \mathcal{C}(Y)$ cannot be uniformly approximated by holomorphic functions in $\mathcal{O}(Y)$:

$$f(w,z) = \begin{cases} z & \text{if } w = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose there exists a real number $\epsilon > 0$ and a holomorphic function $g \in \mathcal{O}(Y)$ such that $|f - g| < \epsilon$ on Y. We automatically have that $g(w,z)$ is bounded, holomorphic and constant on each complex line $\{\frac{1}{j}\} \times \mathbb{C}$, $j = 2, 3, \ldots$. Hence, the holomorphic function $\frac{\partial g}{\partial z}$ vanishes on each complex line $\{\frac{1}{j}\} \times \mathbb{C}$, $j = 2, 3, \ldots$ as well. Since the zero set of $\frac{\partial g}{\partial z}$ is an analytic set, this derivative must be zero in a neighbourhood of $\{0\} \times \mathbb{C}$ and hence on the connected set Y. The last statement is a contradiction to the fact that $|g(1, z) - z| < \epsilon$ for every $z \in \mathbb{C}$.

On the other hand, to see that $\widehat{Y} = Y$, notice that $Y = \bigcup_{r>0} Y_r$, where $Y_r = (\mathcal{I} \times \{0\}) \cup (K \times \Delta_r)$ and $\Delta_r \subset \mathbb{C}$ are closed disks of radius r. The set $K \times \Delta_r$ is polynomially convex because it is the Cartesian product of two polynomially convex sets in $\mathbb{C}$; and so Y_r is polynomially convex because of Theorem 2.1.

Although connectivity, as we have emphasized, plays a crucial role in this paper, similar results can be obtained for sets whose connected components form a locally finite family. Finally, we remark that, on a Stein manifold, analogous results also hold by simply embedding the Stein manifold into some $\mathbb{C}^n$. A possible exception is Proposition 3.2, since $\Re(z)$ is not well-defined on a manifold.

4 Historical notes

In this section we recapitulate and supplement some of the historical remarks which are dispersed throughout this paper.

Of course, the foundation of approximation theory is the Weierstrass theorem (1885), which affirms that each closed interval is a set of uniform approximation by polynomials. This is essentially a *real* result. In the complex setting, the most beautiful approximation theorem is a deep theorem of Walsh [24, in 1926] which lifts the Weierstrass theorem to the complex domain by asserting that each Jordan arc (homeomorphic image of a closed interval) in the complex plane is a set of uniform approximation by polynomials. For a survey on this result of Walsh and its impact, see [12].

Just as Walsh's theorem is the most beautiful result of uniform approximation in the complex plane $\mathbb{C}$, the outstanding open problem in complex approximation is to extend Walsh's theorem to higher dimensions ($\mathbb{C}^n$). Any compact set firstly needs to be polynomially convex (see [21]) in order to be a set of uniform approximation by polynomials. In high dimensions, Wermer [25, in 1955] and Rudin [19, in 1956] gave examples of Jordan arcs which are not polynomially convex, and hence they are not sets of uniform polynomial approximation. The main problem can be then formulated more precisely: Is it true that each polynomially convex Jordan arc is a set of polynomial approximation? This problem has remained open for over half a century.

The following Jordan arcs are known to be sets of uniform approximation: analytic arcs (Wermer [26], [27] and [28] in 1958), $\mathcal{C}^1$-smooth arcs (Stolzenberg [22] in 1966), rectifiable arcs (Alexander [2] in 1971); and in the present paper we allow arcs which are of finite *length* at each point, except perhaps at a finite set of points.

One can also consider rational approximation and; here again, any compact set firstly needs to be rationally convex in order to be a set of uniform rational approximation. It is known that any compact set of area zero is a set of rational approximation. Bagby and one of the authors [6] have given an example of an arc of finite area which is not rationally convex and, *a fortiori*, it is not a set of rational approximation.

We have seen, on one side, that we have polynomial approximation on Jordan arcs whose *length* is locally finite except perhaps at a finite subset of points. On the other hand, we do not have rational approximation on a certain Jordan arc of finite area. It is quite natural to ask about the intermediate cases, namely, Jordan arcs whose dimension lies between 1 and 2. This question was in fact posed by Gamelin [11].

As mentioned earlier in this paper, the Weierstrass theorem was also extended in a different way by Carleman [7, in 1927], who showed that the real-line in $\mathbb{C}$ is a set of tangential approximation by entire functions in $\mathbb{C}$. This result was also generalized to several complex variables in two ways. First of all, Scheinberg [20, in 1976] showed that the real part of $\mathbb{C}^n$ is a set of Carleman approximation by entire functions in $\mathbb{C}^n$. Secondly, and this is the generalization which concerns us in the present paper, Carleman himself had conjectured and Keldysh proved that each unbounded Jordan arc in $\mathbb{C}$ is a set of tangential approximation by entire functions as well. This result was extended by Alexander [2, in 1979] to unbounded Jordan arcs in $\mathbb{C}^n$ which are piecewise $\mathcal{C}^1$-smooth. We have shown that Alexander's result also holds for unbounded Jordan arcs which are of locally finite *length*. This had been conjectured by Aupetit [1, in 1978] and announced by Alexander [2]. But we showed a stronger result, by allowing a discrete subset of exceptional points, and also by allowing more general sets than only unbounded arcs.

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