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Existence of Nash equilibria in some Markov games with discounted payoff *

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Abstract

This work considers $N-$person stochastic game models with a discounted payoff criterion, under two different structures. First, we consider games with finite state and action spaces, and infinite horizon. Second, we consider games with Borel state space, compact action sets, and finite horizon. For each of these games, we give conditions that ensure the existence of a Nash equilibrium, which is a stationary strategy in the former case, and a Markovian strategy in the latter.

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1 Introduction

In this paper we study the existence of Nash equilibria for stochastic games with a discounted payoff criterion. First we consider a game with finite state and action spaces, and infinite horizon. In a more general framework, we study games with a Borel state space, compact action sets, and finite horizon. The purpose of this work is to present in a clear, self-contained manner the proofs of these results. Our main source was the paper by Dutta and Sundaram [7].

The first studies of games in the economics literature were the papers by Cournot [6], Bertrand [3], and Edgeworth [8] on oligopoly pricing and

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production. The idea of a general theory of games was introduced by John von Neumann and Oskar Morgenstern in their famous 1944 book *Theory of Games and Economic Behavior* [25], which proposed that most economic questions should be analyzed as games. Nash [15] introduced what came to be known as "Nash equilibrium" as a way of extending game-theoretic analyses to nonzero-sum games. Stochastic games with discounted payoffs have been widely studied. This class includes the two-person zero-sum stochastic games, for which Nash equilibria are known to exist under a variety of assumptions; see, for instance, Filar and Vrieze [10], Nowak [16] or Ramírez-Reyes [19].

In this work we study nonzero-sum games under two different sets of hypotheses. The first result (for games with finite state spaces) was proved by Rogers [21] and Sobel [24], with an extension to countable state spaces by Parthasarathy [17]. The second result was proved by Rieder [20] (as an approximation to what he calls an $\varepsilon-$equilibrium) under some special assumptions on the structure of the game. The general result (that is, games with Borel state and action spaces, and infinite horizon) is an open problem, even with compact action sets. However, it has been solved imposing an additive structure in the reward functions and the transition law; see Hernández-Hernández [12] or Parthasarathy and Sinha [17], for instance.

The remainder of this work is organized as follows. Section 2 presents standard material on stochastic games, including the discounted optimality criteria and the definition of a Nash equilibrium. Sections 3 and 4 are devoted to proving the two main results, Theorems 3.1 and 4.1 respectively, that is, the existence of Nash equilibria for the games mentioned in the first paragraph. An appendix is included with some useful facts needed in the proofs of Theorems 3.1 and 4.1.

2 The stochastic game model

In this section we introduce the $N-$person stochastic game model. We start with the following remark on terminology and notation (for further details see Bertsekas and Shreve [4], chapter 7).

Remark 2.1 a) A Borel subset X of a complete and separable metric space is called a Borel space, and its Borel $\sigma-$algebra is denoted by $\mathcal{B}(X)$. A Borel subset of a Borel space is itself a Borel space.

b) Let X and Y be Borel spaces. A stochastic kernel on X given Y, is a function $P(\cdot \mid \cdot)$ such that

b.1) $P(\cdot \mid y)$ is a probability measure on X for each fixed $y \in Y$, and

b.2) $P(D \mid \cdot)$ is a measurable function on Y for each fixed $D \in \mathcal{B}(X)$.

c) The set of all stochastic kernels on X given Y is denoted by $\mathcal{P}(X \mid Y)$. Moreover, $\mathcal{P}(X)$ denotes the set of probability measures on X.

Definition 2.2 A stochastic game model is described by

$$GM := \{\mathcal{N}, S, (A_i, \Phi_i, r_i)_{i\in\mathcal{N}}, Q, T\}, \tag{1}$$

where:

(1) $\mathcal{N} = \{1, ..., N\}$ is the finite set of players.

(2) S is the state space, a Borel space.

(3) Each player $i \in \mathcal{N}$ is characterized by three objects (A_i, Φ_i, r_i), where:

(a) A_i, a Borel space, is the action space of player i. Let $\mathbf{A} = A_1 \times ... \times A_N$ and denote by $\mathbf{a}$ a generic element of $\mathbf{A}$.

(b) Φ_i, a multifunction from S to A_i, defines for each $s \in S$ the set $\Phi_i(s)$ of feasible actions for player i at state s. Let $\mathbf{\Phi}(s) = \Phi_1(s) \times ... \times \Phi_N(s)$ and $\mathcal{K} = \{(s, \mathbf{a}) : s \in S, \mathbf{a} \in \mathbf{\Phi}(s)\}$.

(c) r_i, a bounded measurable function from $\mathcal{K}$ to $\mathbb{R}$, specifies (for each state s and action $\mathbf{a} \in \mathbf{\Phi}(s)$ taken by the players at s) a reward $r_i(s, \mathbf{a})$ for player i.

(4) Q, a stochastic kernel in $\mathcal{P}(S \mid \mathcal{K})$, specifies the game transition law.

(5) $T \in \{0, 1, 2, ...\} \cup \infty$ is the horizon of the game.

If $T = 1$, the game is static, and it is denoted by

$$\{\mathcal{N}, S, (A_i, \Phi_i, r_i)_{i\in\mathcal{N}}\}.$$

The game is played as follows. At each time $t = 0, 1, ...$, each player observes the current state $s \in S$ of the system, and, independently of the other players, chooses an action $a_i \in \Phi_i(s)$. Then each player $i \in \mathcal{N}$ obtains a reward $r_i(s, \mathbf{a})$, and the system moves to a new state according to the probability distribution $Q(\cdot \mid s, \mathbf{a})$. The objective of each player is to win as much as possible.

Histories. A t-history of the game is a complete description of the evolution of the game up to the beginning of period t. Thus, a t-history specifies the state s_r that occurred in each previous period $r \in \{0, 1, ..., t-1\}$, the actions $\mathbf{a}_r = (a_{1,r}, ..., a_{N,r})$ taken by the players in those periods ($a_{i,r}$ denotes de action taken by player i at period r), and the state s_t in the period t. Let H_t be the set of all possible $t-$histories, with h_t denoting a typical element of H_t, i.e.

$$h_t = (s_0, \mathbf{a}_0, s_1, \mathbf{a}_1, ..., s_{t-1}, \mathbf{a}_{t-1}, s_t) \quad \text{with } \mathbf{a}_r \in \mathbf{\Phi}(s_r). \tag{2}$$

Note that $H_0 = S$ and $H_t = \mathcal{K} \times H_{t-1}$ for $t = 1, 2, ...$

Strategies. A strategy π_i for player i is a vector $\{\pi_{it}\}_{t=0}^{T-1}$ (or sequence if $T = \infty$) of stochastic kernels $\pi_{it} \in \mathcal{P}(A_i \mid H_t)$, where for each t and each $t-$history h_t up to t, π_{it} specifies the action $\pi_{it}(h_t) \in \mathcal{P}(A_i)$ such that

$$\pi_{it}(\Phi_i(s_t) \mid h_t) = 1 \quad \forall h_t \in H_t, t = 0, 1, ...$$

A strategy is also called a mixed or randomized strategy, which means that the player chooses an action in a random manner. The set of mixed strategies includes the pure strategies, when the player chooses the actions in a deterministic way.

Let Π_i denote the set of all strategies for player i, and let $\Pi :=$ $\Pi_1 \times ... \times \Pi_N$. A generic element of Π is denoted by π, and it is said to be a multistrategy. A strategy $\pi_i = \{\pi_{it}\}_{t=0}^{T-1}$ for player i is called Markov if $\pi_{it} \in \mathcal{P}(A_i \mid S)$ for each $t = 0, 1, ..., T-1$, meaning that each π_{it} depends only on the current state s_t of the system. The set of all Markov strategies of player i will be denoted Π_{iM}. A Markov strategy $\pi_i = \{\pi_{it}\}_{t=0}^{T-1}$ is said to be stationary if $\pi_{it} = \pi_{i0}$ for each $t = 0, 1, ..., T-1$, where $\pi_{i0} \in \mathcal{P}(A_i \mid S)$. We denote by Π_{iS} the set of all stationary strategies of player i. We have

$$\Pi_{iS} \subset \Pi_{iM} \subset \Pi_i.$$

In a similar manner

$$\Pi_S \subset \Pi_M \subset \Pi,$$

where $\Pi_S := \Pi_{1S} \times ... \times \Pi_{NS}$ is the set of stationary multistrategies, and $\Pi_M := \Pi_{1M} \times ... \times \Pi_{NM}$ is the set of Markov multistrategies.

Optimality criteria. Let δ be a fixed number in $(0,1)$, and define the $\delta-$discounted expected payoff function for player i as

$$J_{i,\delta}(s,\pi) := E_s^{\pi}\left[\sum_{t=0}^{\infty} \delta^t r_i(s_t, \mathbf{a}_t)\right] \tag{3}$$

for each multistrategy π and each initial state s. It represents the expected present value of the rewards of player i under the multistrategy π. The number δ is called a "discount factor".

Definition 2.3 For $n = 1, 2, ...$, we define the T-stage expected discounted payoff function for player i as

$$J_{i,\delta,T}(s,\pi) := E_s^{\pi}\left[\sum_{t=0}^{T-1} \delta^t r_i(s_t, \mathbf{a}_t)\right],$$

where $0 < \delta < 1$ is a discount factor. If $T = \infty$, we write $J_{i,\delta,T}(s,\pi)$ as $J_{i,\delta}(s,\pi)$; see (3).

Now we are in position to define a Nash equilibrium. As usual in the literature, the vector $(\bar{\pi}_i, \pi_{-i})$ will signify the multistrategy π with its strategy π_i replaced by $\bar{\pi}_i$.

Definition 2.4 A multistrategy π is a **Nash Equilibrium** of the T-stage game GM if

$$J_{i,\delta,T}(s,\pi) \geq J_{i,\delta,T}(s,(\bar{\pi}_i, \pi_{-i})) \textit{ for all } s \in S, \bar{\pi}_i \in \Pi_i, i \in \mathcal{N}.$$

Before proceeding we give some notation. First note that $\int_{\mathbf{A}}$ means $\int_{A_1} ... \int_{A_N}$ and that $\sum_{\mathbf{A}}$ means $\sum_{A_1} ... \sum_{A_N}$. Let $\nu : \mathcal{K} \to \mathbb{R}$ be a measurable function, $\pi_0 \in \mathcal{P}(\Phi_1(s)) \times ... \times \mathcal{P}(\Phi_N(s))$ and $\bar{\pi}_{i0} \in \mathcal{P}(\Phi_i(s))$ for some $s \in S$ and some $i \in \mathcal{N}$, then

$$\nu(s,\pi_0) := \int_{\mathbf{A}} \nu(s,\mathbf{a})\pi_0(d\mathbf{a})$$

$$= \int_{\mathbf{A}} \nu(s,\mathbf{a})\pi_{1,0}(da_1)...\pi_{i0}(da_i)...\pi_{N,0}(da_N) \tag{4}$$

and

$$\nu(s,(\bar{\pi}_{i0},\pi_{-i0})) := \int_{\mathbf{A}} \nu(s,\mathbf{a})\pi_{1,0}(da_1)...\bar{\pi}_{i0}(da_i)...\pi_{N,0}(da_N) \tag{5}$$

In particular

$$r_i(s,\pi_0) := \int_{\mathbf{A}} r_i(s,\mathbf{a})\pi_0(d\mathbf{a})$$

$$= \int_{\mathbf{A}} r_i(s,\mathbf{a})\pi_{1,0}(da_1)...\pi_{i0}(da_i)...\pi_{N,0}(da_N),$$

$$r_i(s,(\bar{\pi}_{i0},\pi_{-i0})) := \int_{\mathbf{A}} r_i(s,\mathbf{a})\pi_{1,0}(da_1)...\bar{\pi}_{i0}(da_i)...\pi_{N,0}(da_N),$$

$$Q(\cdot \mid s,\pi_0) := \int_{\mathbf{A}} Q(\cdot \mid s,\mathbf{a})\pi_0(d\mathbf{a})$$

$$= \int_{\mathbf{A}} Q(\cdot \mid s,\mathbf{a})\pi_{1,0}(da_1)...\pi_{i0}(da_i)...\pi_{N,0}(da_N)$$

and

$$Q(\cdot \mid s,(\bar{\pi}_{i0},\pi_{-i0})) := \int_{\mathbf{A}} Q(\cdot \mid s,\mathbf{a})\pi_{1,0}(da_1)...\bar{\pi}_{i0}(da_i)...\pi_{N,0}(da_N).$$

Remark 2.5 If $A_1,...,A_N$ (and hence $\mathbf{A}$) are finite or countable sets, then the integrals are replaced with summations.

The next two sections are devoted to proving the existence of a Nash equilibrium in games with

1) finite state and action spaces, and infinite horizon (Theorem 3.1); and

2) games with Borel state space, compact action sets, and finite horizon (Theorem 4.1).

The proofs are based on a standard procedure; see, for instance, Dutta and Sundaram [7]. The procedure is to introduce a multifunction which is a $K-$mapping (see Definition 5.5) on a nonempty compact convex set. Then we use Kakutani´s or Glicksberg´s fixed-point theorem to ensure the existence of a fixed point (Definition 5.6), which yields a Nash equilibrium.

3 The equilibrium existence in finite spaces

Theorem 3.1 *Suppose S and A_i are finite spaces for each i, and $T = \infty$. Then the stochastic game model GM has a Nash equilibrium in stationary strategies.*

Proof: As was already mentioned, the idea is to prove the existence of a fixed point of a certain multifunction; this fixed point is an equilibrium. To prove the existence of a fixed point of the multifunction, we use Kakutani´s theorem (Theorem 5.7).

The proof is organized as follows:

In Step 0 we define the multifunction. In Step 1 we prove that such a multifunction is defined on a nonempty compact convex subset of $\mathbb{R}^n$. In Step 2 we prove that the multifunction maps points to a nonempty convex set, and, finally, in Step 3, we prove that the multifunction is u.s.c. (see Definition 5.5). The last two steps prove that the multifunction is a $K-$mapping (Definition 5.5). Since Steps 1, 2 and 3 verify the hypotheses of Kakutani´s theorem, a fixed point exists.

Step 0: Definition of the multifunction BR.

Let $\pi := (\pi_1, ..., \pi_n) \in \Pi_S$, and let $BR_i(\pi)$ be the set of *best responses* of player i to π, that is

$$BR_i(\pi) := \left\{ \bar{\pi}_i \in \Pi_{iS} : J_{i,\delta}(s, (\bar{\pi}_i, \pi_{-i})) = \sup_{\alpha \in \Pi_{iS}} J_{i,\delta}(s, (\alpha, \pi_{-i})) \ \forall s \in S \right\}.$$

Let $BR = BR_1 \times .... \times BR_N$. Note that $BR : \Pi_S \twoheadrightarrow \Pi_S$.

Step1: The set Π_S is a nonempty compact convex subset of a normed space.

Because A_i is a finite set, the set $\mathcal{P}(A_i)$ is simply the positive unit simplex of dimension $|A_i| - 1$ ($|\cdot|$ means the cardinality). Note that $\mathcal{P}(A_i) \subset \mathbb{R}^{|A_i|-1}$. Since S is also a finite set, the $|S|-$fold $\mathcal{P}(A_i \mid S) = \mathcal{P}(A_i)^{|S|}$, the Cartesian product of this simplex, is a compact convex subset of a finite-dimensional Euclidean space. Now every stationary strategy for player i can be associated in the obvious way with a unique sequence of elements of

$$I^{|\Phi(s_1)|-1} \times ... \times I^{\left|\Phi(s_{|S|})\right|-1},$$

where $I := [0, 1]$. Hence Π_{iS} is a nonempty compact convex subset of $\mathbb{R}^n$ for some n, and so is $\Pi_S = \Pi_{1S} \times ... \times \Pi_{NS}$.

Step 2: $BR(\pi)$ is a nonempty convex subset of Π_S for each $\pi \in \Pi_S$.

Given $\pi \in \Pi_S$, the best-response problem faced by player i (that is, finding $BR_i(\cdot)$) is a discounted Markov decision problem

$$MDP := \left\{S, A_i, \Phi_i, r'_i, Q', T = \infty\right\}$$

where S is the state space and A_i is the action space. Furthermore, for each $s \in S$, $\Phi_i(s)$ is the set of feasible actions in state s. The reward function is

$$r'_i(s, a_i) := r_i(s, (a_i, \pi_{-i})) \tag{6}$$

Similarly, the transition law is

$$Q'(s' \mid s, a_i) := Q(s' \mid s, (a_i, \pi_{-i})) \tag{7}$$

By Remark 2.5, (6) and (7) represent finite sums. It is well known that there exists a nonempty set $BR_i(\pi)$ of optimal stationary strategies in response to π; see Filar and Vrieze [10], Theorem 2.3.1, for example.

Denote the value function of the MDP as $J_i^*(s)$. To prove that $\mathrm{BR}_i(\pi)$ is convex, let $\alpha, \beta \in BR_i(\pi)$ and $0 \leq \lambda \leq 1$; then we want to prove that $\mu := \lambda\alpha + (1-\lambda)\beta$ is in $BR_i(\pi)$.

Since $\alpha, \beta \in BR_i(\pi)$, we have that the Bellman equations

$$J_i^*(s) = r'_i(s, \alpha) + \delta \int_S J_i^*(s') Q(ds' \mid s, \alpha) \tag{8}$$

and

$$J_i^*(s) = r'_i(s, \beta) + \delta \int_S J_i^*(s') Q(ds' \mid s, \beta) \tag{9}$$

hold. Therefore, by (8) and (9),

$$\begin{aligned} & r'_i(s, \mu) + \delta \int_S J_i^*(s') Q(ds' \mid s, \mu), \\ &= \lambda \left(r'_i(s, \alpha) + \delta \int_S J_i^*(s') Q(ds' \mid s, \alpha) \right) \\ &+ (1-\lambda) \left(r'_i(s, \beta) + \delta \int_S J_i^*(s') Q(ds' \mid s, \beta) \right) \\ &= \lambda J_i^*(s) + (1-\lambda) J_i^*(s) = J_i^*(s). \end{aligned}$$

That is

$$r'_i(s, \mu) + \delta \int_S J_i^*(s') Q(ds' \mid s, \mu) = J_i^*(s),$$

and so μ is in $BR_i(\pi)$. This shows that $BR(\pi)$ is a nonempty convex set for each $\pi \in \Pi_S$.

Step 3: BR is an u.s.c. multifunction on Π_S, and $BR(\pi)$ is compact for each $\pi \in \Pi_S$.

This will be established if we show that it holds for each BR_i. So fix i. Suppose that

$$\pi_n := (\pi_{1n}, ..., \pi_{Nn}) \to \pi := (\pi_1, ..., \pi_N),$$

and $\alpha_{in} \in BR_i(\pi_n)$ is such that $\alpha_{in} \to \alpha_i \in \Pi_{iS}$. We are going to show that $\alpha_i \in BR_i(\pi)$.

Let $J^*_{i,n}(s)$ denote the value function of the player i in a best response to π_n. Since $\delta < 1$, the sequence $J^*_{i,n}(s)$ is uniformly bounded by

$$M = (1-\delta)^{-1} \max \{|r_i(s, \mathbf{a})| : (s, \mathbf{a}) \in S \times \mathbf{A}\}.$$

So, because $J^*_{i,n}(s) \in [-M, M]$ for each $s \in S$ and for each n, $J^*_{i,n}$ converges pointwise (perhaps through a subsequence) to a limit J^*_i.

On the other hand, we know that $J^*_{i,n}(s)$ satisfies the Bellman equation

$$J^*_{i,n}(s) = r_i(s, (\alpha_{in}, \pi_{-i})) + \delta \int_S J^*_{i,n}(s') Q\,(ds' \mid s, (\alpha_{in}, \pi_{-i}))\,, \tag{10}$$

for each n and s. Moreover, for any $\beta \in \mathcal{P}(\Phi_i(s))$, we have

$$J^*_{i,n}(s) \geq r_i(s, (\beta, \pi_{-i})) + \delta \int_S J^*_{i,n}(s') Q\,(ds' \mid s, (\beta, \pi_{-i}))\,. \tag{11}$$

Since the integrals are finite sums (recall our Remark 2.5), when $n \to \infty$ we have

$$r_i(s, (\alpha_{in}, \pi_{-i})) \to r_i(s, (\alpha_i, \pi_{-i})),$$

$$\int_S J^*_{i,n}(s') Q\,(ds' \mid s, (\alpha_{in}, \pi_{-i})) \to \int_S J^*_i(s') Q\,(ds' \mid s, (\alpha_i, \pi_{-i}))\,,$$

and, similarly,

$$\int_S J^*_{i,n}(s') Q\,(ds' \mid s, (\beta, \pi_{-i})) \to \int_S J^*_i(s') Q\,(ds' \mid s, (\beta, \pi_{-i}))\,.$$

Hence, letting $n \to \infty$ in (10) and (11) we get

$$J^*_i(s) = r_i(s, (\alpha_i, \pi_{-i},)) + \delta \int_S J^*_i(s') Q\,(ds' \mid s, (\alpha_i, \pi_{-i}))$$

and

$$J_i^*(s) \geq r_i(s, (\beta, \pi_{-i})) + \delta \int_S J_i^*(s')Q\left(ds' \mid s, (\beta, \pi_{-i})\right),$$

respectively. These expressions establish precisely that $J_i^*(s)$ is the value function in a best response of i to π, and that α_i is a stationary best-response, that is, $\alpha_i \in BR_i(\pi)$. Thus, BR_i is an u.s.c. multifunction for each i, and so is BR.

Also note that the u.s.c. shows that $BR_i(\pi)$ is a closed set in Π_{iS}. Thus, since Π_{iS} is compact, $BR_i(\pi)$ is also compact.

Summarizing, we have that for each π, $BR(\pi)$ is a nonempty compact convex subset, and that $BR(\cdot)$ is u.s.c.; hence $BR(\cdot)$ is a $K-$mapping. An appeal to Kakutani´s fixed-point theorem (Theorem 5.7) yields the existence of $\pi^* \in \Pi_S$ such that $\pi^* \in BR(\pi^*)$, completing the proof of Theorem 3.1. $\square$

4 The equilibrium existence in Borel spaces

Consider the stochastic game model GM in (1) with the following assumptions.

Assumption 0 S and A_i are Borel spaces, and A_i is compact for each i.

Assumption 1 For all i, $\Phi_i : S \twoheadrightarrow A_i$ is a compact–valued multifunction on S.

Assumption 2 For all i, r_i is bounded and jointly measurable in $(s, \mathbf{a})$, and it is continuous in $\mathbf{a}$ for each fixed $s \in S$.

Assumption 3 For each Borel subset B of S, $Q(B \mid s, \mathbf{a})$ is jointly measurable in $(s, \mathbf{a})$, and setwise continuous in $\mathbf{a}$ for each fixed s; that is, if $\mathbf{a}_n \to \mathbf{a}$ then $Q(B \mid s, \mathbf{a}_n)$ converges to $Q(B \mid s, \mathbf{a})$.

Theorem 4.1 *Suppose the assumptions **0,1,2,3** hold and T is finite. Then the stochastic game model GM has a Nash equilibrium in Markovian strategies (possibly nonstationary).*

The result is an easy consequence of the following lemmata. The first lemma essentially shows that the theorem is true for $T = 1$; the combination of the lemmata, together with a selection theorem establishes the result for general $T < \infty$ through an induction argument.

Lemma 4.2 *For some fixed* $s \in S$, *consider a* N*-player stochastic game model in which the action sets are the compact metric spaces* $\Phi_1(s)$,..., $\Phi_N(s)$ *and the reward functions are* $r_1(s,\cdot),...,r_N(s,\cdot)$ *defined on* $\mathbf{\Phi}(s) = \Phi_1(s) \times ... \times \Phi_N(s)$. *If* $r_i(s,\cdot)$ *is continuous on* $\mathbf{\Phi}(s)$ *for each* $i \in \mathcal{N}$, *the stochastic game model* $\{\mathcal{N}, \{s\}, (\Phi_i(s), r_i)_{i\in\mathcal{N}}\}$ *admits a Nash equilibrium. That is, there exist* $\pi^* := (\pi_1^*, ..., \pi_N^*) \in \mathcal{P}(\Phi_1(s)) \times ... \times \mathcal{P}(\Phi_N(s))$ *such that, for each* i,

$$r_i(s,\pi^*) \geq r_i(s,(\pi_i,\pi_{-i}^*)) \ \forall \pi_i \in \mathcal{P}(\Phi_i(s)).$$

In the following proof we consider

$$\Pi(s) := \mathcal{P}(\Phi_1(s)) \times ... \times \mathcal{P}(\Phi_N(s)).$$

Proof: The idea is to prove the existence of a fixed point of a certain multifunction; this fixed point is an equilibrium. To prove the existence of such a fixed point, we use Glicksberg´s theorem (Theorem 5.8).

The proof is organized as follows: In Step 0 we define a multifunction BR. In Step 1 we prove that BR is defined on a nonempty compact convex subset of a locally convex Hausdorff space. In Step 2 and Step 3 we prove that BR is a $K-$mapping (Definition 5.5). By the Steps 1, 2 and 3 and Glicksberg's theorem, a fixed point exists.

Step 0: We define the multifunction BR as in the **Step 0** of the proof of the Theorem 3.1.

For each $\pi := (\pi_1, ..., \pi_N) \in \Pi(s)$, let $BR_i(\pi)$ be the set of best responses of player i to π, that is

$$BR_i(\pi) := \left\{ \bar{\pi}_i \in \mathcal{P}(\Phi_i(s)) : r_i(s,(\bar{\pi}_i,\pi_{-i})) = \sup_{\mu_i \in \mathcal{P}(\Phi_i(s))} r_i(s,(\mu_i,\pi_{-i})) \right\}.$$

Let $BR := BR_1 \times .... \times BR_N$. Note that $BR : \Pi(s) \twoheadrightarrow \Pi(s)$.

Step1: $\Pi(s)$ is a nonempty compact convex space.

Convexity is obvious. Morever, by Theorem 5.4, $\mathcal{P}(\Phi_i(s))$ equipped with the topology of weak convergence is a compact metric space; hence so is $\Pi(s)$.

Step 2: $BR(\pi)$ is a nonempty convex set for each $\pi \in \Pi(s)$.

Given $\pi := (\pi_1, ..., \pi_N) \in \Pi(s)$, the best-response problem faced by player i is a discounted Markov decision problem $MDP := \{\{s\}, \Phi_i, r_i'\}$ where $\{s\}$ is the state space, $\Phi_i(s)$ is the action space, and

$$r_i'(s,a_i) := r_i(s,(a_i,\pi_{-i})), \tag{12}$$

which is continuous in a_i. It is well known that the set $BR_i(\pi)$ is nonempty (see Puterman [18], Theorem 6.2.10).

Let $r_i^*(s)$ be the value function of the MDP and let $\mu_1, \mu_2 \in BR_i(\pi)$, $0 \leq \lambda \leq 1$ and $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ (i.e. $\mu(B) = \lambda\mu_1(B) + (1-\lambda)\mu_2(B)$ for each $B \in \mathcal{B}(\Phi_i(s))$). Then

$$r_i^{'}(s,\mu) = \lambda r_i^{'}(s,\mu_1) + (1-\lambda)r_i^{'}(s,\mu_2). \tag{13}$$

Since $\mu_1, \mu_2 \in BR_i(\pi)$, the expression (13) is the same as

$$\lambda r_i^*(s) + (1-\lambda)r_i^*(s) = r_i^*(s),$$

i.e.

$$r_i^{'}(s,\mu) = r_i^*(s).$$

Therefore, $\mu \in BR_i(\pi)$, and so $BR_i(\pi)$ is convex.

Step 3: BR is u.s.c. and $BR(\pi)$ is compact for each $\pi \in \Pi(s)$.

Let $\pi_n := (\pi_{1n}, ..., \pi_{Nn}) \in BR(\pi)$ such that

$$\pi_n \to \pi := (\pi_1, ..., \pi_N) \tag{14}$$

in the weak topology, and $\alpha_{in} \in BR_i(\pi_n)$ with

$$\alpha_{in} \to \alpha_i \in \mathcal{P}(\Phi_i(s)). \tag{15}$$

We are going to show that $\alpha_i \in BR_i(\pi)$. With this in mind, note that $\alpha_{in} \in BR_i(\pi_n)$ gives

$$r_i(s,(\alpha_{in},\pi_{-in})) \geq r_i(s,(\beta,\pi_{-in}))\ \forall\beta \in \mathcal{P}(\Phi_i(s)) \text{ and } n. \tag{16}$$

Since $r_i(s,\mathbf{a})$ is continuous in $\mathbf{a}$, by Corollary 5.3, as $n \to \infty$ (14) and (15) give

$$r_i(s,(\alpha_{in},\pi_{-in})) \to r_i(s,(\alpha_i,\pi_{-i}))$$

and

$$r_i(s,(\beta,\pi_{-in})) \to r_i(s,(\beta,\pi_{-i}))\ \forall\beta \in \mathcal{P}(\Phi_i(s)).$$

Then, as $n \to \infty$, the inequality (16) yields

$$r_i(s,(\alpha_i,\pi_{-i})) \geq r_i(s,(\beta,\pi_{-i}))\ \forall\beta \in \mathcal{P}(\Phi_i(s));$$

hence $\alpha_i \in BR_i(\pi)$, which shows that BR_i is u.s.c.

The u.s.c. proves that $BR_i(\pi)$ is a closed set in $\mathcal{P}(\Phi_i(s))$. Thus, since $\mathcal{P}(\Phi_i(s))$ is compact, $BR_i(\pi)$ is compact.

We have that for each π, $BR(\pi)$ is a compact convex nonempty set and $BR(\cdot)$ is u.s.c., so $BR(\cdot)$ is a $K-$mapping. Finally, Glicksberg´s fixed-point theorem implies the existence of $\pi^* \in \Pi(s)$ such that $\pi^* \in BR(\pi^*)$, completing the proof of Lemma 4.2, because π^* is a Nash equilibrium. $\square$

Lemma 4.3 *For $i = 1, ..., N$, let $v_i : S \to \mathbb{R}^N$ be a bounded measurable function. For each $s \in S$, $i \in \mathcal{N}$, $\mathbf{a} \in \Phi_1(s) \times ... \times \Phi_N(s)$, define*

$$H_i(s, \mathbf{a}) := r_i(s, \mathbf{a}) + \delta \int_S v_i(s')Q(ds' \mid s, \mathbf{a}).$$

Then the stochastic game model $\{\mathcal{N}, S, (A_i, \Phi_i(s), H_i(s, \cdot))_{i \in N}\}$ admits a Nash equilibrium. That is, there exists $\pi^* := (\pi_1^*, ..., \pi_N^*) \in \mathcal{P}(\Phi_1(s)) \times ... \times \mathcal{P}(\Phi_N(s))$ such that, for each $i \in \mathcal{N}$,

$$H_i(s, \pi^*) \geq H_i(s, (\pi_i, \pi_{-i}^*)) \ \forall \pi_i \in \mathcal{P}(\Phi_i(s)), \quad s \in S.$$

In the following proof we consider

$$\Pi := \mathcal{P}(A_1) \times ... \times \mathcal{P}(A_N)$$

and

$$\Pi(s) := \mathcal{P}(\Phi_1(s)) \times ... \times \mathcal{P}(\Phi_N(s)).$$

Proof: The idea is, using Lemma 4.2, to prove the existence of Nash equilibria for each $s \in S$ (Step 0), and then use a selection theorem to get a Nash equilibrium (Step 1).

Step 0: It is easy to show that thet continuity condition in Assumption 3 is equivalent to the following: If $\mathbf{a}_n \to \mathbf{a}$, then

$$\int_S f(s')Q(ds' \mid s, \mathbf{a}_n) \to \int_S f(s')Q(ds' \mid s, \mathbf{a})$$

for each bounded measurable function f. Then

$$H_i(s, \mathbf{a}_n) \to H_i(s, \mathbf{a})$$

if $\mathbf{a}_n \to \mathbf{a}$. This gives that $H_i(s, \mathbf{a})$ is continuous in $\mathbf{a}$. By Lemma 4.2, for each $s \in S$ the stochastic game model

$$\{\mathcal{N}, \{s\}, (\Phi_i(s), H_i(s, \cdot))_{i \in \mathcal{N}}\}$$

has a Nash equilibrium.

Since it is possible to find equilibria for each $s \in S$, consider the multifunction $\Theta : S \twoheadrightarrow \Pi$ which assigns the set of equilibria points to each $s \in S$, i.e.

$$(17) \qquad \Theta(s) := \{\pi^* \in \Pi(s) : H_i(s, \pi^*) \geq H_i(s, (\pi_i, \pi^*_{-i})) \\ \forall\ \pi_i \in \mathcal{P}(\Phi_i(s)),\ i = 1, ..., N\}$$

or equivalently

$$(18) \qquad \Theta(s) := \{\pi^* \in \Pi(s) : H_i(s, \pi^*) = \sup_{\pi_i \in \mathcal{P}(\Phi_i(s))} H_i(s, (\pi_i, \pi^*_{-i})), \\ i = 1, ..., N\}$$

Before proceeding with the next step, we make the following remark: without loss of generality we may assume that

$$r_i(s, \mathbf{a}) = \theta - 1 \quad \text{if} \quad \mathbf{a} \notin \mathbf{\Phi}(s) \text{ for each}\, i \in \mathcal{N},$$

where $\theta := \inf_{i,s,\mathbf{a}} r_i(s, \mathbf{a})$. So, if $\pi^* := (\pi^*_1, ..., \pi^*_N) \in \Theta(s)$ then

$$(19) \qquad r_i(s, \pi^*) \geq r_i(s, (\mu_i, \pi^*_{-i}))\, \forall\, \mu_i \in \mathcal{P}(A_i),$$

and, moreover, if $\theta = \min(\inf_{i,s,\mathbf{a}} r_i(s, \mathbf{a}), \inf_{i,s} v_i(s))$, then

$$(20) \qquad H_i(s, \pi^*) \geq H_i(s, (\mu_i, \pi^*_{-i}))\, \forall\, \mu_i \in \mathcal{P}(A_i),$$

even if support$(\mu_i) \cap \Phi_i(s)^c \neq \emptyset$. So $H_i(s, \pi)$ is well defined for every $\pi \in \Pi$.

We use this remark in the next step.

Step 1: There is a measurable selector for Θ, i.e. a measurable function $\xi : S \to \Pi$, such that $\xi(s) \in \Theta(s)$ for each $s \in S$.

We want to use Theorem 5.11to show the existence of a measurable selection. To use such theorem, we need to prove that

(i) Π satisfies the property $\mathbb{S}$ (Remark 5.10),

(ii) $\Theta(s)$ is compact for each $s \in S$, and

(iii) $\Theta^{-1}(F)$ is a Borel set in S for every closed set F in Π.

Since Π is separable metric space (by Theorem 5.4), it is easy to see (i) holds; it suffices to take a countable dense subset of Π and the family of closed balls with radius a rational number and center an element of the countable dense set.

In order to prove (ii), let $\{\pi_n^*\} \subset \Theta(s)$ (where $\pi_n^* := (\pi_{1n}^*, ..., \pi_{Nn}^*)$) be such that $\pi_n^* \to \pi^*$. Since $\pi_n^* \in \Theta(s)$, for each n we have

$$H_i(s, \pi_n^*) \geq H_i(s, (\mu_i, \pi_{-in}^*)) \quad \forall\, \mu_i \in \mathcal{P}(A_i), \tag{21}$$

and, therefore, by Corollary 5.3, as $n \to \infty$ we obtain

$$H_i(s, \pi^*) \geq H_i(s, (\mu_i, \pi_{-i}^*)) \quad \forall\, \mu_i \in \mathcal{P}(A_i). \tag{22}$$

It follows that $\pi^* \in \Theta(s)$, and so $\Theta(s)$ is a closed subset of the compact space $\Pi(s)$. Hence Θ is a compact-valued multifunction.

To prove (iii), we use (18).

First note that $H_i(\cdot, \cdot)$ and $\sup_{\pi_i \in \mathcal{P}(\Phi_i(s))} H_i(\cdot, (\pi_i, \cdot))$ are jointly measurable, then the function

$$F : S \times \Pi \to \mathbb{R}^N, \tag{23}$$

defined by

$$F(s, \pi) := \left(H_i(s, \pi) - \sup_{\mu_i \in \mathcal{P}(\Phi_i(s))} H_i(s, (\mu_i, \pi_{-i})) \right)_{i=1,...,N}$$

is jointly measurable.

Then, the set

$$F^{-1}((0, ..., 0)), \text{ with } (0, ..., 0) \in \mathbb{R}^N$$

is a Borel set. Finally, note that $F^{-1}((0, ..., 0)) = Gr(\Theta)$; hence, by Theorem 5.9, (iii) holds.

Since the assumptions of Theorem 5.11 hold, a measurable selector for Θ exists. $\square$

Proof: [Proof of Theorem 4.1] Consider $T = 1$. By assumption, $r_i(s, \cdot)$ is continuous on $\mathbf{\Phi}(s)$ for each s, and r_i is measurable on $S \times \mathbf{A}$. Thus, by Lemma 4.3 , there exists a strategy $\pi :=(\pi_1, ..., \pi_N) \in \Pi$ such that for each $s \in S$,

$$r_i(s, \pi) \geq r_i(s, (\beta, \pi_{-i}))\ \forall \beta \in \mathcal{P}(\Phi_i(s)) \quad \text{and} \quad i \in \mathcal{N}.$$

Denote π by π_1, and let $\nu_i(s) = r_i(s, \pi_1)$ $(i = 1, ..., N)$. Then π_1 is a Nash equilibrium of the one period game $\{\mathcal{N}, S, (A_i, \Phi_i, r_i)\}$, with ν_i

$(i = 1, ..., N)$ the corresponding equilibrium payoffs. Now, let $H_i(s, \mathbf{a})$ as in Lemma 4.3, then the stochastic game

$$\{\mathcal{N}, S, (A_i, \Phi_i(s), H_i(s, \cdot))_{i\in\mathcal{N}}\}$$

has an equilibrium. Denote π_2 the equilibrium of this game, then (π_1, π_2) is the equilibrium of

$$\{\mathcal{N}, S, (A_i, \Phi_i, r_i)_{i\in\mathcal{N}}, Q, 2\}.$$

We now proceed by induction. Suppose the existence of a Nash equilibrium strategy $(\pi_{T-1}, ..., \pi_1)$ of the stochastic game

$$\{\mathcal{N}, S, (A_i, \Phi_i, r_i)_{i\in\mathcal{N}}, Q, T-1\}$$

and let $\nu_i(s)$ be the corresponding discounted equilibrium payoff of player i $(i = 1, ..., N)$ and initial state s. By Lemma 4.3, the game

$$\{\mathcal{N}, S, (A_i, \Phi_i(s), H_i(s, \cdot))_{i\in\mathcal{N}}\}$$

admits an equilibrium $\pi_T \in \Pi$. It follows that $(\pi_T, \pi_{T-1}, ..., \pi_1)$ specifies an equilibrium of the stochastic game $\{\mathcal{N}, S, (A_i, \Phi_i, r_i)_{i\in\mathcal{N}}, Q, T\}$, and that the discounted equilibrium payoff is given by

$$\nu_i(s, \pi_T) = \int_{\mathbf{A}} H_i(s, \mathbf{a})\pi_T(d\mathbf{a} \mid s), \ i = 1, ..., N.$$

Note that if $\pi = (\pi_T, \pi_{T-1}, ..., \pi_1)$, then

$$\nu_i(s, \pi_T) = J_{i,\delta,T}(s, \pi).$$

Also note that $\pi \in \Pi_M$. This completes the proof. $\square$

5 Appendix

In this section we summarize some facts used in the proof of Theorems 3.1 and 4.1.

The topology of weak convergence

Definition 5.1 Let $\mathcal{P}(X)$ be the set of all probability measures on $(X, \mathcal{B}(X))$ where X is a general metric space. The topology of weak convergence is the topology in the space $\mathcal{P}(X)$ which has the following basic neighborhoods of any element $\mu \in \mathcal{P}(X)$,

$$U_\epsilon(\mu, \{f_1, ..., f_k\}) := \left\{ \lambda \in \mathcal{P}(X) : \left| \int_S f_i d\lambda - \int_S f_i d\mu \right| < \epsilon, i = 1, ..., k \right\}$$

where ε is positive and $f_1, ..., f_k$ are elements of $C(X)$ (the space of continuous bounded functions on X).

The following are some useful results.

Lemma 5.2 *Let (X, d) be a metric space. Let $\{\mu_n\} \subset \mathcal{P}(X)$ and $\mu \in \mathcal{P}(X)$. Then $\mu_n \to \mu$ in the topology of weak convergence if and only if $\int f d\mu_n \to \int f d\mu$ for all $f \in C(X)$.*

For a proof of Lemma 10 see, for instance, Proposition 7.21 in Bertsekas and Shreve [10].

Corollary 5.3 *Let $X_1, ..., X_N$ be metric spaces. Let $\{\mu_{in}\}$ be a sequence in $\mathcal{P}(X_i)$ and $\mu_i \in \mathcal{P}(X_i)$ for $i = 1, ..., N$. Each $\mathcal{P}(X_i)$ has the topology of weak convergence. If*

$$\mu_{in} \to \mu_i \quad \text{for } i = 1, ..., N,$$

then

$$\int f d\mu_{1n} ... d\mu_{Nn} \to \int f d\mu_1 ... d\mu_N \ \forall f \in C(X_1 \times ... \times X_N).$$

Next, we have a result used in sections 3 and 4; for a proof see Proposition 7.22 in Bertsekas and Shreve [4].

Theorem 5.4 *If (X, d) is a compact separable metric space, then the topology of weak convergence in $\mathcal{P}(X)$ is compact, separable and metrizable.*

Kakutani´s Theorem

If each point x of a space X is mapped into a nonempty set $T(x)$ of a space Y, we call T a set-valued mapping, also known as a multifunction or correspondence. We write $T : X \twoheadrightarrow Y$ to specify that it is a multifunction.

Definition 5.5 Let T be a multifunction from a topological space X to a topological space Y. We say that T is a $K-$mapping of X into Y if

i) for each x in X, $T(x) \subset Y$ is a compact convex set; and

ii) the graph of T, which defined as

$$Gr(T) = \{(x,y) : y \in T(x)\},$$

is closed in $X \times Y$.

If i) holds, then the condition ii) is equivalent to the upper semicontinuity (u.s.c.) condition, i.e. if $x_n \to x$ in X, $y_n \in T(x_n)$ and $y_n \to y$, then $y \in T(x)$.

Definition 5.6 A fixed point for a multifunction $T : X \twoheadrightarrow X$ is a point x such that $x \in T(x)$.

Theorem 5.7 (Kakutani theorem) *If $T : X \twoheadrightarrow X$ is a $K-$mapping, where X is a nonempty compact convex subset of $\mathbb{R}^m$, then T has a fixed point.*

See Smart [23], Chap. 9 for further details. Kakutani´s theorem was extended to Banach spaces by Bohnenblust and Karlin [5], and to locally convex spaces by Ky Fan [9] and Glicksberg [11].

Theorem 5.8 (Glicksberg theorem) *Let $T : X \twoheadrightarrow X$ be a K-mapping where X is a nonempty compact convex subset of a locally convex Hausdorff space. Then T has a fixed point.*

The proof is in Corollary 16.51 in Aliprantis and Border [1].

Correspondence with measurable graph

Let $T : X \twoheadrightarrow Y$ be a multifunction and $A \subset Y$, then the lower inverse T^{-1} is defined by

$$T^{-1}(A) := \{x \in X : T(x) \cap A \neq \emptyset\}.$$

Theorem 5.9 *Let X and Y be nonempty Borel spaces. Let $T : X \twoheadrightarrow Y$ be a compact-valued multifunction, then the following statements are equivalent:*

a) $Gr(T)$ is a Borel subset of $X \times Y$;

b) $T^{-1}(F)$ is a Borel subset of X for every closed set $F \subset Y$.

The proof is in Aliprantis and Border [1] Theorem 14.84 (see Proposition D.4 in Hernandez-Lerma and Lasserre [13] for reference).

A selection theorem

Remark 5.10 A topological space X is said to satisfy condition $\mathbb{S}$ if there is a countable family $\{F_n\}$ of closed sets which separates the points of X, that is, if x and y are any distinct points of X, there is a set F_n which contains one of them but not both.

Condition $\mathbb{S}$ is trivially satisfied when X is a Borel space.

Theorem 5.11 (Selection theorem) *Let $(S, \mathcal{A})$ be a measurable space, and X be a topological space which satisfies condition $\mathbb{S}$. If $\Theta : S \twoheadrightarrow X$ is a multifunction such that $\Theta(s)$ is a nonempty compact set of X and $\Theta^{-1}(F) \in \mathcal{A}$ for every closed set F in X, then there is a measurable selector ξ for Θ (that is, a measurable function from S to X with $\xi(s) \in \Theta(s)$ for each $s \in S$).*

See Leese [14] for the proof. For further details see Wagner [26].

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