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On Anosov energy levels that are of contact type

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Abstract

In this work we prove that given an autonomous Lagrangian L on a closed manifold M, if an Anosov energy level k can be reparametrized to make it of contact type, then $k > c_0(L)$, the critical value of L associated with the abelian covering.

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1 Introduction

Let M be a closed connected manifold, TM its tangent bundle. An autonomous Lagrangian is a smooth function, $L : TM \to \mathbf{R}$ convex and superlinear. This means that L restricted to each T_xM has positive definite Hessian and for some Riemannian metric we have

$$\lim_{|v|\to\infty} \frac{L(x,v)}{|v|} = \infty,$$

uniformly on x. Since M is compact, the Euler-Lagrange equation defines a complete flow φ_t on TM. Recall that the energy $E : TM \to \mathbf{R}$ is defined by

$$E(x,v) := \frac{\partial L}{\partial v}(x,v)v - L(x,v).$$

Since L is autonomous, E is a first integral of the flow φ_t. Let us set

$$e := \max_{x\in M} E(x,0) = -\min_{x\in M} L(x,0).$$

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Note that the energy level $E^{-1}\{k\}$ projects onto the manifold M if and only if $k \geq e$.

We shall denote by $\mathcal{L}:TM \to T^*M$ the Legendre transform which is defined by $(x,v) \to \frac{\partial L}{\partial v}(x,v)$. Our hypotheses on L assure that $\mathcal{L}$ is a diffeomorphism. Let $H : T^*M \to \mathbf{R}$ be the Hamiltonian associated to L:

$$H(x,v) := \max_{v \in T_xM}\{pv - L(x,v)\}.$$

If θ denotes the canonical 1-form on T^*M, then the Euler-Lagrange flow of L can be also obtained as the Hamiltonian flow of E with respect to the symplectic form on TM given by $-\mathcal{L}^*d\theta$ thus, if X denotes the vector field associated with the Euler-Lagrange flow then $i_X\mathcal{L}^*d\theta = -dE$. In other words, the energy function satisfies $E = H \circ \mathcal{L}$, so that energy levels for L are sent to level sets of H.

Definition: An energy level $\Sigma = H^{-1}\{k\}$ is of *contact type* if there exists a 1-form λ on Σ such that $d\lambda = \omega(= -d\theta)$ and $\lambda(X) \neq 0$ on every point of Σ.

An Anosov energy level, is a regular energy level $E^{-1}\{k\}$ on which the flow φ_t is an Anosov flow.

In [6] G. Paternain shows that if an Anosov energy level k on a surface can be reparametrized to make it of contact type then $k > c_0(L)$ the critical value of L associated with the abelian covering. Our goal in this note is to generalize this result, we shall prove the following:

Theorem A. *Given an autonomous Lagrangian L on a closed manifold M with dim $M \geq 2$, If an Anosov energy level k can be reparametrized to make it of contact type then $k > c_0(L)$.*

This completes the previous result.

2 Preliminaries and proofs

Let $\mathcal{M}(L)$ be the set of probabilities on the Borel σ-algebra on TM that have compact support and are invariant under the flow φ_t. Let $H_1(M,\mathbf{R})$ be the first real homology group of M. Given a closed one-form ω on M and $\rho \in H_1(M,\mathbf{R})$, let $\langle [w], \rho\rangle$ denote the integral of ω on any closed curve in the homology class ρ. If $\mu \in \mathcal{M}(L)$, its homology is defined as the unique $\rho(\mu) \in H_1(M,\mathbf{R})$ such that

$$\langle [w], \rho(\mu)\rangle = \int_{TM} \omega d\mu,$$

for all closed 1-form on M. The integral on the right-hand side is with respect to μ with ω considered as the function $\omega : TM \to \mathbf{R}$.

Let μ be a φ_t-invariant probability supported on the energy level $\Sigma = E^{-1}\{k\}$. The Schwartzman's asymptotic cycle $\mathcal{S}(\mu) \in H_1(\Sigma, \mathbf{R})$ of μ is defined by

$$\langle [\Omega], \mathcal{S}(\mu) \rangle = \int_\Sigma \Omega(X) d\mu,$$

for every closed 1-form Ω on Σ, where X is the Lagrangian field on Σ. The homology $\rho(\mu)$ of the measure μ is the projection of its asymptotic cycle by $\pi_* : H_1(\Sigma, \mathbf{R}) \to H_1(M, \mathbf{R})$.

Recall that the L-action of an absolutely continuous curve $\gamma : [a, b] \to M$ is defined by

$$A_L(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt.$$

Given two points $x_1, x_2 \in M$ and some $T > 0$ denoted by $C(x_1, x_2)$ the set of absolutely continuous curves $\gamma : [a, b] \to M$ with $\gamma(0) = x_1$ and $\gamma(T) = x_2$. For each $k \in \mathbf{R}$, we define

$$\Phi_k(x_1, x_2; T) := \inf\{A_{L+k}(\gamma) \mid \gamma \in C(x_1, x_2)\ \}.$$

The action potential $\Phi_k\ :\ M \times M \to \mathbf{R} \cup \infty$ of L is defined by

$$\Phi_k(x_1, x_2) := \inf_{T>0} \Phi_k(x_1, x_2; T).$$

Definition (Mañe): The critical value of L is the real number

$$c = c(L) := \inf\{k \mid \Phi_k(x, x) > -\infty \text{ for some } x \in M\}.$$

Note that if $k > c(L)$ actually $\Phi_k(x, x) > -\infty$ for all $x \in M$. Since L is convex and superlinear, and M is compact, such a number exists. We can also consider the critical value of the lift of the Lagrangian L to a covering of the compact manifold M. Suppose that $p : N \to M$ is a covering space and consider the Lagrangian $\mathbf{L} : TN \to \mathbf{R}$ given by $\mathbf{L} := \mathbf{L} \circ dp$, for each $k \in \mathbf{R}$ we can define an action potential Φ_k in $N \times N$ just as above and similarly we obtain a critical value $c(\mathbf{L})$ for $\mathbf{L}$. It can be easily checked that if N_1 and N_2 are coverings of M such that N_1 covers N_2 then

$$c(\mathbf{L_1}) \leq c(\mathbf{L_2}),$$

where $\mathbf{L_1}$ and $\mathbf{L_2}$ denote the lifts of the Lagrangian L to N_1 and N_2 respectively.

Among all possible coverings of M there are two distinguished ones; the universal covering which we shall denote by $\widetilde{M}$, and the abelian covering which we shall denote by $\overline{M}$. The latter is defined as the covering of M whose fundamental group is the kernel of the Hurewicz homomorphis $\pi_1(M) \to H_1(M, \mathbf{R})$ these coverings give rise to the critical values

$$c_u(L) := c(\widetilde{L}) \text{ and } c_a(L) = c_0(L) := c(\overline{L})$$

where $\widetilde{L}$ and $\overline{L}$ denote the lifts of the Lagrangian L to $\widetilde{M}$ and $\overline{M}$ respectively. Therefore we have $c_u(L) \leq c_0(L)$, but in general the inequality may be strict as it was shown in [5].

2.1 Contact and Anosov energy levels

We begin by introducing some concepts related to Euler-Lagrange flow restricted on energy levels.

Definition: An energy level $\Sigma = H^{-1}\{k\}$ is of *contact type* if there exists a 1-form λ on Σ such that $d\lambda = \omega(= -d\theta)$ and $\lambda(X) \neq 0$ on every point of Σ.

Equivalently, if there exists a vector field Y based on Σ, such that the Lie derivative $L_Y\omega = \omega$. The correspondence is given by $i_Y\omega = \lambda$. The vector field Y must be tranverse to Σ because if it is tangent to Σ one has that $\lambda(X) = \omega(Y, X) = dH(Y) = 0$.

Lemma 2.2.1 *The set $\{k \in \mathbf{R} \mid H^{-1}\{k\}$ is of contact type$\}$ is open.*

Proof: Suppose that $\Sigma = H^{-1}\{k\}$ is of contact type, then k is a regular point of H, for otherwise the Hamiltonian flow contains a singularity on Σ and that violates the condition $\lambda(X) \neq 0$. If λ is a contact form for λ, since $d\lambda = \omega$ then $\lambda = pdx|_\Sigma + \tau$, where τ is a closed 1-form on Σ. We can extend λ as follows. Let $\pi : U \to \Sigma$ be the projection of an open neighbourhood U of Σ onto Σ. Let $\overline{\lambda} := pdx + \pi^*(\tau)$ then $d\overline{\lambda} = \omega$ and for m near k $d\overline{\lambda}|_{H^{-1}\{m\}} \neq 0$. □

The following criterion for contact type appears in [2]

Proposition 2.2.2 *If L is a convex Lagrangian then an energy level $E^{-1}\{k\}$ is of contact type if and only if $\int_{TM}(L+k)d\mu > 0$ for any invariant measure μ supported $E^{-1}\{k\}$ with zero asymptotic cycle $\mathcal{S}(\mu) = 0$.*

We shall need the following result:

Lemma 2.2.3 *Suppose M a closed connected manifold with dim $M \geq 2$ and $M \neq T^2$. If $k > e$ then $\pi_* : H_1(E^{-1}\{k\}, \mathbf{R}) \to H_1(M, \mathbf{R})$ is an isomorphism.*

Proof: Since $k > e$ and dim $M \geq 2$ then the energy level $E^{-1}\{k\}$ is isomorphic to the unit tangent bundle of M with the canonical projection. Using the Gysin exact sequence of the circle bundle $\pi : E^{-1}\{k\} \to M$ one can show that (see [3], lemma 1.45) the lemma follows if M is orientable.

If M is not orientable and dim $M \geq 3$, using the exact homotopy sequence of the circle bundle $\pi : E^{-1}\{k\} \to M$:

$$0 = \pi_1(S^{n-1}) \to \pi_1(E^{-1}\{k\}) \xrightarrow{\pi_*} \pi_1(M) \to \pi_0(S^{n-1}) = 0,$$

thus we obtain that $\pi_* : \pi_1(E^{-1}\{k\}) \to \pi_1(M)$ is an isomorphism, which in turn implies that $\pi_* : H_1(E^{-1}\{k\}, \mathbf{R}) \to H_1(M, \mathbf{R})$ is a isomorphism. In the case that M is not orientable and dim $M = 2$, the proof is a minor modification of the above arguments. □

An *Anosov energy level*, is a regular energy level $E^{-1}\{k\}$ on which the flow φ_t is an Anosov flow. In [1] was shown

Proposition 2.2.4 *If the energy level k is Anosov, then*

$$k > c_u(L).$$

In [5] G. Paternain and M. Paternain gave examples of Anosov energy levels k with $k < c_0(L)$ on surface of genus greater or equal than two. These examples give a negative answer to a question raised by Mañe.

2.3 Proof of theorem A

Now we shall prove the theorem A, for this we use the next result of Paternain [4] and following his ideas we shall prove this result

Proposition 2.3.1 *If $c_u(L) < k < c_0(L)$, there exists an invariant measure μ supported in the energy level k, such that $\rho(\mu) = 0$ and*

$$\int_{E^{-1}\{k\}} (L + k) d\mu \leq 0.$$

Proof of theorem A: It follows from a result of Margulis that the energy levels of T^2 does not support Anosov flows thus in the case of T^2 the theorem is valid trivially. Therefore we can assume that $M \neq T^2$. Now as the flow is Anosov, by the proposition 2.2.4 we have that $k > c_u(L)$. But if the energy level $k \in (c_u(L), c_0(L))$ then applying the proposition 2.3.1, there exists an invariant measure μ such that $\rho(\mu) = 0$ and

$$\int_{E^{-1}\{k\}} (L+k)d\mu \leq 0.$$

therefore the lemma 2.2.3 and proposition 2.2.2 implies that, the energy level k is not of contact type. Finally by lemma 2.2.1 the energy $k = c_0(L)$ cannot be of contact type then, we must have that $k > c_0(L)$. □

Proof of proposition 2.3.1 : Since $k < c_0(L) = c_a(L)$ there exists $T > 0$ and an absolutely continuous closed curve $\gamma : [0, T] \to M$ homologous to zero such that

$$A_{L+k}(\gamma) < 0. \tag{1}$$

For $n \geq 1$, let us denote by $\gamma^n : [0, nT] \to M$ the curve γ wrapped up n times. Since $k > c_u(L)$, by (1) γ^n cannot be homotopic to zero. Let $p : \widetilde{M} \to M$ the covering projection and take y such that $p(y) = \gamma(0) = \gamma(T)$, and let $\widetilde{\gamma}^n : [0, nT] \to \widetilde{M}$ be the unique lift of γ^n with $\widetilde{\gamma}^n(0) = y$. As $k > c_u(L)$ for each n there exists a solution $x_n(t)$ of Euler-Lagrange with energy k and some $T_n > 0$ such that $x_n(0) = y$ and $x_n(T_n) = \widetilde{\gamma}^n(nT)$.

Let μ_n denote the probability measure in TM uniformly distributed along $p \circ x_n|_{[0,T_n]}$ and take μ a point of accumulation of μ_n, this measure μ has the required properties of the proposition 2.3.1. □

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