Morfismos, Vol. 8, No. 2, 2004, pp. 1–25

Homotopy triangulations of a manifold triple *

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Abstract

The set of homotopy triangulations of a given manifold fits into a surgery exact sequence which is the main tool for the classification of manifolds. In the present paper we describe relations between homotopy triangulations of different manifolds for a given manifold triple and its connection to surgery theory. We introduce a group of obstructions to split a homotopy equivalence along a pair of submanifolds and study its properties. The main results are given by commutative diagrams of exact sequences.

2000 Mathematics Subject Classification: 57R67, 57Q10, 19J25, 19G24, 18F25.
Keywords and phrases: surgery on manifolds, surgery and splitting obstruction groups, surgery exact sequence, the set of homotopy triangulations.

1 Introduction

Let X^n be a closed n–dimensional CAT ($CAT = TOP, PL, Diff$) manifold with fundamental group $\pi = \pi_1(X)$. A fundamental problem of geometric topology is to describe all closed n–dimensional CAT–manifolds which are homotopy (simple homotopy) equivalent to X. The main tool for such investigation is the surgery exact sequence (see [23, 20])

$$\cdots \to L_{n+1}(\pi) \to \mathcal{S}_n(X) \to [X, G/CAT] \xrightarrow{\sigma} L_n(\pi) \to \cdots \qquad (1.1)$$

*Invited article. Partially supported by Russian Foundation for Fundamental Research Grant no. 02–01–00014, CONACyT, DGAPA–UNAM, Fulbright–García Robles and UW–Madison.

The elements of the set $[X, G/CAT]$ are called normal invariants [23, §10]. In the present paper we shall work in the category of topological manifolds ($CAT = TOP$), and consider the groups $L_*(\pi) = L^s_*(\pi)$ which give obstructions to simple homotopy equivalence. The elements of $\mathcal{S}_n(X) = \mathcal{S}^s_n(X)$ are called homotopical triangulations or s–triangulations of the manifold X (see [23, §10],[20]). The structure set $\mathcal{S}^s_n(X)$ is the set of s–cobordism classes of CAT–manifolds which are simple homotopy equivalent to X^n (see [23, §10] and [20, p. 542]).

Let $Y \subset X$ be a submanifold of codimension q in X. A simple homotopy equivalence $f : M \to X$ splits along the submanifold Y if it is homotopy equivalent to a map g transversal to Y, such that for $N = g^{-1}(Y)$ the restrictions

$$g|_N : N \to Y, \quad g|_{(M\setminus N)} : M \setminus N \to X \setminus Y$$

are simple homotopy equivalences. Let ∂U be the boundary of a tubular neighborhood U of the submanifold Y in X. There exists a group $LS_{n-q}(F)$ of obstructions to splitting (see [23, 20]) which depends only on $n - q$ mod 4 and a pushout square

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(U) & \to & \pi_1(X) \end{pmatrix} \tag{1.2}$$

of fundamental groups with orientations.

We consider the group $LP_{n-q}(F)$ of obstructions to surgery on pairs of manifolds (X, Y) as defined in [23, 20]. This group depends as well only on $n-q$ mod 4 and the square F. In [20] Ranicki introduced a set $\mathcal{S}_{n+1}(X, Y, \xi)$ of homotopy triangulations of a pair of manifolds (X, Y), where ξ denotes the normal bundle of Y in X. This set consists of concordance classes of maps $f : (M, N) \to (X, Y)$ which are splitted along Y and fits into a commutative braid of exact sequences ([20, Proposition 7.2.6])

$$\begin{array}{ccccccccccc}
\to & \mathcal{S}_{n+1}(X,Y,\xi) & & \longrightarrow & & H_n(X,\mathbf{L}_\bullet) & & \longrightarrow & & L_n(\pi_1(X)) & \to \\
 & \nearrow \quad \searrow & & & & \nearrow \quad \searrow & & & & \nearrow \quad \searrow & \\
 & & & \mathcal{S}_{n+1}(X) & & & & LP_{n-q}(F) & & & \\
 & \searrow \quad \nearrow & & & & \searrow \quad \nearrow & & & & \searrow \quad \nearrow & \\
\to & L_{n+1}(\pi_1(X)) & & \longrightarrow & & LS_{n-q}(F) & & \to & & \mathcal{S}_n(X,Y,\xi) & \to,
\end{array} \tag{1.3}$$

where $H_n(X, \mathbf{L}_\bullet) \simeq [X, G/TOP]$ and the spectrum $\mathbf{L}_\bullet$ is a one–connected cover of the Ω–spectrum $\mathbb{L}(\mathbb{Z})$ with $\mathbf{L}_{\bullet 0} \simeq G/TOP$ [21] (see also [23, 20]).

Consider a triple $Z^{n-q-q'} \subset Y^{n-q} \subset X^n$ of closed topological manifolds. We shall suppose that every submanifold is locally flat in an ambient manifold, and equipped with the structure of normal topological bundle (see [20, p. 562–563]). The groups $LT_{n-q-q'}(X, Y, Z)$ of obstructions to surgery on manifold triples were recently introduced in [18]. The groups LT_* are natural generalizations of the obstruction groups LP_* to surgery on manifold pairs.

The set $\mathcal{S}_{n+1}(X, Y, Z)$ of homotopy triangulations of the triple (X, Y, Z) is the natural generalization of the structure sets $\mathcal{S}_{n+1}(X)$ and $\mathcal{S}_{n+1}(X, Y, \xi)$. This set fits in the exact sequence

$$\cdots \to LT_{k+1}(X, Y, Z) \to \mathcal{S}_{n+1}(X, Y, Z) \to H_n(X, \mathbf{L}_\bullet) \to LT_k(X, Y, Z) \to \cdots \tag{1.4}$$

where $k = n - q - q'$. The relations between $\mathcal{S}_*(X, Y, Z)$ and $\mathcal{S}_*(X, Y, \xi)$ are given by the following braid of exact sequences [18]

$$\begin{tikzcd}
{} \arrow[r] & \mathcal{S}_{n+1}(X,Y,Z) \arrow[rr] \arrow[dr] & & H_n(X,\mathbf{L}_\bullet) \arrow[rr] \arrow[dr] & & LP_{n-q}(F) \arrow[r] \arrow[dr] & {} \\
{} \arrow[ur] \arrow[dr] & & \mathcal{S}_{n+1}(X,Y,\xi) \arrow[ur] \arrow[dr] & & LT_k(X,Y,Z) \arrow[ur] \arrow[dr] & & {} \\
{} \arrow[r] & LP_{n-q+1}(F) \arrow[rr] \arrow[ur] & & LS_k(\Psi) \arrow[rr] \arrow[ur] & & \mathcal{S}_n(X,Y,Z) \arrow[r] \arrow[ur] & {}
\end{tikzcd} \tag{1.5}$$

where Ψ is the square of fundamental groups in the splitting problem for the pair (Y, Z). Remark that the map

$$\mathcal{S}_{n+1}(X, Y, Z) \longrightarrow \mathcal{S}_{n+1}(X, Y, \xi)$$

in (1.5) is a natural forgetful map.

We have the following topological normal bundles: ξ for the submanifold Y in X, η for the submanifold Z in Y, and ν for the submanifold Z in X. Let U_ξ be a space of normal bundle ξ. We shall suppose that the space U_ν of the normal bundle ν is identified with the space V_ξ of restriction of bundle ξ on a space U_η of normal bundle η in such way that $\partial U_\nu = \partial U_\xi|_{U_\eta} \cup U_\xi|_{\partial U_\eta}$.

In the present paper we describe various relations between sets of homotopy triangulations $\mathcal{S}_*(X)$, $\mathcal{S}_*(Y)$, $\mathcal{S}_*(Z)$, $\mathcal{S}_*(X, Y, \xi)$, $\mathcal{S}_*(X, Z, \nu)$, $\mathcal{S}_*(Y, Z, \eta)$, and $\mathcal{S}_*(X, Y, Z)$ which arise naturally for a triple of embedded manifolds. The main results are given by commutative diagrams of exact sequences. We also introduce a group LSP_* of obstructions to split a simple homotopy equivalence $f : M \to X$ along a pair of embedded submanifolds $(Z \subset Y) \subset X$, and describe its relations to the classical obstruction groups in surgery theory. The group LSP_* is a natural straightforward generalization of the group LS_* if we consider the pair of submanifolds $Z \subset Y$ instead of the submanifold Y.

2 Preliminaries

In this section we recall some definitions and results used in the paper (see [23, 20, 10, 11, 18, 21, 22, 8]).

The definition of a topological normal map $(f, b) : M \to X$ is given in [20, p. 36] (see also [21, 19]). For $n \geq 5$ the set of concordance classes of topological normal maps into a manifold X coincides with the set $[X, G/TOP] \cong H_n(X, \mathbf{L}_\bullet)$ (see [20, 21, 19]). We shall use the definition of topological manifold pair (X, Y, ξ) as given in [20, §7.2]. Here Y is the submanifold of codimension q for which the topological normal bundle $(D^q, S^{q-1}) \to (E(\xi), S(\xi)) \to Y$ is defined. A topological normal map

$$((f, b), (g, c)) : (M, N, \xi_1) \to (X, Y, \xi)$$

with $N = f^{-1}(Y)$ is defined in [20, §7.2]. For this map the restrictions $(f, b)|_N = (g, c) : N \to Y$ and $(f, b)|_P = (h, d) : (P, S(\xi_1)) \to (Z, S(\xi))$ are topological normal maps where $P = \overline{M \setminus E(\xi_1)}$, $Z = \overline{X \setminus E(\xi)}$. Additionally, the restriction $(h, d)|_{S(\xi_1)} : S(\xi_1) \to S(\xi)$ coincides with the induced map $(g, c)^! : S(\xi_1) \to S(\xi)$, and we have $(f, b) = (g, c)^! \cup (h, d)$. It follows from [20, Proposition 7.2.3] that the set of concordance classes of normal maps to the pair (X, Y, ξ) coincides with the set of concordance classes of normal maps to the manifold X.

A normal map $((f, b), (g, c)) : (M, N) \to (X, Y)$ represents an element of $\mathcal{S}_{n+1}(X, Y, \xi)$ if the maps

$$f : M \to X, \ g : N \to Y, \text{ and } h : (P, S(\nu)) \to (Z, S(\xi))$$

are s–triangulations (see [20, p. 571]). It follows from the definition of s–triangulation of the pair (X, Y, ξ) that the forgetful maps

$$\begin{array}{ll} \mathcal{S}_{n+1}(X, Y, \xi) \to \mathcal{S}_{n+1}(X), & ((f, b), (g, c)) \to (f, b); \\ \mathcal{S}_{n+1}(X, Y, \xi) \to \mathcal{S}_{n-q+1}(Y), & ((f, b), (g, c)) \to (g, c) \end{array}$$

are well defined. In general, the map $\mathcal{S}_{n+1}(X, Y, \xi) \to \mathcal{S}_{n+1}(X)$ is not an epimorphism or a monomorphism [20, p. 571].

Recall that if (X, Y) is a pair of topological spaces, equipped with an orientation homomorphism, then the relative groups $\mathcal{S}_*(X, Y)$ are well defined [20, p. 560]. These groups fit into the following exact sequences

$$\begin{aligned} &\cdots \to \mathcal{S}_n(Y) \to \mathcal{S}_n(X) \to \mathcal{S}_n(X, Y) \to \mathcal{S}_{n-1}(X, Y) \to \cdots, \\ &\cdots \to H_n(X, Y; \mathbf{L}_\bullet) \to L_n(\pi_1(Y) \to \pi_1(X)) \to \mathcal{S}_n(X, Y) \to \cdots. \end{aligned} \tag{2.1}$$

For a manifold pair (X, Y) the groups $\mathcal{S}_*(X, Y)$ differ from the groups $\mathcal{S}_*(X, Y, \xi)$.

A topological normal map

$$((f, b), (g, c), (h, d)) : (M, N, K) \to (X, Y, Z)$$

of a triple of manifolds (X, Y, Z) is given by topological normal maps of manifold pairs $((f, b), (g, c)) : (M, N) \to (X, Y)$ and $((g, c), (h, d)) : (N, K) \to (Y, Z)$. A topological normal map $(f, b) \in [X, G/TOP]$ with $f : M \to X$ (see [20, 18]) defines the topological normal map $((f, b), (g, c), (h, d)) : (M, N, K) \to (X, Y, Z)$ as follows from topological transversality (see [20, Proposition 7.2.3]). A topological normal map

$$((f, b), (g, c), (h, d)) : (M, N, K) \to (X, Y, Z)$$

is an s–triangulation of the triple (X, Y, Z) if the constituent normal maps $((f, b), (g, c)) : (M, N) \to (X, Y)$, $((g, c), (h, d)) : (N, K) \to (Y, Z)$, and $((f, b), (h, d)) : (M, K) \to (X, Z)$ are s–triangulations. The set of concordance classes of s–triangulations of the triple (X, Y, Z) is denoted by $\mathcal{S}_{n+1}(X, Y, Z)$. As follows from [18] this set has a group structure and fits into the commutative braid of exact sequences (1.5).

In [18] the groups $LT_*(X, Y, Z)$ and the map

$$\Theta_*(f, b) : [X, G/TOP] = H_n(X, L_\bullet) \to LT_{n-q-q'}(X, Y, Z)$$

are defined in such a way that the normal map (f, b) is normally bordant to an s–triangulation of the triple (X, Y, Z) if and only if $\Theta_*(f, b) = 0$ (for $n - q - q' \geq 5$).

Proposition 2.1 *Suppose that a topological normal map*

$$((f, b), (g, c), (h, d)) : (M, N, K) \to (X, Y, Z)$$

gives s–triangulations of manifold pairs

$$((f, b), (g, c)) : (M, N) \to (X, Y)$$

and

$$((g,c),(h,d)) : (N,K) \to (Y,Z).$$

Then the map

$$((f,b),(h,d)) : (M,K) \to (X,Z)$$

is an s–triangulation of the pair (X,Z)*.*

Proof: The maps $f : M \to X$ and $h : K \to Z$ are simple homotopy equivalences by definition. Denote by V_ξ the space of restriction of the bundle ξ on the space U_η. We can identify the space V_ξ with the space U_ν of the bundle ν. The map g already splitted along the submanifold Z. Hence its restriction is a simple homotopy equivalence on $Y \setminus Z$ and, therefore, we have a simple homotopy equivalence $f|_H$ where H is $\overline{U_\xi \setminus U_\nu}$. By definition the restriction of the map f gives a simple homotopy equivalence on $E = \overline{X \setminus U_\xi}$. The map $f|_{H\cap E}$ will be a simple homotopy equivalence since g is a simple homotopy equivalence on $Y \setminus Z$ and on the boundary of the tubular neighborhood. Hence the map $f|_{H\cup E}$ will be a simple homotopy equivalence (see [8, §23]). We can identify $\overline{X \setminus U_\nu}$ with $H \cup E$ and the proposition is proved. □

Proposition 2.2 *For the triple* $Z \subset Y \subset X$ *the natural forgetful maps fit in the following commutative diagrams*

$$\begin{array}{ccc} \mathcal{S}_{n+1}(X,Y,Z) & \to & \mathcal{S}_{n+1}(X,Y,\xi) \\ \downarrow & & \downarrow \\ \mathcal{S}_{n-q+1}(Y,Z,\eta) & \to & \mathcal{S}_{n-q+1}(Y), \end{array}$$

$$\begin{array}{ccc} \mathcal{S}_{n+1}(X,Y,Z) & \to & \mathcal{S}_{n+1}(X,Y,\xi) \\ \downarrow & & \downarrow \\ \mathcal{S}_{n+1}(X,Z,\nu) & \to & \mathcal{S}_{n+1}(X), \end{array}$$

and

$$\begin{array}{ccc} \mathcal{S}_{n+1}(X,Y,Z) & \to & \mathcal{S}_{n+1}(X,Z,\nu) \\ \downarrow & & \downarrow \\ \mathcal{S}_{n-q+1}(Y,Z,\eta) & \to & \mathcal{S}_{n-q-q'+1}(Z). \end{array}$$

Proof: The result follows from the definition of s–triangulation of triple of manifolds. □

In the present paper we shall use realizations of the various groups and natural maps in surgery theory on the spectra level (see [23, 20, 18, 21, 1, 8]). The surgery exact sequence (1.1) in the topological category is realized on the spectra level [21]. The commutative diagrams (1.3) and (1.5) are realized on the spectra level (see [20, 18]), too. We recall here that transfer and induced maps of L–groups are realized on the spectra level. A homomorphism of oriented groups $f : \pi \to \pi'$ induces a cofibration of Ω–spectra [8]

$$\mathbb{L}(\pi) \longrightarrow \mathbb{L}(\pi') \longrightarrow \mathbb{L}(f),$$

where $\pi_n(\mathbb{L}(\pi)) = L_n(\pi)$ and similarly for the other spectra. The homotopy long exact sequence of this cofibration gives a relative exact sequence of L_*–groups of the map f

$$\ldots \to L_n(\pi) \to L_n(\pi') \to L_n(f) \to L_{n-1}(\pi) \to \ldots.$$

Let $p : E \to X$ be a bundle over an n–dimensional manifold X whose fiber is an m–dimensional manifold M^m. Then the transfer map (see [23, 10, 11]) $p^* : L_n(\pi_1(X)) \to L_{n+m}(\pi_1(E))$ is well defined. This map is realized on the spectra level by a map

$$p^! : \mathbb{L}(\pi_1(X)) \to \Sigma^{-m}\mathbb{L}(\pi_1(E)).$$

For the pair of manifolds (X, Y) consider a homotopy commutative diagram of spectra (see [23, 20, 1])

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \to & \Sigma^{-q}\mathbb{L}(\pi_1(\partial U) \to \pi_1(Y)) & \to & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) \\ & \searrow & \downarrow & & \downarrow \\ & & \Sigma^{1-q}\mathbb{L}(\pi_1(\partial U)) & \to & \Sigma^{1-q}\mathbb{L}(\pi_1(X \setminus Y)), \end{array} \tag{2.2}$$

where the left maps are the transfer maps on the spectra level. The spectrum $\mathbb{L}S(F)$ is defined as a homotopy cofiber of the map

$$\Sigma\mathbb{L}(\pi_1(Y)) \to \Sigma^{-q-1}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)),$$

and the spectrum $\mathbb{L}P(F)$ is defined as a hmotopy cofiber of the map

$$\Sigma^{-1}\mathbb{L}(\pi_1(Y)) \to \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y))$$

(see [1, 13, 17, 12, 8]). Then the homotopy groups of these spectra coincide with the splitting obstruction groups $\pi_n(\mathbb{L}S(F)) \cong LS_n(F)$ and the surgery obstruction groups for the manifold pair $\pi_n(\mathbb{L}P(F)) \cong LP_n(F)$.

It follows from [20, 18, 21] that the sets of s–triangulations are realized on the spectra level. we shall denote by $\mathbb{S}(X)$ the corresponding spectrum for the structure set $\mathcal{S}_{n+1}(X)$, and similarly for the other structure sets.

For the triple $Z^{n-q-q'} \subset Y^{n-q} \subset X^n$ denote by j the natural inclusion $(X \setminus Z, Y \setminus Z) \to (X, Y)$ of CW–pairs of codimension q (see [20, §7.2]). Let F_Z be the square of fundamental groups for the pair $(X \setminus Z, Y \setminus Z)$. Let $W = (X \setminus Z)$. The map j induces the map of squares $F \to F_Z$ and therefore a commutative diagram

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\ldots \to & & LS_{n-q}(F_Z) & \to & L_{n-q}(\pi_1(Y \setminus Z)) & \to & \\
 & & \downarrow & & \downarrow & & \\
\ldots \to & & LS_{n-q}(F) & \to & L_{n-q}(\pi_1(Y)) & \to & \\
 & & \downarrow & & \downarrow & & \\
\ldots \to & & LNS_k & \to & L_{n-q}(\pi_1(Y \setminus Z) \to \pi_1(Y)) & \xrightarrow{tr^{rel}} & \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

$$
\begin{array}{cccc}
 & \vdots & & \\
 & \downarrow & & \\
\to & L_n(\pi_1(X \setminus Y) \to \pi_1(W)) & \to \cdots & \\
 & \downarrow & & \\
\to & L_n(\pi_1(X \setminus Y) \to \pi_1(X)) & \to \cdots & \\
 & \downarrow & & \\
\xrightarrow{tr^{rel}} & L_n(\pi_1(W) \to \pi_1(X)) & \to \cdots & \\
 & \downarrow & & \\
 & \vdots & &
\end{array} \tag{2.3}
$$

where $k = n-q-q'$. Two right upper horizontal maps induce the map of the relative groups of two right upper vertical maps with relative groups LNS_* (see [7, 15]). Diagram (2.3) is realized on the spectra level.

3 Homotopy triangulations of a triple of manifolds

In this section we describe relations between various structure sets which arise naturally for a triple of manifolds. Denote by Φ the square of fundamental groups in the splitting problem for the pair (X, Z). Recall here that F, F_Z, and Ψ denote the similar squares for the pairs (X, Y), $(X \setminus Z, Y \setminus Z)$, and (Y, Z), respectively.

Theorem 3.1 *The natural forgetful maps of Proposition 2.2 fit in the following commutative diagrams of exact sequences*

$$
\begin{array}{ccccccccc}
\to \mathcal{S}_n(X \setminus Y) & & \longrightarrow & & \mathcal{S}_n(X, Y, \xi) & & \longrightarrow & & LS_{k-1}(\Psi) \;\to \\
\nearrow \quad \searrow & & & & \nearrow \quad \searrow & & & & \nearrow \quad \searrow \\
 & & \mathcal{S}_n(X, Y, Z) & & & & \mathcal{S}_{n-q}(Y) & & \\
\searrow \quad \nearrow & & & & \searrow \quad \nearrow & & & & \searrow \quad \nearrow \\
\to \; LS_k(\Psi) & & \longrightarrow & & \mathcal{S}_{n-q}(Y, Z, \eta) & & \longrightarrow & & \mathcal{S}_{n-1}(X \setminus Y) \to,
\end{array}
\tag{3.1}
$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & & LS_{n-q}(F_Z) & \to & LS_{n-q}(F) & \to & LNS_k & & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & & \mathcal{S}_n(X, Y, Z) & \to & \mathcal{S}_n(X, Y, \xi) & \to & LS_{k-1}(\Psi) & & \to \cdots, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & & \mathcal{S}_n(X, Z, \nu) & \to & \mathcal{S}_n(X) & \to & LS_{k-1}(\Phi) & & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
\tag{3.2}
$$

and

$$
\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & LS_{n-q}(F_Z) & \to & \mathcal{S}_n(X, Y, Z) & \to & \mathcal{S}_n(X, Z, \nu) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathcal{S}_{n-q}(Y \setminus Z) & \to & \mathcal{S}_{n-q}(Y, Z, \eta) & \to & \mathcal{S}_k(Z) & \to \cdots, \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathcal{S}_{n-1}(X \setminus Z, X \setminus Y) & \to & \mathcal{S}_{n-1}(X \setminus Y) & \to & \mathcal{S}_{n-1}(X \setminus Z) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots &
\end{array}
\tag{3.3}
$$

where $k = n - q - q'$. The diagrams are realized on the spectra level.

Proof: Diagram (3.1) was obtained in [18]. The group $H_n(X, \mathbf{L}_\bullet)$ maps to the groups of a commutative diagram of forgetful maps

$$
\begin{array}{ccc}
LT_{n-q-q'}(X, Y, Z) & \to & LP_{n-q}(F) \\
\downarrow & & \downarrow \\
LP_{n-q-q'}(\Phi) & \to & L_n(\pi_1(X)),
\end{array}
\tag{3.4}
$$

which is realized on the spectra level (see [18, 15]). We obtain a commutative diagram in the form of pyramid with the group $H_n(X, \mathbf{L}_\bullet)$ in the top. This diagram is realized on the spectra level. The cofibres of the maps which correspond to the side edges give a homotopy commutative diagram of spectra of structure sets

$$\begin{array}{ccc} \mathbb{S}(X,Y,Z) & \to & \mathbb{S}(X,Y,\xi) \\ \downarrow & & \downarrow \\ \mathbb{S}(X,Z,\nu) & \to & \mathbb{S}(X), \end{array}$$

which realizes the second commutative diagram from Proposition 2.2. It is easy to see that diagram (3.2) follows from the diagram above and diagrams (1.3), (2.3). Now consider a commutative diagram

$$\begin{array}{ccc} H_n(X,\mathbf{L}_\bullet) & = & H_n(X,\mathbf{L}_\bullet) \\ \downarrow & & \downarrow \\ H_{n-q}(Y,\mathbf{L}_\bullet) & \to & H_{n-q-q'}(Z,\mathbf{L}_\bullet), \end{array} \tag{3.5}$$

in which the vertical maps and the lower horizontal map are given by compositions of transfer maps and isomorphisms (see [20, p. 579])

$$\begin{gathered} H_{n-q}(Y,\mathbf{L}_\bullet) \cong H_n(X, X\setminus Y;\mathbf{L}_\bullet), \\ H_{n-q-q'}(Z,\mathbf{L}_\bullet) \cong H_n(X, X\setminus Z;\mathbf{L}_\bullet), \\ H_{n-q-q'}(Z,\mathbf{L}_\bullet) \cong H_{n-q}(Y, Y\setminus Z;\mathbf{L}_\bullet). \end{gathered}$$

Consider a natural map of (3.5) to the commutative diagram of forgetful maps (see [18, 15])

$$\begin{array}{ccc} LT_{n-q-q'}(X,Y,Z) & \to & LP_{n-q-q'}(\Phi) \\ \downarrow & & \downarrow \\ LP_{n-q-q'}(\Psi) & \to & L_{n-q-q'}(\pi_1(Z)). \end{array} \tag{3.6}$$

Diagrams (3.5) and (3.6) and the obtained maps between them are realized on the spectra level. Cofibres give a homotopy commutative diagram of spectra of structure sets

$$\begin{array}{ccc} \mathbb{S}(X,Y,Z) & \to & \mathbb{S}(X,Z,\nu) \\ \downarrow & & \downarrow \\ \Sigma^q\mathbb{S}(Y,Z,\eta) & \to & \Sigma^{q+q'}\mathbb{S}(Z). \end{array} \tag{3.7}$$

Diagram (3.7) realizes on the spectra level commutative diagram (3.3). Now the result follows similarly to the previous case. $\square$

Theorem 3.2 *Let Φ and Ψ be the squares of fundamental groups concerning with a splitting problem for the manifold pairs (X,Z) and (Y,Z),* *respectively. Then there exists a commutative braid of exact sequences*

$$
\begin{array}{ccccccccccc}
\to & LNS_{k-1} & & \longrightarrow & & LS_{k-1}(\Psi) & & \to & & \mathcal{S}_{k-1}(Z) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow & \\
& & & \mathcal{S}_l(Y, Y\setminus Z) & & & & LS_{k-1}(\Phi) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & \\
\to & \mathcal{S}_k(Z) & & \longrightarrow & & \mathcal{S}_n(X, X\setminus Z) & & \longrightarrow & & LNS_{k-2} & \to,
\end{array}
\tag{3.8}
$$

where $k = n - q - q', l = n - q$.

Proof: The transfer maps give isomorphisms

$$
\begin{tikzcd}
H_{n-q-q'}(Z, \mathbf{L}_\bullet) \arrow[r, "\cong"] \arrow[dr, "\cong"'] & H_{n-q}(Y, Y\setminus Z; \mathbf{L}_\bullet) \arrow[d, "\cong"] \\
& H_n(X, X\setminus Z; \mathbf{L}_\bullet).
\end{tikzcd}
\tag{3.9}
$$

Consider a commutative triangle (see [7, 15])

$$
\begin{tikzcd}
L_{n-q-q'}(\pi_1(Z)) \arrow[r] \arrow[dr] & L_{n-q}(\pi_1(Y\setminus Z) \to \pi_1(Y)) \arrow[d, "tr^{rel}"] \\
& L_n(\pi_1(X\setminus Z) \to \pi_1(X)).
\end{tikzcd}
\tag{3.10}
$$

The right vertical map in (3.9) is a relative transfer map from (2.3) and the two other maps are compositions of transfer maps and maps induced by inclusions. For the pair (X, Y) the corresponding map on the spectra level is described in (2.2). Taking the obstruction to surgery we obtain maps from (3.9) into (3.10) for which homotopy cofibers give a homotopy commutative triangle of the spectra of structure sets

$$
\begin{array}{ccc}
\Sigma^{q+q'}\mathbb{S}(Z) & \to & \Sigma^q\mathbb{S}(Y, Y\setminus Z) \\
& \searrow & \downarrow \\
& & \mathbb{S}(X, X\setminus Z).
\end{array}
\tag{3.11}
$$

The cofiber of horizontal map in (3.11) is $\Sigma^{k+1}\mathbb{L}S(\Psi)$ and the cofiber of the sloping map is $\Sigma^{k+1}\mathbb{L}S(\Phi)$ as follows from [20, Proposition 7.2.6,ii]. Thus we have a push–out square of spectra

$$
\begin{array}{ccc}
\Sigma^q\mathbb{S}(Y, Y\setminus Z) & \to & \Sigma^{k+1}\mathbb{L}S(\Psi) \\
\downarrow & & \downarrow \\
\mathbb{S}(X, X\setminus Z) & \to & \Sigma^{k+1}\mathbb{L}S(\Phi),
\end{array}
\tag{3.12}
$$

where the right vertical maps fit into (3.2). Now the homotopy long exact sequences of maps in (3.12) give diagram (3.8). □

Theorem 3.3 *There exists the following braid of exact sequences*

$$\begin{array}{ccccccccccc}
\to & H_k(Z,\mathbf{L}_\bullet) & & \longrightarrow & & L_n(C\to D) & & \longrightarrow & & LNS_{k-1} & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow & \\
& & & L_l(A\to B) & & & & \mathcal{S}_n(X, X\setminus Z) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & \\
\to & LNS_k & & \longrightarrow & & \mathcal{S}_l(Y, Y\setminus Z) & & \longrightarrow & & H_{k-1}(Z,\mathbf{L}_\bullet) & \to,
\end{array} \qquad (3.13)$$

where $k=n-q-q', l=n-q, \pi_1(Y\setminus Z)=A, \pi_1(Y)=B, \pi_1(X\setminus Z)=C, \pi_1(X)=D$*. Diagram (3.13) is realized on the spectra level.*

Proof: The proof is similar to that of Theorem 3.2. It is necessary to use the definition of the map $tr^{rel}: L_l(A\to B)\to L_n(C\to D)$ from (2.3) and isomorphisms (3.9). □

Theorem 3.4 *There exists a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots\to & \mathcal{S}_l(Y\setminus Z) & \to & \mathcal{S}_n(X\setminus Z, X\setminus Y) & \to & LS_{l-1}(F_Z) & \to\cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots\to & \mathcal{S}_l(Y) & \to & \mathcal{S}_n(X, X\setminus Y) & \to & LS_{l-1}(F) & \to\cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots\to & \mathcal{S}_l(Y, Y\setminus Z) & \to & \mathcal{S}_n(X, X\setminus Z) & \to & LNS_{k-1} & \to\cdots, \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots &
\end{array} \qquad (3.14)$$

where $l=n-q$ *and* $k=n-q-q'$*. Diagram (3.14) is realized on spectra level.*

Proof: The proof is similar to that of Theorem 3.2. □

Theorem 3.5 *There exists a commutative braid of exact sequences*

$$\begin{array}{ccccccccccc}
\to & H_n(X,\mathbf{L}_\bullet) & & \longrightarrow & & LP_k(\Phi) & & \longrightarrow & & LS_{l-1}(F_Z) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow & \\
& & & LT_k(X,Y,Z) & & & & \mathcal{S}_n(X,Z,\nu) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & \\
\to & LS_l(F_Z) & & \longrightarrow & & \mathcal{S}_n(X,Y,Z) & & \longrightarrow & & H_{n-1}(X,\mathbf{L}_\bullet) & \to,
\end{array} \qquad (3.15)$$

where $k=n-q-q'$ *and* $l=n-q$*. Diagram (3.15) is realized on the spectra level.*

Proof: Consider the maps from $H_n(X, \mathbf{L}_\bullet)$ to the groups $LT_k(X, Y, Z)$ and $LP_k(\Phi)$ obtained by taking the obstructions to surgery. We obtain a commutative triangle in which the third map is the natural forgetful map $LT_k(X, Y, Z) \to LP_k(\Phi)$. Now the result follows from the definitions of the groups $\mathcal{S}_n(X, Z, \nu)$ and $\mathcal{S}_n(X, Y, Z)$ and from commutative diagram (3.2). $\square$

Theorem 3.6 *There exists a commutative diagram of exact sequences*

$$
\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_n(X \setminus Y, \mathbf{L}_\bullet) & \to & H_n(X, \mathbf{L}_\bullet) & \to & H_{n-q}(Y, \mathbf{L}_\bullet) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & L_n(\pi_1(X \setminus Y)) & \to & LT_k(X, Y, Z) & \to & LP_k(\Psi) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathcal{S}_n(X \setminus Y) & \to & \mathcal{S}_n(X, Y, Z) & \to & \mathcal{S}_l(Y, Z, \eta) & \to \cdots, \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots &
\end{array}
\tag{3.16}
$$

where $l = n - q$ and $k = n - q - q'$. Diagram (3.16) is realized on the spectra level.

Proof: The right upper square in (3.16) is commutative and it is realized on the spectra level by [20, 18]. Now diagram (3.16) is obtained by considering the homotopy long exact sequences of maps from this square. $\square$

4 Splitting a homotopy equivalence along a submanifold pair

In this section we introduce the obstruction groups $LSP_* = LSP_*(X, Y, Z)$ for a triple of embedded manifolds $Z \subset Y \subset Z$. These groups fit into an exact sequence

$$
\begin{aligned}
\ldots \to LSP_{n-q-q'} \to LT_{n-q-q'}(X, Y, Z) \to \\
\to L_n(\pi_1(X)) \to LSP_{n-q-q'-1} \to \ldots .
\end{aligned}
\tag{4.1}
$$

The groups $LSP_*(X, Y, Z)$ are a natural straightforward generalization of the splitting obstruction groups $LS_*(F)$ for the manifold X with a submanifold Y to the case when the manifold X contains a pair of embedded submanifolds $(Z \subset Y) \subset X$. Hence the groups $LSP_*(X, Y, Z)$ are obstruction groups for doing surgery on the manifold pair (Y, Z) inside the manifold X. In particular, there is a natural forgetful map $LSP_*(X, Y, Z) \to LP_*(\Psi)$ forgetting the ambient manifold X. We also describe relations between the introduced groups and the sets of homotopy triangulations which arise for the triple of manifolds.

Recall here that in [18] the spectrum $\mathbb{L}T(X, Y, Z)$ with homotopy groups $LT_n(X, Y, Z) = \pi_n(\mathbb{L}T(X, Y, Z))$ is defined as a homotopy cofiber of the map

$$\Sigma^{-q'-1}\mathbf{v} : \Sigma^{-q'-1}\mathbb{L}P(F) \to \mathbb{L}S(\Psi),$$

where Σ denotes the suspension functor. The map of homotopy groups induced by $\mathbf{v}$ coincides with the composition

$$LP_{n-q+1}(F) \to \mathcal{S}_{n+1}(X, Y, \xi) \to \mathcal{S}_{n-q+1}(Y) \to LS_{n-q-q'}(\Psi), \qquad (4.2)$$

where the middle map is the natural forgetful map and the other maps are described in (1.3). Hence, from the cofibration sequence of the map $\mathbf{v}$, we obtain the map $\mathbf{t} : \Sigma^{q+q'}\mathbb{L}T(X, Y, Z) \to \Sigma^q\mathbb{L}P(F)$. The composition of the map $\mathbf{t}$ with the natural forgetful map $LP_{n-q}(F) \to L_n(\pi_1(X))$ on spectra level provides a map $\mathbf{s} : \Sigma^{q+q'}\mathbb{L}T(X, Y, Z) \to \mathbb{L}(\pi_1(X))$. We define a spectrum $\mathbb{L}SP(X, Y, Z)$ as the spectrum fitting in the cofibration

$$\mathbb{L}SP(X, Y, Z) \to \mathbb{L}T(X, Y, Z) \to \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)). \qquad (4.3)$$

Let $LSP_n = LSP_n(X, Y, Z)$ denote the homotopy group $\pi_n(\mathbb{L}SP(X, Y, Z))$. As follows from the definition, these groups fit into the long exact sequence in (4.1).

Theorem 4.1 *There exists a commutative braid of exact sequences*

$$\begin{tikzcd}
{} \arrow[r] & \mathcal{S}_{n+1}(X,Y,Z) \arrow[rr] \arrow[dr] & & H_n(X,\mathbf{L}_\bullet) \arrow[rr] \arrow[dr] & & L_n(\pi_1(X)) \arrow[r] \arrow[dr] & {} \\
{} \arrow[ur] \arrow[dr] & & \mathcal{S}_{n+1}(X) \arrow[ur] \arrow[dr, "\alpha"] & & LT_{n-q-q'} \arrow[ur] \arrow[dr] & & {} \\
{} \arrow[r] & L_{n+1}(\pi_1(X)) \arrow[rr] \arrow[ur] & & LSP_{n-q-q'} \arrow[rr] \arrow[ur] & & \mathcal{S}_n(X,Y,Z) \arrow[r] \arrow[ur] & {}
\end{tikzcd} \qquad (4.4)$$

which is realized on the spectra level.

Proof: The definition of the spectrum $\mathbb{L}T$ yields a homotopy commutative right square of a homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \Sigma^{-1}\mathbb{S}(X) & \to & X \wedge \mathbf{L}_\bullet & \to & \mathbb{L}(\pi_1(X)) \\ \downarrow & & \downarrow & & \downarrow = \\ \Sigma^{q+q'}\mathbb{L}SP & \to & \Sigma^{q+q'}\mathbb{L}T & \to & \mathbb{L}(\pi_1(X)) \end{array} \tag{4.5}$$

where the existence of the left vertical map follows from [22]. The cofibers of the two horizontal maps of the left square in (4.5) coincide. Hence the left square is a pull–back square and the homotopy long exact sequences of this square give diagram (4.4). □

Commutative diagram (4.4) is a natural generalization of diagram (1.3) in the case of a triple of embedded manifolds. The left vertical map in (4.5) induces a map $\alpha : \mathcal{S}_{n+1}(X) \to LSP_{n-q-q'}(X, Y, Z)$ in (4.4). The geometric meaning of this map is explained in the following theorem.

Theorem 4.2 *Let $f : M \to X$ be a simple homotopy equivalence which represents an element of $\mathcal{S}_{n+1}(X)$. Then $\alpha(f) = 0$ if and only if the homotopy class of the map f contains an s–triangulation of the triple (X, Y, Z) (f splits along the pair $Z \subset Y$).*

Proof: Let the homotopy class of the map f contain an s–triangulation of the triple (X, Y, Z). Then the element f lies in the image of the forgetful map $\mathcal{S}_{n+1}(X, Y, Z) \to \mathcal{S}_{n+1}(X)$. The composition $\mathcal{S}_{n+1}(X, Y, Z) \to \mathcal{S}_{n+1}(X) \to LSP_{n-q-q'}$ is trivial as follows from (4.4). Hence $\alpha(f) = 0$. Conversely, let $\alpha(f) = 0$. Then the same exact sequence shows that f lies in the image $\mathcal{S}_{n+1}(X, Y, Z) \to \mathcal{S}_{n+1}(X)$, and the result follows. □

Suppose that the pairs of manifolds (X, Y) and (Y, Z) are Browder–Livesay pairs (see [3]). In this case the spectrum $\Sigma^2\mathbb{L}T(X, Y, Z)$ coincides with the third member of the filtration in the construction of the surgery spectral sequence of Hambleton and Kharshiladze (see [18, 6]). Then the map $r_p : L_n(\pi_1(X)) \to LSP_{n-q-q'-1}(X, Y, Z)$ from (4.1) is a natural generalization of the Browder–Livesay invariant $r : L_n(\pi_1(X)) \to LS_{n-q-1}(F) = LN_{n-q-1}(\pi_1(X \setminus Y) \to \pi_1(X))$ (see [3, 5]). Recall here (see [3]), that if $r(x) \neq 0$, then the element $x \in L_n(\pi_1(X))$ is not realized by a normal map of closed manifolds. In fact, the map r_p gives an information which is equivalent to consider the first and second Browder–Livesay invariants (see [15, 6, 5]).

Proposition 4.3 *For a triple of manifolds (X, Y, Z), let (X, Y) and (Y, Z) be Browder–Livesay pairs. If $r_p(x) \neq 0$, then the element $x \in L_n(\pi_1(X))$ is not realized by a normal map of closed manifolds.*

Proof: The result follows from [15, Proposition 3] and [5]. $\square$

Recall here the diagram (see [18])

$$\begin{array}{ccccccccccc}
\to & L_n(\pi_1(X\setminus Y)) & & \longrightarrow & & LP_{n-q}(F) & & \longrightarrow & & LS_{k-1}(\Psi) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & LT_k & & & & L_{n-q}(\pi_1(Y)) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & LS_k(\Psi) & & \longrightarrow & & LP_k(\Psi) & & \longrightarrow & & L_{n-1}(\pi_1(X\setminus Y)) & \to,
\end{array} \tag{4.6}$$

that gives the relations of LT_*–groups to splitting obstruction groups and to surgery obstruction groups for manifold pairs. Diagram (4.6) is realized on spectra level (see [18]).

The relations of LSP_* to classical surgery obstruction groups for the triple (X, Y, Z) is given by the following result.

Theorem 4.4 *There exist braids of exact sequences*

$$\begin{array}{ccccccccccc}
\to & L_n(C) & & \longrightarrow & & L_n(D) & & \longrightarrow & & LSP_{k-1} & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & LT_k(X,Y,Z) & & & & L_n(C\to D) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & LSP_k & & \longrightarrow & & LP_k(\Psi) & & \longrightarrow & & L_{n-1}(C) & \to,
\end{array} \tag{4.7}$$

$$\begin{array}{ccccccccccc}
\to & LS_{n-q}(F_Z) & & \longrightarrow & & LT_k(X,Y,Z) & & \longrightarrow & & L_n(D) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & LSP_k & & & & LP_k(\Phi) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & L_{n+1}(D) & & \longrightarrow & & LS_k(\Phi) & & \longrightarrow & & LS_{n-q-1}(F_Z) & \to,
\end{array} \tag{4.8}$$

$$\begin{array}{ccccccccccc}
\to & LS_{n-q}(F_Z) & & \longrightarrow & & LS_{n-q}(F) & & \longrightarrow & & LS_{k-1}(\Psi) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & LSP_k & & & & LNS_k & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & LS_k(\Psi) & & \longrightarrow & & LS_k(\Phi) & & \longrightarrow & & LS_{n-q-1}(F_Z) & \to,
\end{array} \tag{4.9}$$

and

$$\begin{array}{ccccccccccc}
\to & LS_{n-q+1}(F) & & \longrightarrow & & LS_k(\Psi) & & \longrightarrow & & LT_k & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & LP_{n-q+1}(F) & & & & LSP_k & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & LT_{k+1} & & \longrightarrow & & L_{n+1}(D) & & \longrightarrow & & LS_{n-q}(F) & \to,
\end{array} \tag{4.10}$$

where $C=\pi_1(X\setminus Y), D=\pi_1(X), k=n-q-q'$. Diagrams (4.7)–(4.10) are realized on the spectra level.

Proof: Consider a homotopy commutative diagram

$$\begin{array}{ccc} LP_{n-q-q'}(\Psi) & \to & L_n(\pi_1(X\setminus Y)\to\pi_1(X)) \\ \downarrow & & \downarrow \\ L_{n-1}(\pi_1(X\setminus Y)) & \overset{=}{\to} & L_{n-1}(\pi_1(X\setminus Y)) \end{array}$$

in which the upper horizontal map and the left map are compositions of the natural forgetful map $LP_{n-q-q'}(\Psi)\to L_{n-q}(\pi_1(Y))$ and maps of L–groups induced from (2.2). This diagram is realized on spectra level and by [22] we obtain a map of cofibration sequences

$$\begin{array}{ccccc} \mathbb{L}T(X,Y,Z) & \to & \mathbb{L}P(\Psi) & \to & \Sigma^{-q-q'+1}\mathbb{L}(\pi_1(X\setminus Y)) \\ \downarrow & & \downarrow & & \downarrow= \\ \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)) & \to & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X\setminus Y)\to\pi_1(X)) & \to & \Sigma^{-q-q'+1}\mathbb{L}(\pi_1(X\setminus Y)) \end{array}$$

where the left square is a pull–back square of spectra. The homotopy long exact sequences of this square provide the commutative braid of exact sequences (4.7) if we use the definition of LSP_*–groups.

The natural forgetful maps (see [18, 15]) $LT_{n-q-q'}\to LP_{n-q-q'}(\Phi)$ $\to L_n(\pi_1(X))$ provide a map of cofibration sequence

$$\begin{array}{ccccc} \mathbb{L}SP(X,Y,Z) & \to & \mathbb{L}T(X,Y,Z) & \to & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)) \\ \downarrow & & \downarrow & & \downarrow= \\ \mathbb{L}S(\Phi) & \to & \mathbb{L}P(\Phi) & \to & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)). \end{array}$$

Now, similarly to the previous result, we obtain diagram (4.8), since the map $LT_*\to LP_*(\Phi)$ fits into (3.15).

Transfer maps and diagram (2.3) provide a map of the commutative diagram

$$\begin{array}{ccc} LP_{n-q-q'}(\Psi) & \to & L_{n-q}(\pi_1(Y)) \\ \downarrow & & \downarrow \\ L_{n-q-q'}(\pi_1(Z)) & \to & L_{n-q}(\pi_1(Y\setminus Z)\to\pi_1(Y)) \end{array} \tag{4.11}$$

to the commutative diagram

$$\begin{array}{ccc} L_n(\pi_1(X\setminus Y)\to\pi_1(X)) & \overset{=}{\to} & L_n(\pi_1(X\setminus Y)\to\pi_1(X)) \\ \downarrow & & \downarrow \\ L_n(\pi_1(X\setminus Z)\to\pi_1(X)) & \to & L_n(\pi_1(X\setminus Z)\to\pi_1(X)). \end{array} \tag{4.12}$$

All these maps are realized on the spectra level. Hence, on the spectra level, the cofibers of the maps from (4.11) to (4.12) provide a homotopy commutative diagram of spectra

$$\begin{array}{ccc} \mathbb{L}SP & \to & \Sigma^{-q'}\mathbb{L}S(F) \\ \downarrow & & \downarrow \\ \mathbb{L}S(\Phi) & \to & \mathbb{L}NS \end{array} \tag{4.13}$$

as follows from (2.3) and (4.6). The realizations on spectra level of diagrams (4.11) and (4.12) give pull–back squares. Hence the homotopy commutative square (4.13) is a pull–back and diagram (4.9) is obtained from the homotopy long exact sequence of (4.13).

The natural forgetful maps $LT_{n-q-q'} \to LP_{n-q}(F) \to L_n(\pi_1(X))$ from (4.6) provide a homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}T(X,Y,Z) & \to & \Sigma^{-q'}\mathbb{L}P(F) & \to & \Sigma\mathbb{L}S(\Psi) \\ \downarrow = & & \downarrow & & \downarrow \\ \mathbb{L}T(X,Y,Z) & \to & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)) & \to & \Sigma\mathbb{L}SP(X,Y,Z), \end{array} \tag{4.14}$$

where the rows are cofibrations and the right vertical map is defined by [22]. Hence the right square in (4.14) is a pull–back and its homotopy long exact sequences give diagram (4.10). □

Corollary 4.5 *There exist exact sequences*

$$\cdots \to LSP_k \to LS_{n-q}(F) \to LS_{k-1}(\Psi) \to \cdots,$$

$$\cdots \to LSP_k \to LS_k(\Phi) \to LS_{n-q-1}(F_Z) \to \cdots,$$

and

$$\cdots \to LSP_k \to LP_k(\Psi) \to L_{n-1}(\pi_1(X \setminus Y) \to \pi_1(X)) \to \cdots,$$

in which the left maps are natural forgetful maps.

Now we describe some relations between the introduced groups LSP_* and various structure sets which arise for the triple (X, Y, Z).

Theorem 4.6 *There exist braids of exact sequences*

$$\begin{array}{ccccccccccc} \to & \mathcal{S}_n(X) & & \longrightarrow & & LSP_{k-1} & & \longrightarrow & & \mathcal{S}_{l-1}(Y,Z,\eta) & \to \\ & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & & & \mathcal{S}_n(X, X\setminus Y) & & & & \mathcal{S}_{n-1}(X,Y,Z) & & & \\ & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\ \to & \mathcal{S}_l(Y,Z,\eta) & & \longrightarrow & & \mathcal{S}_{n-1}(X\setminus Y) & & \longrightarrow & & \mathcal{S}_{n-1}(X) & \to, \end{array} \tag{4.15}$$

$$\begin{array}{ccccccccc}
\to H_l(Y,\mathbf{L}_\bullet) & & \longrightarrow & L_n(\pi_1(X\setminus Y)\to\pi_1(X)) & & \longrightarrow & LSP_{k-1} & & \to \\
\nearrow & \searrow & & \nearrow & \searrow & & \nearrow & \searrow & \\
& & LP_k(\Psi) & & & \mathcal{S}_n(X,X\setminus Y) & & & \\
\searrow & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & \\
\to LSP_k & & \longrightarrow & \mathcal{S}_l(Y,Z,\eta) & & \longrightarrow & H_{l-1}(Y,\mathbf{L}_\bullet) & & \to,
\end{array} \tag{4.16}$$

$$\begin{array}{ccccccccc}
\to LS_l(F_Z) & & \longrightarrow & \mathcal{S}_n(X,Y,Z) & & \longrightarrow & \mathcal{S}_n(X) & & \to \\
\nearrow & \searrow & & \nearrow & \searrow & & \nearrow & \searrow & \\
& & LSP_k & & & \mathcal{S}_n(X,Z,\nu) & & & \\
\searrow & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & \\
\to \mathcal{S}_{n+1}(X) & & \longrightarrow & LS_k(\Phi) & & \longrightarrow & LS_{l-1}(F_Z) & & \to,
\end{array} \tag{4.17}$$

and

$$\begin{array}{ccccccccc}
\to LS_{l+1}(F) & & \longrightarrow & LS_k(\Psi) & & \longrightarrow & \mathcal{S}_n(X,Y,Z) & & \to \\
\nearrow & \searrow & & \nearrow & \searrow & & \nearrow & \searrow & \\
& & \mathcal{S}_{n+1}(X,Y,\xi) & & & LSP_k & & & \\
\searrow & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & \\
\to \mathcal{S}_{n+1}(X,Y,Z) & & \longrightarrow & \mathcal{S}_{n+1}(X) & & \longrightarrow & LS_l(F) & & \to,
\end{array} \tag{4.18}$$

where $l = n-q, k = n-q-q'$. Diagrams (4.15)–(4.18) are realized on the spectra level.

Proof: Transfer maps give a commutative diagram (see [20])

$$\begin{array}{ccc}
H_{n-q}(Y,\mathbf{L}_\bullet) & \xrightarrow{\cong} & H_n(X,X\setminus Y;\mathbf{L}_\bullet) \\
& \searrow & \downarrow \\
& & H_{n-1}(X\setminus Y;\mathbf{L}_\bullet).
\end{array} \tag{4.19}$$

Consider the commutative triangle

$$\begin{array}{ccc}
LP_{n-q-q'}(\Psi) & \to & L_n(\pi_1(X\setminus Y)\to\pi_1(X)) \\
& \searrow & \downarrow \\
& & L_{n-1}(\pi_1(X\setminus Y))
\end{array} \tag{4.20}$$

which follows from the commutative diagram obtained in the proof of Theorem 4.4.

The results of [20, Proposition 7.2.6] provide the map from (4.19) to (4.20). On the spectra level cofibres of this map give a homotopy commutative triangle of spectra of structure sets

$$\begin{array}{ccc}
\mathbb{S}(Y,Z,\eta) & \to & \Sigma^{-q}\mathbb{S}(X,X\setminus Y) \\
& \searrow & \downarrow \\
& & \Sigma^{-q+1}\mathbb{S}(X\setminus Y).
\end{array} \tag{4.21}$$

By [22] diagram (4.21) induces a map of cofibration sequences

$$\begin{array}{ccccc} \mathbb{S}(Y,Z,\eta) & \to & \Sigma^{-q}\mathbb{S}(X,X\setminus Y) & \to & \Sigma^{q'+1}\mathbb{L}SP \\ \downarrow= & & \downarrow & & \downarrow \\ \mathbb{S}(Y,Z,\eta) & \to & \Sigma^{-q+1}\mathbb{S}(X\setminus Y) & \to & \Sigma^{-q+1}\mathbb{S}(X,Y,Z). \end{array}$$

where the left square is a pull–back square of spectra. The homotopy long exact sequences of this square provide the commutative braid of exact sequences in (4.15). In a similar way the map from (4.19) to (4.20) provides a pull–back square

$$\begin{array}{ccc} \Sigma^{q'}\mathbb{L}P(\Psi) & \to & \Sigma^{-q}\mathbb{L}(\pi_1(X\setminus Y)\to\pi_1(X)) \\ \downarrow & & \downarrow \\ \mathbb{S}(Y,Z,\eta) & \to & \Sigma^{-q}\mathbb{S}(X,X\setminus Y) \end{array}$$

where the cofibers of the vertical maps are homotopy equivalent to the spectrum $Y_+\wedge\mathbf{L}_\bullet$. From this the braid of exact sequences (4.16) follows. The diagram (4.17) is obtained in a similar way if we consider on the spectra level the homotopy commutative triangle of the cofibers of the map from $H_n(X,\mathbf{L}_\bullet)$ to the triangle of natural forgetful maps

$$\begin{array}{ccc} LT_{n-q-q'} & \to & LP_{n-q-q'}(\Phi) \\ & \searrow & \downarrow \\ & & L_n(\pi_1(X)). \end{array} \tag{4.22}$$

We obtain diagram (4.18) in a similar way to the construction of diagram (4.17). To do this we have to consider the commutative triangle

$$\begin{array}{ccc} LT_{n-q-q'} & \to & LP_{n-q}(F) \\ & \searrow & \downarrow \\ & & L_n(\pi_1(X)). \end{array}$$

So the proof is complete. □

5 Examples

Now we give examples how to compute some LSP–groups.

Consider the triple

$$(Z\subset Y\subset X)=(\mathbb{RP}^n\subset\mathbb{RP}^{n+1}\subset\mathbb{RP}^{n+2})$$

of real projective spaces with $n \geq 5$.

The orientation homomorphism

$$w : \pi_1(\mathbb{RP}^k) = \mathbb{Z}/2 \to \{\pm 1\}$$

is trivial for k odd and nontrivial for k even.

We have the following table for surgery obstruction groups (see [23, 9])

	$n=0$	$n=1$	$n=2$	$n=3$
$L_n(1)$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0
$L_n(\mathbb{Z}/2^+)$	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$L_n(\mathbb{Z}/2^-)$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

where superscript ”+” denotes the trivial orientation of the corresponding group and superscript ”−” denotes the nontrivial orientation.

We have two squares for codimension one splitting problems which appear for different pairs $\mathbb{RP}^k \subset \mathbb{RP}^{k+1}$ of the considered triple.

We denote by

$$F^{\pm} = \begin{pmatrix} 1 & \to & 1 \\ \downarrow & & \downarrow \\ \mathbb{Z}/2^{\mp} & \to & \mathbb{Z}/2^{\pm} \end{pmatrix}$$

the oriented square F of fundamental groups in accordance with the orientation $\{\pm\}$ of the ambient manifold.

We have the following isomorphisms (see [9, p. 15] and [23])

$$LS_n(F^+) = LN_n(1 \to \mathbb{Z}/2^+) = BL_{n+1}(+) = L_{n+2}(1)$$

and

$$LS_n(F^-) = LN_n(1 \to \mathbb{Z}/2^-) = BL_{n+1}(-) = L_n(1).$$

Now we recall intermediate computations of obstruction groups $LP_*(F^{\pm})$ and $LT^*(X,Y,Z)$ from [18].

The computation of LP_*–groups for a pair $Y \subset X$ is based on the following braid of exact sequences [23]

$$\begin{array}{ccccccccccc}
\to & L_{n+1}(C) & & \longrightarrow & & L_{n+1}(D) & & \xrightarrow{\partial} & & LS_{n-q}(F) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & LP_{n-q+1}(F) & & & & L_{n+1}(C\to D) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & LS_{n-q+1}(F) & & \longrightarrow & & L_{n-q+1}(B) & & \longrightarrow & & L_n(C) & \to
\end{array} \qquad (4.23)$$

where $A = \pi_1(\partial U)$, $B = \pi_1(Y)$, $C = \pi_1(X \setminus Y)$, and $D = \pi_1(X)$.

In the cases of squares $F^{\pm}$ we have $q = 1$, and the natural map that forget the ambient manifold

$$LS_n(F^{\pm}) \to L_n(\mathbb{Z}/2^{\mp})$$

coincides with the map

$$l_n : BL_n(\pm) \to L_{n-1}(\mathbb{Z}/2^{\mp})$$

which is described in [9, p. 35].

Using this result and a diagram chasing in diagram (4.23) we obtain surgery obstruction groups (see also [16])

$$LP_n(F^+) = LP_{n-1}(F^-) = \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}$$

for $n = 0,\ 1,\ 2,\ 3 \pmod 4$, respectively.

Now a diagram chasing in diagram (4.6) provides the following results.

Proposition 5.1 *[18] Let M^{n-k} be a closed simply connected topological manifold. For the triple of manifolds*

$$(Z^n \subset Y^{n+1} \subset X^{n+2}) = (M^{n-k} \times \mathbb{RP}^k \subset M^{n-k} \times \mathbb{RP}^{k+1} \subset M^{n-k} \times \mathbb{RP}^{k+2})$$

with $n \geq 5$ we have the following results.

For k odd the groups LT_n are isomorphic to

$$\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively.

For k even $LT_0 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $LT_1 \cong \mathbb{Z}/2$. The groups LT_3 and LT_2 fit into the exact sequence

$$0 \to LT_3 \to \mathbb{Z} \to \mathbb{Z} \to LT_2 \to \mathbb{Z}/2 \to 0.$$

Now we apply these results to compute the LSP_*–groups in the considered cases.

Theorem 5.2 *Under assumptions of Proposition 5.1 we have the following:*

For k odd the groups LSP_n are isomorphic to

$$\mathbb{Z}, \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2$$

for $n = 0, 1, 2, 3$ (mod 4), respectively.

For k even we have isomorphisms $LSP_0 \cong LSP_1 \cong \mathbb{Z}/2$. The groups LSP_3 and LSP_2 fit into the exact sequence

$$0 \to LSP_3 \to \mathbb{Z} \to \mathbb{Z} \to LSP_2 \to 0.$$

Proof: Consider the case when k is odd. From diagram (4.6) in the considered case we conclude that all maps $LT_n \to LP_{n+1}(F^+)$ are epimorphisms (see also [18]). Now it is easy to describe the maps $LP_n(F^+) \to L_{n+1}(\mathbb{Z}/2^+)$ from diagram (4.23). For $n = 1 \bmod 4$ and $n = 2 \bmod 4$ these maps are isomorphisms $\mathbb{Z}/2 \to \mathbb{Z}/2$ as follows considering exact sequences lying in diagram (4.23). For $n = 0 \bmod 4$ the map is trivial since the group $L_1(\mathbb{Z}/2^+)$ is trivial.

The map

$$\mathbb{Z} = LP_3(F^+) \to L_0(\mathbb{Z}/2^+) = \mathbb{Z} \oplus \mathbb{Z}$$

is an inclusion on a direct summand. The image of this map coincides with the image of the map $L_0(1) \to L_0(\mathbb{Z}/2^+)$ that is induced by the inclusion $1 \to \mathbb{Z}/2^+$. This follows from the commutative triangle

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[d, equal] & \\
 & LP_3(F^+) \arrow[dr] & \\
L_0(1) \arrow[ur, "\cong"] \arrow[rr, "mono"] \arrow[d, equal] & & L_0(\mathbb{Z}/2^+) \arrow[d, equal] \\
\mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z}
\end{tikzcd}$$

which lies in diagram (4.23). From diagram (4.7) we obtain an exact sequence

$$\cdots \to LT_n \xrightarrow{\tau} L_{n+2}(\mathbb{Z}/2^+) \to LSP_{n-1} \to LT_{n-1} \to \cdots \tag{4.24}$$

The map τ in (4.24) is a composition

$$LT_n \to LP_{n+1}(F^+) \to L_{n+2}(\mathbb{Z}/2^+)$$

of maps that we already know.

From this we obtain that τ is trivial for $n = 3$, an isomorphism $\mathbb{Z}/2 \to \mathbb{Z}/2$ for $n = 1$, an epimorphism $\mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z}/2$ with a kernel $\mathbb{Z}$ for $n = 0$, and a map $\mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z} \oplus \mathbb{Z}$ with kernel $\mathbb{Z}/2$ and cokernel $\mathbb{Z}$ for $n = 2$. Now considering the exact sequence (4.24) we get the result of the theorem for k odd. We get the result for k the even case in a similar way. $\square$

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