Morfismos, Vol. 9, No. 1, 2005, pp. 21-34

Linear programming relaxations of the mixed postman problem

Francisco Javier Zaragoza Martínez [1]

Abstract

The mixed postman problem consists of finding a minimum cost tour of a connected mixed graph traversing all its vertices, edges, and arcs at least once. We prove in two different ways that the linear programming relaxations of two well-known integer programming formulations of this problem are equivalent. We also give some properties of the extreme points of the polyhedra defined by one of these relaxations and its linear programming dual.

2000 Mathematics Subject Classification: 05C45, 90C35.
Keywords and phrases: Eulerian graph, integer programming formulation, linear programming relaxation, mixed graph, postman problem.

1 Introduction

We study a class of problems collectively known as *postman* problems [6]. As the name indicates, these are the problems faced by a postman who needs to deliver mail to all streets in a city, starting and ending his labour at the city's post office, and minimizing the length of his walk. In graph theoretical terms, a postman problem consists of finding a minimum cost tour of a graph traversing all its arcs (one-way streets) and edges (two-way streets) at least once. Hence, we can see postman problems as generalizations of *Eulerian* problems.

The postman problem when all streets are one-way, known as the *directed* postman problem, can be solved in polynomial time by a network

[1]This work was partly funded by UAM Azcapotzalco research grant 2270314 and CONACyT doctoral grant 69234.

flow algorithm, and the postman problem when all streets are two-way, known as the *undirected* postman problem, can be solved in polynomial time using Edmonds' matching algorithm, as shown by Edmonds and Johnson [3]. However, Papadimitriou showed that the postman problem becomes NP-hard when both kinds of streets exist [9]. This problem, known as the *mixed postman problem*, is the central topic of this paper.

We study some properties of the linear programming relaxations of two well-known integer programming formulations for the mixed postman problem — described in Section 3. We prove that these linear programming relaxations are equivalent (Theorem 4.1.1). In particular, we show that the polyhedron defined by one of them is essentially a projection of the other (Theorem 4.1.2). We also give new proofs of the half-integrality of one of these two polyhedra (Theorem 4.2.1) and of the integrality of the same polyhedron for mixed graphs with vertices of even degree (Theorem 4.2.2). Finally, we prove that the corresponding dual polyhedron has integral optimal solutions (Theorem 4.3.1).

2 Preliminaries

A *mixed graph* M is an ordered triple $(V(M), E(M), A(M))$ of three mutually disjoint sets $V(M)$ of *vertices*, $E(M)$ of *edges*, and $A(M)$ of *arcs*. When it is clear from the context, we simply write $M = (V, E, A)$. Each edge $e \in E$ has two *ends* $u, v \in V$, and each arc $a \in A$ has a *head* $u \in V$ and a *tail* $v \in V$. Each edge can be *traversed* from one of its ends to the other, while each arc can be *traversed* from its tail to its head. The *associated directed graph* $\vec{M} = (V, A \cup E^+ \cup E^-)$ of M is the directed graph obtained from M by replacing each edge $e \in E$ with two oppositely oriented arcs $e^+ \in E^+$ and $e^- \in E^-$.

Let $S \subseteq V$. The *undirected cut* $\delta_E(S)$ determined by S is the set of edges with one end in S and the other end in $\bar{S} = V \setminus S$. The *directed cut* $\delta_A(S)$ determined by S is the set of arcs with tails in S and heads in $\bar{S}$. The *total cut* $\delta_M(S)$ determined by S is the set $\delta_E(S) \cup \delta_A(S) \cup \delta_A(\bar{S})$. For single vertices $v \in V(M)$ we write $\delta_E(v)$, $\delta_A(v)$, $\delta_M(v)$ instead of $\delta_E(\{v\})$, $\delta_A(\{v\})$, $\delta_M(\{v\})$, respectively. We also define the *degree of* S as $d_E(S) = |\delta_E(S)|$, and the *total degree of* S as $d_M(S) = |\delta_M(S)|$.

A *walk* from v_0 to v_n is an ordered tuple $W = (v_0, e_1, v_1, \ldots, e_n, v_n)$ on $V \cup E \cup A$ such that, for all $1 \leq i \leq n$, e_i can be traversed from v_{i-1} to v_i. If $v_0 = v_n$, W is said to be a *closed* walk. If, for any two vertices u and v, there is a walk from u to v, we say that M is *strongly* *connected*. If W is closed and uses all vertices of M, we call it a *tour*, and if it traverses each edge and arc exactly once, we call it *Eulerian*. If $e_1, \ldots, e_n$ are pairwise distinct, W is called a *trail*. If W is a closed trail, and $v_1, \ldots, v_n$ are pairwise distinct, we call it a *cycle*.

Given a matrix $A \in \mathbb{Q}^{n\times m}$ and a vector $b \in \mathbb{Q}^n$, the *polyhedron* determined by A and b is the set $P = \{x \in \mathbb{R}^m : Ax \leq b\}$. A vector $x \in P$ is called an *extreme point* of P if x is not a convex combination of vectors in $P \setminus \{x\}$. For our purposes, P is *integral* if all its extreme points have integer coordinates, and it is *half-integral* if all its extreme points have coordinates which are integer multiples of $\frac{1}{2}$.

Let S be a set, and let $T \subseteq S$. If $x \in \mathbb{R}^S$, we define $x(T) = \sum_{t\in T} x_t$. The *characteristic* vector χ^T of T with respect to S is defined by the entries $\chi^T(t) = 1$ if $t \in T$, and $\chi^T(t) = 0$ otherwise. If $T = S$ we write $\mathbf{1}_S$ or $\mathbf{1}$ instead of χ^S, if T consists of only one element t we write $\mathbf{1}_t$ instead of $\chi^{\{t\}}$, and if T is empty we write $\mathbf{0}_S$ or $\mathbf{0}$ instead of $\chi^\emptyset$. If $x \in \mathbb{R}^n$, the *positive support* of x is the vector $y \in \mathbb{R}^n$ such that $y_i = 1$ if $x_i > 0$, and $y_i = 0$ otherwise, and it is denoted by $\operatorname{supp}_+(x)$. The *negative support* $\operatorname{supp}_-(x)$ is defined similarly.

3 Integer programming formulations

Let $M = (V, E, A)$ be a strongly connected mixed graph, and let $c \in \mathbb{Q}_+^{E\cup A}$. A *postman tour* of M is a tour that traverses all edges and arcs of M at least once. The *cost* of a postman tour is the sum of the costs of all edges and arcs traversed, counting repetitions. The *mixed postman problem* is to find the minimum cost of a postman tour. We present two integer programming formulations of the mixed postman problem.

3.1 First formulation

The first integer programming formulation we give is due to Kappauf and Koehler [7], and Christofides et al. [1]. Similar formulations were given by other authors [3, 5, 10]. All these formulations are based on the following characterization of mixed Eulerian graphs.

Theorem 3.1.1 (Veblen [11]) *A connected, mixed graph M is Eulerian if and only if M is the disjoint union of some cycles.*

Let $\vec{M} = (V, A \cup E^+ \cup E^-)$ be the associated directed graph of M. For every $e \in E$, let $c_{e^+} = c_{e^-} = c_e$. A nonnegative integer *circulation*

x of $\vec{M}$ (a vector on $A \cup E^+ \cup E^-$ such that $x(\vec{\delta}(\bar{v})) = x(\vec{\delta}(v))$ for every $v \in V$, for more on the theory of *flows* see [4]) is the incidence vector of a postman tour of M if and only if $x_e \geq 1$ for all $e \in A$, and $x_{e^+} + x_{e^-} \geq 1$ for all $e \in E$. Therefore, we obtain the integer program:

$$\begin{aligned}
&(1) & \text{MMPT1}(M, c) &= \min c_A^\top x_A + c_E^\top x_E^+ + c_E^\top x_E^- \\
& & \text{subject to} & \\
&(2) & x(\vec{\delta}(\bar{v})) - x(\vec{\delta}(v)) &= 0 \text{ for all } v \in V, \\
&(3) & x_a &\geq 1 \text{ for all } a \in A, \\
&(4) & x_{e^+} + x_{e^-} &\geq 1 \text{ for all } e \in E, \text{and} \\
&(5) & x_a &\geq 0 \text{ and integer for all } a \in A \cup E^+ \cup E^-.
\end{aligned}$$

Let $\mathcal{P}^1_{MPT}(M)$ be the convex hull of the feasible solutions to the integer program above, and let $\mathcal{Q}^1_{MPT}(M)$ be the set of feasible solutions to its linear programming relaxation:

$$\begin{aligned}
&(6) & \text{LMMPT1}(M, c) &= \min c_A^\top x_A + c_E^\top x_E^+ + c_E^\top x_E^- \\
& & \text{subject to} & \\
&(7) & x(\vec{\delta}(\bar{v})) - x(\vec{\delta}(v)) &= 0 \text{ for all } v \in V, \\
&(8) & x_a &\geq 1 \text{ for all } a \in A, \\
&(9) & x_{e^+} + x_{e^-} &\geq 1 \text{ for all } e \in E, \text{and} \\
&(10) & x_a &\geq 0 \text{ for all } a \in A \cup E^+ \cup E^-.
\end{aligned}$$

3.2 Second formulation

The second integer programming formulation we give is due to Nobert and Picard [8]. The approach they use is based on the following characterization of mixed Eulerian graphs.

Theorem 3.2.1 (Ford and Fulkerson [4, page 60]) *Let M be a connected, mixed graph. Then M is Eulerian if and only if, for every subset S of vertices of M, the number of arcs and edges from $\bar{S}$ to S minus the number of arcs from S to $\bar{S}$ is a nonnegative even number.*

The vector $x \in \mathbb{Z}_+^{E \cup A}$ is the incidence vector of a postman tour of M if and only if $x_e \geq 1$ for all $e \in E \cup A$, $x(\delta_{E \cup A}(v))$ is even for all

$v \in V$, and $x(\delta_A(\bar{S})) + x(\delta_E(S)) \geq x(\delta_A(S))$ for all $S \subseteq V$. Therefore we obtain the integer program:

$$
\begin{aligned}
&(11) \quad & \text{MMPT2}(M,c) &= \min c^\top x \\
& & \text{subject to} & \\
&(12) & x(\delta_{E\cup A}(v)) &\equiv 0 \text{ (mod 2) for all } v \in V, \\
&(13) & x(\delta_A(\bar{S})) + x(\delta_E(S)) &\geq x(\delta_A(S)) \text{ for all } S \subseteq V, \text{and} \\
&(14) & x_e &\geq 1 \text{ and integer for all } e \in E \cup A.
\end{aligned}
$$

Note that the parity constraints (12) are not in the required form for integer programming; however, this can be easily solved by noting that, for all $v \in V$,

$$(15) \qquad x(\delta_{E\cup A}(v)) \equiv x(\delta_A(\bar{v})) + x(\delta_E(v)) - x(\delta_A(v)) \text{ (mod 2)},$$

and introducing a *slack* variable $s_v \in \mathbb{Z}_+$ to obtain the equivalent constraint

$$(16) \qquad x(\delta_A(\bar{v})) + x(\delta_E(v)) - x(\delta_A(v)) - 2s_v = 0 \text{ for all } v \in V.$$

Let $\mathcal{P}^2_{MPT}(M)$ be the convex hull of the feasible solutions to the integer program above, and let $\mathcal{Q}^2_{MPT}(M)$ be the set of feasible solutions to its linear programming relaxation:

$$
\begin{aligned}
&(17) \quad & \text{LMMPT2}(M,c) &= \min c^\top x \\
& & \text{subject to} & \\
&(18) & x(\delta_A(\bar{S})) + x(\delta_E(S)) - x(\delta_A(S)) &\geq 0 \text{ for all } S \subseteq V \text{and} \\
&(19) & x_e &\geq 1 \text{ for all } e \in E \cup A.
\end{aligned}
$$

Note that the constraints (12) were relaxed to $x(\delta_{E\cup A}(v)) \geq 0$ for all $v \in V$, but these constraints are redundant in the linear program LMMPT2(M,c). We reach the same conclusion if we use the formulation with slacks and we discard them.

4 Linear programming relaxations

In the previous section we gave two integer programming formulations for the mixed postman problem, as well as their linear relaxations. One of the first questions we might ask is whether one of the relaxations is better than the other or they are in fact equivalent. We answer this question by showing in two rather different ways that the relaxations are equivalent. A third, different proof is due to Corberán et al [2]. With this result in hand, we study some of the properties of the extreme points of the set $\mathcal{Q}^1_{MPT}(M)$ of solutions to our first formulation.

4.1 Equivalence

We give two proofs that LMMPT1(M, c) and LMMPT2(M, c) are essentially equivalent. Our first result says that solving both linear programs would give the same objective value.

Theorem 4.1.1 *For every $x^1 \in \mathcal{Q}^1_{MPT}(M)$ there exists $x^2 \in \mathcal{Q}^2_{MPT}(M)$ such that $c^\top x^1 = c^\top x^2$, and conversely, for every $x^2 \in \mathcal{Q}^2_{MPT}(M)$ there exists $x^1 \in \mathcal{Q}^1_{MPT}(M)$ such that $c^\top x^1 = c^\top x^2$. Moreover, in both cases, $x^1_a = x^2_a$ for all $a \in A$ and $x^1_{e^+} + x^1_{e^+} = x^2_e$ for all $e \in E$.*

Proof: First note that $x^1_e = x^2_e$ for all $e \in A$, and $x^1_{e^+} + x^1_{e^-} = x^2_e$ for all $e \in E$ imply $c^\top x^1 = c^\top x^2$ for every vector of costs c. ($\Rightarrow$) Let $x^1 \in \mathcal{Q}^1_{MPT}(M)$ and define x^2 as above. It is clear that $x^2 \in \mathbb{R}^{E \cup A}_+$, so we only have to prove (18). Let $S \subseteq V$, then

$$
\begin{aligned}
(20) \qquad 0 &\leq 2x^1(\vec{\delta}_B(S)) \\
(21) \qquad &= \sum_{v \in S} \Big(x^1(\vec{\delta}(\bar{v})) - x^1(\vec{\delta}(v)) \Big) + 2x^1(\vec{\delta}_B(S)) \\
(22) \qquad &= x^1(\delta_A(\bar{S})) + x^1(\vec{\delta}_B(S)) + x^1(\vec{\delta}_B(\bar{S})) - x^1(\delta_A(S)) \\
(23) \qquad &= x^2(\delta_A(\bar{S})) + x^2(\delta_E(S)) - x^2(\delta_A(S)).
\end{aligned}
$$

($\Leftarrow$) Let $x^2 \in \mathcal{Q}^2_{MPT}(M)$ and assume x^2 is rational. Let N be a positive integer such that each component of $x = Nx^2$ is an even integer. Consider the graph M^N that contains x_e copies of each $e \in E \cup A$. Note that M^N is Eulerian, and $x_e \geq N$ for all $e \in E \cup A$. Hence we can direct some of the copies of $e \in E$ in one direction and the rest in the other (say x_{e^+} and x_{e^-}, respectively) to obtain an Eulerian tour of M^N. Therefore, $x \in \mathcal{Q}^1_{MPT}(M^N)$, $x_e \geq N$ for all $e \in A$, and $x_{e^+} + x_{e^-} \geq N$ for all $e \in E$, and hence $x^1 = \frac{1}{N} x \in \mathcal{Q}^1_{MPT}(M)$. Note that x^1 satisfies the properties in the statement. $\square$

Theorem 4.1.1 implies that, for every vector c, LMMPT1(M, c) = LMMPT2(M, c), that is, it is equivalent to optimize over either polyhedron. Our second result goes a bit further: we show that $\mathcal{Q}^2_{MPT}(M)$ is essentially a projection of $\mathcal{Q}^1_{MPT}(M)$. Let $\mathcal{A}$ be the incidence matrix of the directed graph $D = (V, A)$, and let $\mathcal{D}$ be the incidence matrix of the directed graph $D^+ = (V, E^+)$. Let $\mathcal{Q}^3_{MPT}(M)$ be the set of solutions $x \in \mathbb{R}^{A \cup E \cup E^+ \cup E^-}$ of the system:

$$\begin{array}{lrcl}
(24) & \mathcal{A}x_A + \mathcal{D}(x_{E^+} - x_{E^-}) & = & \mathbf{0}_V \\
(25) & x_E - x_{E^+} - x_{E^-} & = & \mathbf{0}_E \\
(26) & x_A & \geq & \mathbf{1}_A \\
(27) & x_E & \geq & \mathbf{1}_E \\
(28) & x_{E^+} & \geq & \mathbf{0}_E \\
(29) & x_{E^-} & \geq & \mathbf{0}_E
\end{array}$$

Note that this system is a reformulation of (7)–(10) where all the constraints have been written in vector form, and we have included an additional variable x_e for each edge e. The following is a consequence of Theorem 4.1.1, but we give a different proof.

Theorem 4.1.2 *The projection of the polyhedron* $\mathcal{Q}^3_{MPT}(M)$ *onto* $x_{E^+} = \mathbf{0}_E$ *and* $x_{E^-} = \mathbf{0}_E$ *is* $\mathcal{Q}^2_{MPT}(M)$.

Proof: Let Q be the projection of $\mathcal{Q}^3_{MPT}(M)$ onto $x_{E^+} = \mathbf{0}_E$ and $x_{E^-} = \mathbf{0}_E$ (which can be obtained with an application of the Fourier-Motzkin elimination procedure), that is, let

$$Q = \{x \in \mathbb{R}^{A \cup E} : (\mathcal{A}^\top z_V + z_A)^\top x_A + (z_B + z_E)^\top x_E \geq z_A^\top \mathbf{1}_A + z_E^\top \mathbf{1}_E, \forall z \in R\},$$

where

$$R = \{(z_V, z_B, z_A, z_E) \in \mathbb{R}^{V \cup E^+ \cup A \cup E} : z_A \geq \mathbf{0}_A, z_E \geq \mathbf{0}_E \text{ and } z_B \geq |\mathcal{D}^\top z_V|\}.$$

We verify first that (18) and (19) are valid inequalities for Q:

(18) Let $S \subseteq V$, and consider the element of R given by $z_V = \chi^S$, $z_B = \chi^{\delta_E(S)}$, $z_A = \mathbf{0}_A$, and $z_E = \mathbf{0}_E$. This implies the constraint $(\chi^S)^\top \mathcal{A}x_A + (\chi^{\delta_E(S)})^\top x_E \geq 0$, that is, $x(\delta_E(S)) + x(\delta_A(\bar{S})) - x(\delta_A(S)) \geq 0$.

(19) Let $a \in A$, and consider the element of R given by $z_V = \mathbf{0}_V$, $z_B = \mathbf{0}_E$, $z_A = \mathbf{1}_a$, and $z_E = \mathbf{0}_E$. This implies the constraint $\mathbf{1}_a^\top x_A \geq \mathbf{1}_a^\top \mathbf{1}_A$, that is, $x_a \geq 1$. Let $e \in E$, and consider the element of R given by $z_V = \mathbf{0}_V$, $z_B = \mathbf{0}_E$, $z_A = \mathbf{0}_A$, and $z_E = \mathbf{1}_e$. This implies the constraint $\mathbf{1}_e^\top x_E \geq \mathbf{1}_e^\top \mathbf{1}_E$, that is, $x_e \geq 1$.

Now we verify that every element of R can be written as a nonnegative linear combination of the following elements of R:

(S1) For $S \subseteq V$, let $z_V = \chi^S$, $z_B = \chi^{\delta_E(S)}$, $z_A = \mathbf{0}_A$, and $z_E = \mathbf{0}_E$.

(S2) For $S \subseteq V$, let $z_V = -\chi^S$, $z_B = \chi^{\delta_E(S)}$, $z_A = \mathbf{0}_A$, and $z_E = \mathbf{0}_E$.

(A) For $a \in A$, let $z_V = \mathbf{0}_V$, $z_B = \mathbf{0}_E$, $z_A = \mathbf{1}_a$, and $z_E = \mathbf{0}_E$.

(E1) For $e \in E$, let $z_V = \mathbf{0}_V$, $z_B = \mathbf{0}_E$, $z_A = \mathbf{0}_A$, and $z_E = \mathbf{1}_e$.

(E2) For $e \in E$, let $z_V = \mathbf{0}_V$, $z_B = \mathbf{1}_e$, $z_A = \mathbf{0}_A$, and $z_E = \mathbf{0}_E$.

If any component of z_A or z_E is positive, we can use **(A)** or **(E1)** to reduce it to zero, so we only consider the set of solutions of $z_B \geq |\mathcal{D}^\top z_V|$ with z_B and z_V free. Let $S_+ = \mathrm{supp}_+(z_V)$, and let $S_- = \mathrm{supp}_-(z_V)$. If both S_+ and S_- are empty, then we can reduce the components of z_B using **(E2)**. Otherwise, assume that S_+ is nonempty and that the minimal positive component of z_V is 1. For every edge $e \in \delta_E(S_+)$ with endpoints $u \in S_+$, $v \notin S_+$ we have

$$(z_B)_e \geq |(\mathcal{D}^\top z_V)_e| = |(z_V)_u - (z_V)_v| \geq |(z_V)_u| = (z_V)_u \geq 1. \tag{30}$$

Therefore, the vectors

$$z_B^* \equiv z_B - \chi^{\delta_E(S_+)} \text{ and } z_V^* \equiv z_V - \chi^{S_+} \tag{31}$$

satisfy $z_B^* \geq |\mathcal{D}^\top z_V^*|$ and have fewer nonzero components. So we can reduce (z_B, z_V) using **(S1)**. Similarly, if S_- is nonempty, we can reduce (z_B, z_V) using **(S2)**. □

4.2 Half-integrality

Now we explore the structure of the extreme points of $\mathcal{Q}^1_{MPT}(M)$. To start, we offer a simple proof of the following result due independently to several authors. We say that $e \in E$ is *tight* if $x_{e^+} + x_{e^-} = 1$.

Theorem 4.2.1 (Kappauf and Koehler [7], Ralphs [10], Win [12]) *Every extreme point x of the polyhedron $\mathcal{Q}^1_{MPT}(M)$ has components whose values are either $\frac{1}{2}$ or a nonnegative integer. Moreover, fractional components occur only on tight edges.*

Proof: Let x be an extreme point of $\mathcal{Q}^1_{MPT}(M)$. We say that $a \in A$ is *fractional* if x_a is not an integer. Similarly, we say that $e \in E$ is *fractional* if at least one of x_{e^+} or x_{e^-} is not an integer. Let $F = \{e \in E \cup A : e \text{ is fractional}\}$. We will show that $F \subseteq E$, and that each $e \in F$ is tight. Assume that for some $v \in V$, $d_F(v) = 1$. Let e be the unique element of F incident to v. Since the total flow into v is integral the only possibility is that $e \in E$. Moreover, both x_{e^+} and x_{e^-} must be fractional. If e is not tight, the vectors x^1 and x^2 obtained from x replacing the entries in e^+ and e^- by

$$\begin{array}{rclrcl} x^1_{e^+} & = & x_{e^+} + \epsilon & x^1_{e^-} & = & x_{e^-} + \epsilon, \\ x^2_{e^+} & = & x_{e^+} - \epsilon & x^2_{e^-} & = & x_{e^-} - \epsilon \end{array} \tag{32}$$

(where $\epsilon = \min\{x_{e^+}, x_{e^-}, 2(x_{e^+} + x_{e^-} - 1)\} > 0$) would be feasible, with $x = \frac{1}{2}(x^1 + x^2)$, contradicting the choice of x. Hence e is a tight edge, and satisfies $x_{e^+} = x_{e^-} = \frac{1}{2}$. Delete e from F and repeat the above argument until F is empty or F induces an undirected graph with minimum degree 2. (Deletion of e does not alter the argument since it contributes 0 flow into both its ends.) Suppose F contains a cycle C. Assign an arbitrary orientation (say, positive) to C. We say that an arc in C is *forward* if it has the same orientation as C, and we call it *backward* otherwise. Partition C as follows:

$$C_A^+ = \{e \in C \cap A : e \text{ is forward}\}, \tag{33}$$

$$C_A^- = \{e \in C \cap A : e \text{ is backward}\}, \tag{34}$$

$$C_E^= = \{e \in C \cap E : e \text{ is tight}\}, \tag{35}$$

$$C_E^> = \{e \in C \cap E : e \text{ is not tight}\}, \tag{36}$$

and define

$$\epsilon^+ = \min_{e \in C_A^+} \lceil x_e \rceil - x_e, \tag{37}$$

$$\epsilon^- = \min_{e \in C_A^-} x_e - \lfloor x_e \rfloor, \tag{38}$$

$$\epsilon^= = \min_{e \in C_E^=} \{x_{e^+}, x_{e^-}\}, \tag{39}$$

$$\epsilon^> = \min_{e \in C_E^>} \{\lceil x_e \rceil - x_e, x_e - \lfloor x_e \rfloor\}, \tag{40}$$

$$\epsilon^1 = \min\{\epsilon^+, \epsilon^-, 2\epsilon^=, \epsilon^>\}. \tag{41}$$

The choice of C implies $\epsilon^1 > 0$. Now we define a new vector x^1 as follows:

$$(42)\quad x^1_e = \begin{cases} x_e + \epsilon^1 & \text{if } e \in C_A^+ \text{ or } e \text{ is forward in } C_E^> \\ x_e - \epsilon^1 & \text{if } e \in C_A^- \text{ or } e \text{ is backward in } C_E^> \\ x_e + \frac{1}{2}\epsilon^1 & \text{if } e \text{ is the forward copy of an edge in } C_E^= \\ x_e - \frac{1}{2}\epsilon^1 & \text{if } e \text{ is the backward copy of an edge in } C_E^= \\ x_e & \text{otherwise.} \end{cases}$$

This is equivalent to pushing ϵ^1 units of flow in the positive direction of C, and therefore it is easy to verify that $x^1 \in \mathcal{Q}^1_{MPT}(M)$. Similarly, define ϵ^2 and a vector x^2 using the other (negative) orientation of C. But now x is a convex combination of x^1 and x^2 (in fact, by choosing $\epsilon = \min\{\epsilon^1, \epsilon^2\}$ and pushing ϵ units of flow in both directions we would have $x = \frac{1}{2}(x^1 + x^2)$) contradicting the choice of x. Therefore F is empty. □

A similar idea allows us to prove a sufficient condition for $\mathcal{Q}^1_{MPT}(M)$ to be integral. A mixed graph $M = (V, E, A)$ is *even* if the total degree $d_{E\cup A}(v)$ is even for every $v \in V$.

Theorem 4.2.2 (Edmonds and Johnson [3]) *If M is even, then the polyhedron $\mathcal{Q}^1_{MPT}(M)$ is integral. Therefore the mixed postman problem can be solved in polynomial time for the class of even mixed graphs.*

Proof: Let x be an extreme point of $\mathcal{Q}^1_{MPT}(M)$. We say that $a \in A$ is *even* if x_a is even. We say that $e \in E$ is *even* if $x_{e^+} - x_{e^-}$ is even. For a contradiction, assume x is not integral, and define F as in the proof of Theorem 4.2.1. Let $N = \{e \in E \cup A : e \text{ is even}\}$. Note that by Theorem 4.2.1, $F \subseteq N$. Hence N is not empty. We show now that $M[N]$ has minimum degree 2, and hence contains a cycle C. Let $v \in V$. If $d_F(v) \geq 2$ then certainly $d_N(v) \geq 2$. If $d_F(v) = 1$ then

$$(43)\qquad x(\vec{\delta}(v)) - x(\vec{\delta}(\bar{v})) = \sum_{a\in\delta_A(v)\cup\delta_A(\bar{v})} \pm x_a + \sum_{e\in\delta_E(v)} \pm(x_{e^+} - x_{e^-})$$

is the sum of an even number of integer terms (one term per arc $a \in \delta_A(v)\cup\delta_A(\bar{v})$ and one term per edge $e \in \delta_E(v)$), and one of them is equal to zero (the one in $\delta_F(v)$); therefore another term must be even. The same argument works for a vertex v not in $V(F)$, that is, $d_F(v) = 0$, with at least one element of N incident to it, that is, $d_N(v) \geq 1$.

As before, assign an arbitrary (positive) orientation to C and partition it into the classes $C_A^+, C_A^-, C_E^=, C_E^>$. Note that all $e \in C \setminus C_E^=$ satisfy $x_e \geq 2$. Hence the vector x^1 defined as

$$(44) \quad x_e^1 = \begin{cases} x_e + 1 & \text{if } e \in C_A^+ \text{ or } e \text{ is forward in } C_E^>, \\ x_e - 1 & \text{if } e \in C_A^- \text{ or } e \text{ is backward in } C_E^>, \\ x_e + \frac{1}{2} & \text{if } e \text{ is the forward copy of an edge in } C_E^=, \\ x_e - \frac{1}{2} & \text{if } e \text{ is the backward copy of an edge in } C_E^=, \\ x_e & \text{otherwise,} \end{cases}$$

as well as the vector x^2 obtained from the negative orientation of C, belong to $\mathcal{Q}_{MPT}^1(M)$ and satisfy $x = \frac{1}{2}(x^1 + x^2)$. This contradiction implies that F must be empty. □

4.3 Dual integrality

Now we consider the dual of the linear relaxation LMMPT1 (6-10):

$$\begin{array}{lrcl}
(45) & \text{DMMPT1}(M, c) & = & \mathbf{1}^\top z \\
& \text{subject to} & & \\
(46) & y_u - y_v + z_a & \leq & c_a \text{ for all } a \in A \text{ with tail } u \text{ and head } v, \\
(47) & y_u - y_v + z_e & \leq & c_e \text{ for all } e \in E \text{ with ends } u \text{ and } v, \\
(48) & -y_u + y_v + z_e & \leq & c_e \text{ for all } e \in E \text{ with ends } u \text{ and } v, \\
(49) & y_v & & \text{free for all } v \in V, \text{ and} \\
(50) & z_e & \geq & 0 \text{ for all } e \in A \cup E.
\end{array}$$

Theorem 4.3.1 *Let $M = (V, E, A)$ be strongly connected, and let $c \in \mathbb{Z}_+^{E \cup A}$. Then DMMPT1 has an integral optimal solution (y^*, z^*).*

Proof: Since LMMPT1 is feasible and bounded, then DMMPT1 is also feasible and bounded. Furthermore, both problems have optimal solutions. Choose an extreme point optimal solution x^* of the primal. Without loss of generality, we can assume that not both $x_{e^+}^*, x_{e^-}^*$ are positive, unless $e \in E$ is tight. We construct an integral solution (y^*, z^*) to the dual satisfying the complementary slackness conditions:

1. for all $(u, v) = a \in A$, $x_a^* > 0$ implies $y_u^* - y_v^* + z_a^* = c_a$,
2. for all $\{u, v\} = e \in E$, $x_{e^+}^* > 0$ implies $y_u^* - y_v^* + z_e^* = c_e$,
3. for all $\{u, v\} = e \in E$, $x_{e^-}^* > 0$ implies $-y_u^* + y_v^* + z_e^* = c_e$,

4. for all $a \in A$, $z_a^* > 0$ implies $x_a^* = 1$, and
5. for all $e \in E$, $z_e^* > 0$ implies $x_{e^+}^* + x_{e^-}^* = 1$.

Note that $x_a^* > 0$ for all $a \in A$, hence condition (1) implies that $y_u^* - y_v^* + z_a^* = c_a$ for all $(u,v) = a \in A$. Also note that, for any $e \in E$, at least one of $x_{e^+}^* > 0$ and $x_{e^-}^* > 0$ holds. Moreover, the only case in which both hold is when e is a fractional tight edge. In this case, conditions (2) and (3) imply that $y_u^* = y_v^*$ and $z_e^* = c_e$. Hence, to obtain a feasible solution to the dual satisfying complementary slackness, we can set $z_e^* = c_e$ for each fractional tight edge e, and then contract each connected component (V_i, F_i) of the *fractional* graph (V, F) into a single super-vertex v_i, creating a new dual variable y_{v_i} for it. Once we are done with the rest of the construction, we set $y_v^* = y_{v_i}^*$ for each vertex $v \in V_i$.

At this point, all remaining edges e satisfy that either $x_{e^+} = 0$ or $x_{e^-} = 0$. Delete the arc whose variable is zero, and let $D = (V', A')$ the directed graph thus obtained. Observe that the restriction x of x^* to the arcs of D is an optimal integer circulation of D with costs c restricted to the arcs of D. But the minimum cost circulation problem has integral optimal dual solutions. Let $(y, z) \in \mathbb{Z}^{V' \cup E'}$ be one such solution. Let y^* be the extension of y as described in the previous paragraph. Let z^* be the extension of z obtained as follows. For each $a \in A \setminus A'$ let z_a^* have the integer value implied by condition (1). For each $e \notin F$ let z_e^* have the integer value implied by either condition (2) or (3).

Now, using the interpretation of (y, z) as a potential in D, it is not hard to verify that the vector (y^*, z^*) satisfies (4) and (5), and hence it is an integral optimal solution to DMMPT1. $\square$

5 Open problems

One of the most interesting open problems is that of a full characterization of integrality of the polyhedron $\mathcal{Q}_{MPT}^1(M)$. Another interesting option is to add a set of valid inequalities to obtain a tighter relaxation. For example, we can add the well-known *odd-cut constraints* to obtain another polyhedron $\mathcal{O}_{MPT}^1(M)$, and ask again for a full characterization of integrality of this polyhedron. Finally, we may ask whether our knowledge about the extreme points of the primal and dual polyhedra could lead us to a primal-dual approximation algorithm for the mixed postman problem.

Acknowledgements

The author would like to thank Bill Cunningham, Joseph Cheriyan, Jim Geelen, Bertrand Guenin, Miguel Anjos, and Antoine Vella at the University of Waterloo for their continuous support and their insightful comments during countless discussions.

Francisco Javier Zaragoza Martínez
Departmento de Sistemas,
Universidad Autónoma Metropolitana, Unidad Azcapotzalco
Av. San Pablo 180, Edificio H 2do Piso,
Col. Reynosa Tamaulipas,
Deleg. Azcapotzalco, 02200, México, D.F.
franz@correo.azc.uam.mx

References

[1] Christofides N.; Benavent E.; Campos V.; Corberán Á.; Mota E., *An optimal method for the mixed postman problem*, Lecture Notes in Control and Inform. Sci. **59** (1984), 641–649.

[2] Corberán Á.; Mota E.; Sanchis J. M., *A comparison of two different formulations for arc routing problems on mixed graphs*, available online in Comput. Oper. Res., 2005.

[3] Edmonds J.; Johnson E. L., *Matching, Euler tours and the Chinese postman*, Math. Programming **5** (1973), 88–124.

[4] Ford L.R. Jr.; Fulkerson D. R., Flows in Networks, Princeton University Press, Princeton, N.J., 1962.

[5] Grötschel M.; Win Z., *A cutting plane algorithm for the windy postman problem*, Math. Programming Series A, **55** No.3 (1992), 339–358.

[6] Guan M. G., *Graphic programming using odd or even points*, Chinese Math **1** (1960), 273–277.

[7] Kappauf C. H.; Koehler G. J.,*The mixed postman problem*, Discrete Appl. Math. **1** No.1-2 (1979), 89–103.

[8] Nobert Y.; Picard J.-C., *An optimal algorithm for the mixed Chinese postman problem*, Networks **27** No.2 (1996), 95–108.

[9] Papadimitriou C. H., *On the complexity of edge traversing*, J. ACM **23** No.3 (1976), 544–554.

[10] Ralphs T. K., *On the mixed Chinese postman problem*, Oper. Res. Lett. **14** No.3 (1993), 123–127.

[11] Veblen O., *An application of modular equations in analysis situs*, Ann. of Math. **2** No.14 (1912/1913), 86–94.

[12] Win Z., *On the windy postman problem on Eulerian graphs*, Math. Programming Series A, **44** No.1 (1989), 97–112.