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An approximation scheme for the mass transfer problem*

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Abstract

This paper presents an approximation scheme for the Monge-Kantorovich mass transfer (MT) problem in separable metric spaces. A sequence of finite-dimensional linear programs (transport problems) are introduced and it is proven that the value of the MT problem is the limit of a subsequence of the optimal values of these programs.

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1 Introduction

The Monge-Kantorovich mass transfer (MT) problem (introduced in Section 3; see (3.1),(3.2)) is among the oldest and most well known problems in probability theory and its applications. It was originally introduced by Gaspar Monge in 1781 [19] but it was posed as a mathematical programming problem by L.V. Kantorovich in 1942 [17]. For comments on the historical development and applications of the MT problem see, for instance, [1, 7, 20, 21].

The MT problem has been studied by many authors with several approaches [1, 8, 9, 10, 14, 16, 18, 22, 23]. In general, there are many results on the MT problem, but how to obtain explicit solutions, however, is still an open problem.

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In this paper we show how to approximate the MT problem by a sequence of finite-dimensional linear programs. These programs are solvable and each one has associated a probability measure (p.m.) with a finite support. Moreover, there exists a subsequence of these p.m.s that converges weakly to the optimal solution of the MT problem. We also prove that the subsequence of optimal values converges to the optimal value of the MT problem.

Actually, approximation schemes for the MT problem have been studied by several authors. For example, in [1],[2] an algorithm is studied in the case in which the underlying spaces are the unit interval $[0, 1]$ (see Section 3). Hernández-Lerma and Lasserre [15] study the problem in compact metric spaces and they give an approximation scheme based on finite-dimensional linear programs. Other schemes are studied in [4, 5, 6, 11, 13].

The remainder of the paper is organized as follows: In Section 2 we study the so-called transport or transportation problem and under very mild conditions we show that the problem is consistent and solvable. In Section 3 we present our main results concerning the approximation scheme described above.

2 The transportation problem

The classical transportation problem (TP) is a linear program defined as follows:

$$\text{(2.1)} \quad \text{TP} \quad \text{minimize} \quad \sum_{k=1}^{M}\sum_{j=1}^{N} c_{kj}\lambda_{kj}$$

$$\text{(2.2)} \quad \text{subject } \ to: \quad \sum_{j=1}^{N}\lambda_{kj} = a_k, \quad 1 \leqslant k \leqslant M,$$

$$\text{(2.3)} \quad \sum_{k=1}^{M}\lambda_{kj} = b_j, \quad 1 \leqslant j \leqslant N,$$

$$\text{(2.4)} \quad \lambda_{kj} \geq 0, \quad 1 \leqslant k \leqslant M, \quad 1 \leqslant j \leqslant N.$$

The decision variables λ_{kj} represent the amounts shipped from source k to destination j. The demand at destination j is b_j, the supply at source k is a_k and c_{kj} is the unit shipping cost from source k to destination j.

The transportation problem is said to be consistent if there exists a matrix $\Lambda = (\lambda_{kj})$ that satisfies (2.2),(2.3) and (2.4). As an example, when the total supply equals the total demand, we have that TP is consistent; see Theorem 2.3.

Let

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & ... & \lambda_{1N} \\ \lambda_{21} & \lambda_{22} & ... & \lambda_{2N} \\ \vdots & & & \\ \lambda_{M1} & \lambda_{M2} & ... & \lambda_{MN} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} & ... & c_{1N} \\ c_{21} & c_{22} & ... & c_{2N} \\ \vdots & & & \\ c_{M1} & c_{M2} & ... & c_{MN} \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}_{1\times N}, \quad B = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}_{1\times M},$$

$$a = \begin{pmatrix} a_1 & a_2 & \dots & a_M \end{pmatrix} \, and \quad b = \begin{pmatrix} b_1 & b_2 & \dots & b_N \end{pmatrix}.$$

We use the following notation:

$$a \geq 0 \quad \text{if} \quad a_k \geq 0 \quad \text{for} \quad \text{all} \quad k = 1, 2, ..., M$$

and

$$a \gg 0 \quad \text{if} \quad a_k > 0 \qquad \text{for} \quad \text{all} \quad k = 1, 2, .., M.$$

We can then express the TP problem in matrix form as:

(2.5) TP minimize $\langle \Lambda, C \rangle$

(2.6) subject *to* : $A\Lambda^T = a, \quad B\Lambda = b, \quad \Lambda \geq 0,$

where $\langle \Lambda, C \rangle = \sum_{k=1}^{M} \sum_{j=1}^{N} \lambda_{kj} c_{kj}$ and Λ^T is the transpose of Λ.

A matrix Λ is said to be a feasible solution for the TP problem if it satisfies (2.6).

Remark 2.1 *The space of $M \times N$ matrices is a normed space, with norm*

$$\|\Lambda\| = \sum_{k=1}^{M} \sum_{j=1}^{N} |\Lambda_{kj}|.$$

Definition 2.2 *A matrix* Λ^* *is said to be an optimal solution to* TP *if it is feasible and*

$$\langle \Lambda^*, C\rangle = \inf\,(\mathrm{TP}) := \inf\{\langle \Lambda, C\rangle | \Lambda \; is \; a feasible \; solution\}.$$

TP *is solvable if it has an optimal solution.*

Theorem 2.3 *If* $\sum_{k=1}^{M} a_k = \sum_{j=1}^{N} b_j$, *$a \gg 0$, $b \gg 0$ and $C \geq 0$, then TP is solvable.*

Proof: Define

$$\mathbb{F} := \{\Lambda | A\Lambda^T = a \quad and \quad B\Lambda = b, \quad \Lambda \geq 0\}$$

and

$$S := \{\langle \Lambda, C\rangle | \Lambda \in \mathbb{F}\}.$$

Let $t := \sum_{k=1}^{M} a_k = \sum_{j=1}^{N} b_j$. Then it can be proved that

$$\lambda_{ij} = \frac{a_i b_j}{t} \text{ for all } i = 1, 2, ..., M \text{ and } j = 1, 2, ..., N$$

is a feasible solution for TP. Therefore $\mathbb{F}$ and S are both nonempty. Moreover, as $c_{kj} \geq 0$, S is bounded from below, which implies that $\inf\,(\mathrm{TP}) \in \mathbb{R}$.

Let $\{\Lambda_n\}$ be a minimizing sequence for TP; that is, each Λ_n is a feasible solution for TP and

$$\langle \Lambda_n, C\rangle \downarrow \inf\,(\mathrm{TP}) = \inf\, S. \tag{2.7}$$

Furthermore $\langle \Lambda_n, C\rangle \leq \langle \Lambda_0, C\rangle$ for all n in $\mathbb{N}$, where Λ_0 is some feasible solution for TP.

Since $a \gg 0$, $b \gg 0$ and Λ_n is feasible for all n, then it can be proved that $\{\Lambda_n\}$ is bounded in $\mathbb{R}^{nm}$. Hence by the Theorem of Bolzano-Weierstrass ([3], pp. 93), there is a subsequence $\{\Lambda_m\}$ of $\{\Lambda_n\}$ and a matrix Λ^* such that $\Lambda_m \to \Lambda^*$.

Clearly, $\Lambda \mapsto \langle \Lambda, C\rangle$ is a continuous function. Hence

$$\langle \Lambda_m, C\rangle \to \langle \Lambda^*, C\rangle \tag{2.8}$$

and, by (2.7), we obtain

$$\langle \Lambda^*, C \rangle = \inf\,(\mathrm{TP}).$$

Thus, to complete the proof it suffices to show that Λ^* is a feasible solution for TP.

Since Λ_m is in $\mathbb{F}$, we have $A\Lambda_m^t = a$, $B\Lambda_m = b$ and $\Lambda_m \geq 0$. Also, $A\Lambda^t$ and $B\Lambda$ are continuous functions in Λ, and so, by (2.8), $A\Lambda^* = a$, $B\Lambda^* = b$ and, in addition, by the convergence of $\{\Lambda_m\}$ we have that $\Lambda^* \geq 0$. Therefore, Λ^* is a feasible solution. □

In next section we introduce the MT problem and show how can we approximate it by transportation problems.

3 The mass-transfer problem

In the MT problem we are concerned with the following data are given:

a) two metric spaces X and Y endowed with the corresponding Borel σ-algebras $\mathbb{B}(X)$ and $\mathbb{B}(Y)$;

b) A nonnegative measurable function $c : X \times Y \to \mathbb{R}$, and

c) A probability measure (p.m.) v_1 on X, and a p.m. v_2 on Y.

Moreover, let $M(X \times Y)$ be the linear space of finite signed measures on $X \times Y$, endowed with the topology of weak convergence, and let $M^+(X \times Y)$ be the convex cone of nonnegative measures in $M(X \times Y)$.

If μ is in $M(X \times Y)$, we denote by $\Pi_1\mu$ and $\Pi_2\mu$ the marginals (or projections) of μ on X and Y, respectively; that is, for all $A \in \mathbb{B}(X)$ and $B \in \mathbb{B}(Y)$

$$\Pi_1\mu(A) := \mu(A \times Y) \quad \Pi_2\mu(B) := \mu(X \times B).$$

Then, with $\langle \mu, c \rangle := \int c d\mu$, the MT problem can be stated as follows:

$$\text{(3.1) MT} \quad \text{minimize} \quad \langle \mu, c \rangle$$

$$\text{(3.2)} \quad \text{subject } to: \; \Pi_1\mu = v_1, \; \Pi_2\mu = v_2, \; \mu \in M^+(X \times Y).$$

A measure $\mu \in M(X \times Y)$ is said to be a feasible solution for the MT problem if it satisfies (3.2) and $\langle \mu, c \rangle$ is finite. The MT problem is called consistent if the set of feasible solutions is nonempty, in which case its (optimal) value is defined as

$$\inf(\mathrm{MT}) := \inf\{\langle \mu, c \rangle | \; \mu \; is \; a \; feasible \; solution \; for \; \mathrm{MT}\}.$$

It is said that the MT problem is solvable if there is a feasible solution μ^* that attains the optimal value. In this case, μ^* is called an optimal solution for the MT problem and the value inf(MT) is written as $\min(\mathrm{MT}) = \langle \mu^*, c\rangle$.

Remark 3.1 **a)** *Since v_1 and v_2 are p.m.'s, a feasible solution for* MT *is also a p.m.*

b) *If c is a bounded function, then the product measure $\mu := v_1 \times v_2$ is a feasible solution.*

The latter fact is not necessarily true if c is unbounded; see Example 1.2 in [12]. However, even for unbounded c, mild assumptions ensure that the MT problem is consistent [12].

We will need either one of the following assumptions.

Assumption 3.2 **a)** *X and Y are separable metric spaces.*

b) *The "cost" function $c(x, y)$ is nonnegative, continuous and inf-compact, which means that, for each $r \in \mathbb{R}$, the set*

$$K_r = \{(x, y)|c(x, y) \leq r\}$$

is compact.

Assumption 3.3 **a)** *X and Y are σ-compact and separable metric spaces.*

b) *The "cost" function $c(x, y)$ is nonnegative and continuous.*

The following Theorem from [14] establishes that the MT problem is solvable.

Theorem 3.4 *If either Assumption 3.2 or 3.3 holds, then the* MT *problem is solvable.*

Now we introduce a sequence of finite-dimensional linear programs in the following way. By the proof of Proposition 6.3 in [13], there are two sequences of probability measures $\{v_1^i\}$ on $\mathbb{B}(X)$ and $\{v_2^i\}$ on $\mathbb{B}(Y)$ with supports in finite sets $\{x_1^i, ..., x_{M_i+1}^i\}$ and $\{y_1^i, ..., y_{N_i+1}^i\}$, respectively, such that $\{v_1^i\}$ converges weakly to v_1 and $\{v_2^i\}$ converges weakly to v_2. In addition, $\{x_1^i, ..., x_{M_i+1}^i\} \subset X_\infty$, $\{y_1^i, ..., y_{N_i+1}^i\} \subset Y_\infty$ with X_∞, Y_∞ denumerable dense sets in X and Y, respectively.

For each positive integer i, consider the following MT problem

$$\text{(3.3)} \qquad \mathrm{MT}_i: \quad \text{minimize} \quad \langle \mu, c_i \rangle$$

$$\text{(3.4)} \qquad \text{subject to}: \ \Pi_1\mu = v_1^i, \ \ \Pi_2\mu = v_2^i, \ \ \mu \geq 0,$$

where $c_i = \min\{c, i\}$.

Proposition 3.5 *If μ is a feasible solution for* MT_i*, then μ has a finite support and, moreover,*

$$\mathrm{supp}(\mu) \subset \{x_1^i, ..., x_{M_i+1}^i\} \times \{y_1^i, ..., y_{N_i+1}^i\},$$

where $\mathrm{supp}(\mu)$ *means the support of μ.*

Proof: Let $S_1 := \mathrm{supp}(v_1^i)$ and $S_2 := \mathrm{supp}(v_2^i)$. Then

$$(S_1 \times S_2)^c \subset (S_1^c \times Y) \cup (X \times S_2^c),$$

which implies

$$0 \leq \mu[(S_1 \times S_2)^c] \leq \mu(S_1^c \times Y) + \mu(X \times S_2^c)$$
$$= v_1^i(S_1^c) + v_2^i(S_2^c) = 0.$$

Therefore $\mu[(S_1 \times S_2)^c] = 0$, i.e., $\mathrm{supp}(\mu) \subset (S_1 \times S_2)$. □

Let μ_i be a feasible solution for the $\mathrm{MT_i}$ problem. Then μ_i is of the form

$$\text{(3.5)} \qquad \mu_i(E) = \sum_{k=1}^{M_i+1} \sum_{j=1}^{N_i+1} \lambda_{kj}^i \delta_{(x_k^i, y_j^i)}(E) \quad \forall \quad E \in \mathbb{B}(X \times Y),$$

where $\delta_{(x,y)}$ denotes the Dirac measure concentrated at $(x, y) \in X \times Y$. Since μ_i is a p.m.,

$$\text{(3.6)} \qquad \sum_{k=1}^{M_i+1} \sum_{j=1}^{N_i+1} \lambda_{kj}^i = 1 \quad \text{and} \quad \lambda_{kj}^i \geq 0$$

$$\forall\, 1 \leq k \leq M_i + 1, \ 1 \leq j \leq N_i + 1.$$

In addition, (3.3) becomes

$$\text{(3.7)} \qquad \langle \mu_i, c_i \rangle = \int c_i d\mu_i = \sum_{k=1}^{M_i+1} \sum_{j=1}^{N_i+1} \lambda_{kj}^i c_i(x_k^i, y_j^i).$$

Let
$$c^i_{kj} := c_i(x^i_k, y^i_j),$$
and let S_1 and S_2 be as in the proof of Proposition 3.5. To calculate the marginals v^i_1 and v^i_2 on sets $A \subset \mathbb{B}(X)$ and $B \subset \mathbb{B}(Y)$, respectively, it suffices to calculate the marginals on $A \cap S_1$ and $B \cap S_2$. In particular, for any fixed $x^i_{k_0} \in X$, by (3.4) and (3.5) we have:

$$\begin{aligned} v^i_1(\{x^i_{k_0}\}) &= \Pi_1\mu_i(\{x^i_{k_0}\}) \\ &= \mu_i(\{x^i_{k_0}\} \times Y) \\ &= \sum_{k=1}^{M_i+1}\sum_{j=1}^{N_i+1} \lambda^i_{kj}\delta_{(x^i_k,y^i_j)}(\{x^i_{k_0}\} \times Y) \\ &= \sum_{j=1}^{N_i+1} \lambda^i_{k_0 j}. \end{aligned} \tag{3.8}$$

We define $a^i_k := v^i_1(\{x^i_k\})$ for all $k = 1, 2, ..., M_i + 1$.

Similarly,

$$\begin{aligned} v^i_2(\{y^i_{j_0}\}) &= \Pi_2\mu_i(\{y^i_{j_0}\}) \\ &= \mu_i(X \times \{y^i_{j_0}\}) \\ &= \sum_{k=1}^{M_i+1}\sum_{j=1}^{N_i+1} \lambda^i_{kj}\delta_{(x^i_k,y^i_j)}(X \times \{y^i_{j_0}\}) \\ &= \sum_{k=1}^{M_i+1} \lambda_{kj_0}. \end{aligned} \tag{3.9}$$

We define $b^i_j := v^i_2(\{y^i_j\})$ for all $j = 1, 2, ..., N_i + 1$.

By (3.6),(3.8) and (3.9) we have that MT_i is equivalent to the transportation problem

$$\mathrm{TP}_i : \quad \text{minimize} \quad \sum_{k=1}^{M_i+1}\sum_{j=1}^{N_i+1} c^i_{kj}\lambda^i_{kj} \tag{3.10}$$

$$\text{subject to :} \quad \sum_{j=1}^{N_i+1} \lambda^i_{kj} = a^i_k, \ {\scriptstyle 1\le k\le M_i+1} \tag{3.11}$$

$$\sum_{k=1}^{M_i+1} \lambda^i_{kj} = b^i_j, \ {\scriptstyle 1\le j\le N_i+1.} \tag{3.12}$$

Since

$$\sum_{k=1}^{M_i+1} a_k^i = \sum_{k=1}^{M_i+1} v_1^i(\{x_k^i\}) = 1 = \sum_{j=1}^{N_i+1} v_2^i(\{y_j^i\}) = \sum_{j=1}^{N_i+1} b_j^i,$$

it follows from Theorem 2.3 that TP_i is solvable for each i in $\mathbb{N}$.

Let μ_i^* be an optimal solution for the TP_i problem, that is,

$$\mu_i^*(\cdot) = \sum_{k=1}^{M_i+1} \sum_{j=1}^{N_i+1} \lambda_{kj}^i \delta_{(x_k^i, y_j^i)}(\cdot).$$

(Recall (3.5).) We can now state our main result as follows

Theorem 3.6 *If either Assumption 3.2 or Assumption 3.3 holds, there exists a subsequence* $\{\mu_{i_n}^*\}$ *of* $\{\mu_i^*\}$ *and a probability measure* μ^* *such that*

a) $\{\mu_{i_n}^*\}$ *converges weakly to* μ^**,*

b) μ^* *is an optimal solution to the* MT *problem and*

c) $\lim_{n\to\infty} \langle \mu_{i_n}^*, c_{i_n}\rangle = \lim_{n\to\infty} \left(\sum_{k=1}^{M_{i_n}+1} \sum_{j=1}^{N_{i_n}+1} c_{kj}^i \lambda_{kj}^i \right)$

$$= \langle \mu^*, c\rangle = \min(\mathrm{MT}).$$

Proof: The hypothesis (Assumption 3.2 or 3.3) implies that the sequence $\{\mu_i^*\}$ is tight; see Lemma 2.4 and Remark 2.5 in [14]. Hence, by Prohorov's Theorem there is a subsequence $\{\mu_{i_n}^*\}$ of $\{\mu_i^*\}$ and a p.m. μ^* on $\mathbb{B}(X \times Y)$, such that $\{\mu_{i_n}^*\}$ converges weakly to μ^*.

Observe that $\mu_{i_n}^*$ is a feasible solution to MT_{i_n}. Hence

$$\Pi_1 \mu_{i_n}^* = v_1^{i_n} \text{ and } \Pi_2 \mu_{i_n}^* = v_2^{i_n},$$

and by Lemma 2.7 in [14], we have that the marginals $\Pi_1\mu_{i_n}^*$ and $\Pi_2\mu_{i_n}^*$ converge to the marginals $\Pi_1\mu^*$ and $\Pi_2\mu^*$, respectively. This implies that $\Pi_1\mu^* = v_1$ and $\Pi_2\mu^* = v_2$, that is, μ^* is a feasible solution for the MT problem.

By Theorem 3.4 there exists an optimal solution μ for the MT problem and by Theorem 6.4 in [13], there is a sequence $\{\mu_{i_n}\}$ of p.m.s on

$\mathbb{B}(X \times Y)$ such that $\{\mu_{i_n}\}$ converges weakly to μ, and μ_{i_n} is a feasible solution for TP_{i_n}. Since μ is an optimal solution for MT and μ^* is feasible for MT, we have

$$\langle \mu, c\rangle \leq \langle \mu^*, c\rangle. \tag{3.13}$$

Now, for each i, we have

$$\langle \mu_i, c_i\rangle \geq \langle \mu_i^*, c_i\rangle,$$

and by Theorem 6.4 in [13], we have

$$\langle \mu, c\rangle = \lim_{n\to\infty} \langle \mu_{i_n}, c_{i_n}\rangle \geq \limsup_{n\to\infty} \langle \mu_{i_n}^*, c_{i_n}\rangle \geq 0 \tag{3.14}$$

Pick an arbitrary $\varepsilon > 0$. Since $c_n \uparrow c$, there exists an integer m such that

$$\langle \mu^*, c\rangle \geq \langle \mu^*, c_m\rangle \geq \langle \mu^*, c\rangle - \varepsilon. \tag{3.15}$$

For each $i_n \geq m$ we have

$$0 \leq \langle \mu_{i_n}^*, c_m\rangle \leq \langle \mu_{i_n}^*, c_{i_n}\rangle,$$

and as $\mu_{i_n}^* \to \mu^*$ we obtain

$$\langle \mu^*, c_m\rangle = \lim_{n\to\infty} \langle \mu_{i_n}, c_m\rangle \leq \limsup_{n\to\infty} \langle \mu_{i_n}^*, c_{i_n}\rangle.$$

Therefore, by (3.15), it follows that

$$\langle \mu^*, c\rangle \leq \lim_{n\to\infty} \langle \mu_{i_n}^*, c_{i_n}\rangle + \varepsilon.$$

Consequently, as ε was arbitrary,

$$\langle \mu^*, c\rangle \leq \limsup_{n\to\infty} \langle \mu_{i_n}^*, c_{i_n}\rangle,$$

which together with (3.14) gives

$$\langle \mu^*, c\rangle \leq \langle \mu, c\rangle. \tag{3.16}$$

Thus, from (3.13) and (3.16) we obtain $\langle \mu, c\rangle = \langle \mu^*, c\rangle$, that is, μ^* is an optimal solution for MT. □

Observe that the same reasoning of the former theorem, can be applied for any subsequence $\{\mu_k^*\}$ of $\{\mu_i^*\}$ and we get that there are a subsequence $\{\mu_l^*\}$ of $\{\mu_k^*\}$ and probability measure μ, such that μ_l^* converges weakly to μ. Therefore μ is also an optimal solution for MT problem.

Then we have following theorem

Theorem 3.7 *If either Assumption 3.2 or 3.3 holds, then*

$$\lim_{i\to\infty}\langle \mu_i^*, c\rangle = \langle \mu^*, c\rangle = \min(\mathrm{MT}).$$

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References

[1] Anderson E. J.; Nash P., Linear Programming in Infinite-Dimensional Spaces, Wiley, Chichester, U.K, 1987.

[2] Anderson E. J.; Philpott A. B., *Duality and an algorithm for a class of continuous transportation prolems,* Math. Oper. Res. **9** (1984), 222-231.

[3] Bartle R. G., Introducción al Análisis Matemático, Limusa, México, 1987.

[4] Benamou J. D., *Numerical resolutions of an "unbalanced" mass transport problem,* ESAIM: Mathematical Modelling and Numerical Analysis **37** (2003), 851-868.

[5] Benamou J. D.; Brenier Y., *A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem,* Numerische Mathematik **84** (2000), 375-393.

[6] Caffarelli L. A.; Feldman M.; McCann R. J., *Constructing optimal maps for Monge's transport problem as a limit of strictly convex costs,* J. Amer. Math. Soc. **15** (2001), 1-26.

[7] Cuesta-Albertos J. A.; Matrán C.; Rachev S. T.; Rüschendorf L. R., *Mass transportation problems in probability theory,* Mathematical Scientist **21** (1996), 37-72.

[8] Cuesta-Albertos J. A.; Tuero-Díaz A., *A caracterization for the solution Monge-Kantorovich mass transfer problem,* Statist. Prob. Lett. **6** (1993), 147-152.

[9] Gabriel J. R.; López-Martínez R. R.; Hernández-Lerma O., *The Lagrange approach to infinite linear programs,* Top **9** (2001), 293-314.

[10] Gangbo W., *The Monge mass transfer problem and its applications,* NSF-CBMS Conference, Contemporary Mathematics **226** (1999), 1-26.

[11] Guittet K, *On the time-continuous mass transport problem and its approximation by augmented Lagrangian techniques,* SINUM **41** (2003), 382-399.

[12] González-Hernández J.; Gabriel J. R., *On the consistency of the mass transfer problem,* Oper. Res. Lett. **34** (2005), 382-386.

[13] González-Hernández J.; Gabriel J. R.; Hernández-Lerma O., *On solutions to the mass transfer problem,* SIAM J. Optim. **17** (2006), 485-499.

[14] Hernández-Lerma O.; Gabriel J. R., *Strong duality of the Monge-Kantorovich mass transfer problem in metric spaces,* Math. Z. **239** (2002), 579-591.

[15] Hernández-Lerma O.; Lasserre J. B., *Approximation schemes for infinite linear programs,* SIAM J. Optim. **8** (1998), 973-988.

[16] Jiménez-Guerra P.; Rodríguez-Salinas B., *A general solution of the Monge-Kantorovich mass-transfer problem,* J. Math. Anal. Appl. **202** (1996), 492-510.

[17] Kantorovich L. V., *On the translocation of masses,* Dokl. Akad. Nauk. SSSR **37** (1942), 227-229.

[18] McCann R. J., *Exact solutions to the transportation problem on the line,* Proc. Royal. Soc. London **455** (1999), 1341-1380.

[19] Monge G., *Mémoire sur la théorie des déblais et rémblais,* Mem. Acad. Sci., Paris, 1781.

[20] Rachev S. T., Probability Metrics and the Stability of Stochastic Models, Wiley, New York, 1991.

[21] Rachev S. T.; Rüschendorf L. R., Mass Transportation Problems, Vol 1 and Vol 2, Springer, New York, 1998.

[22] Ruzankin P. S., *Construction of the optimal joint distribution of two random variables,* Theory Prob. Appl. **42** (2002), 316-334.

[23] Wu S. Y., *Extremal points and algorithm for a class of continuous transportation problems,* J. Inform. and Optim. Sci. **13** (1992), 97-106.