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A note on the rigidity of the stable norm

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Abstract

Given two metrics on a closed manifold, we ask, in analogy with the marked length spectrum problem, if two metrics with negative curvature and the same stable norm are isometric. We give a negative answer to this question; however, we obtain some affirmative results in the case of the n-torus.

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1 Introduction

Let (M, g) be a compact manifold equipped with a smooth Riemannian metric. We define a function called the stable norm on $H_1(M, \mathbb{R})$ by

$$|h|_s := \inf\{\sum |r_i| l(\delta_i)\}, \quad \forall h \in H_1(M, \mathbb{R}),$$

where δ_i are simplexes, $r_i \in \mathbb{R}$, $\sum r_i \delta_i$ is a cycle representing h, and $l(\delta_i)$ is the length element induced by the metric g. This function defines a norm on $H_1(M, \mathbb{R})$. The stable norm has been extensively used in geometry and analysis as the next results show. A conjecture closely related to Hopf's conjecture recently proved in [2] states that the stable norm of a Riemannian n-torus is Euclidean if and only if the metric is flat. This generalized a result of Hopf (see [4]) according to which a 2-torus without conjugate points is flat.

Now we briefly recall Mather's theory [5] of action minimizing measures in case the variational principle is given by a Riemannian metric

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g. Let $\mathcal{M}$ denote the set of Borel probability measures μ on TM that are invariant under the geodesic flow φ of g and satisfy

$$A(\mu) := \frac{1}{2}\int_{TM} g(v,v)d\mu < \infty.$$

If $x : [a,b] \to M$ is an absolutely continuous curve, then the integral

$$A(x) := \int_a^b g(\dot{x}(t), \dot{x}(t))dt,$$

is called the action of x.

The curve x is said to be minimizing if it minimizes the action over all absolutely continuous curves defined over the same interval, with the same endpoints. A curve $x : \mathbb{R} \to M$ is said to be minimizing if its restriction to any compact interval is minimizing.

For a C^∞ function $f : M \to \mathbb{R}$ the invariance of μ under the geodesic flow implies that

$$\int_{TM} df d\mu = 0.$$

Hence, every $\mu \in \mathcal{M}$ defines a homology class $\rho(\mu) \in H_1(M, \mathbb{R})$ by

$$\langle [w], \rho(\mu)\rangle = \int_{TM} \omega d\mu,$$

for every closed 1-form ω on M.

Further, Mather defines a pair of convex function $\beta : H_1(M, \mathbb{R}) \to \mathbb{R}$, $\alpha : H^1(M, \mathbb{R}) \to \mathbb{R}$ by

$$\beta(h) := \min\{A(\mu) \mid \mu \in \mathcal{M} \textit{ and } \rho(\mu) = h\},$$

$$\alpha(\omega) = -\min\{A(\mu) - \langle w, [\mu]\rangle \mid \mu \in \mathcal{M}\}.$$

One easily sees that α and β are conjugate convex functions. It is well-known that $\frac{1}{2}|h|_s^2 = \beta(h)$, for all $h \in H_1(M, \mathbb{R})$. Now, the stable norm on $H_1(M, \mathbb{R})$ induces a stable norm on $H^1(M, \mathbb{R})$, and in this case also the Mather's function α and the stable norm in cohomology are related by $\frac{1}{2}|[\omega]|_s^2 = \alpha([\omega])$. These facts give certain dynamical interest of the function stable norm.

2 Setting the problem and results

Let M be a manifold. We say that two Riemannian metrics g_1 and g_2 have the same marked length spectrum if in each homotopy class of closed curves in M the infimum of g_1-lengths of curves and the infimum of g_2-lengths of curves are the same.

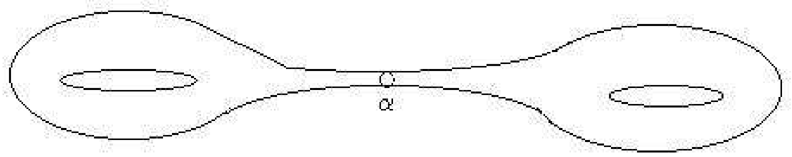
The marked length spectrum problem consists in showing that any two metrics with the same marked length spectrum must be isometric. Of course, this cannot hold for arbitrary metrics (for example, if we allow conjugate points). If the metrics do not have conjugate points, then this problem has a solution as the rigidity theorem in [3], [6] show. The strongest of these results (see [3]) states that

Theorem 1 *Let M be a closed surface and g_1, g_2 be two Riemannian metrics on M, with g_1 non-positive curvature and g_2 without conjugate points. If g_1 and g_2 have the same marked length spectrum, then they are isometric by an isometry homotopic to the identity.*

Inspired by the above results, we can ask if two metrics with the same stable norm are isometric. The next proposition shows that this question has a negative answer.

Proposition 1 *There exists two metrics g, g_1 on a surface of genus two with the same stable norm and such that g has constant curvature -1, g_1 has negative curvature, but g, g_1 are not isometric.*

Proof: We consider a surface of genus two, and take a metric g with constant curvature equal to -1 on this. In this case the notion of minimizing is just that of least lenght with the hyperbolic metric g.



If the length of the neck of this surface (see figure) is sufficiently long with respect to its diameter, then we can find a neighborhood around the closed curve α such that all minimizing curves with respect to g do not cross this neighborhood. This is because any closed curve that crosses the neck will be longer than the closed curves around the holes of the surface, therefore we can make on this neighborhood a smooth perturbation of the metric g and obtain another metric g_1 with negative curvature but not isometric to g. Now, it is not difficult to see that all minimizing curves with respect to g_1 do not cross this neighborhood, therefore g, g_1 have the same stable norm. □

If $\omega \in H^1(T^n, \mathbb{R})$, we denote by the same symbol the unique harmonic form in the cohomology class ω. The integral $\sqrt{\int_{T^n} \omega \wedge \star\omega}$ defines an L^2-norm on the space of differential one-forms on T^n, and a Euclidean norm on $H^1(T^n, \mathbb{R})$. The duality between homology and cohomology transfers the latter to $H_1(T^n, \mathbb{R})$. We will use the notation $\| \cdot \|_2$ for both of them.

As it was commented, we have some positive results in the case of the n-torus. The next result gives an affirmative solution for the question of ridigity in this situation.

Proposition 2 *Suppose that a Riemannian metric g on the n-torus T^n satisfies that for any cohomology class ω we have*

$$|\omega|_s = \| \omega \|_2,$$

and that it has the same stable norm as a flat metric g_0 on T^n, then (T^n, g) and (T^n, g_0) are isometric by an isometry homotopic to the identity.

Proof: The condition $|\omega|_s = \| \omega \|_2$ implies that the constant functions are solutions of the Hamiltonian equation $|d_x u + \omega_x|^2 = c$, with c constant. Thus we have that the unique harmonic representative in $[\omega]$ for any cohomology class has constant Riemanninan norm. Now given a covector $p \in T_x^* T^n$, there exists a unique harmonic 1-form ω such that $\omega_x = p$. As the Hamiltonian is constant on Graph(ω), the Hamilton-Jacobi theorem implies that Graph(ω) is a Lagrangian submanifold invariant under the Hamiltonian flow. Thus, by a theorem of Mañe, T^n is free of conjugate points. By the result in [2], T^n is flat. Therefore the result follows from the next claim.

Claim: For any two flat metrics g_1, g_2 on an n-torus T^n with the same stable norm, we have that (T^n, g_1) and (T^n, g_2) are isometric by an isometry homotopic to the identity.

Proof: For any differential one-form ω and for $x \in T^n$, we denote by $\| \omega(x) \|$ the norm of the corresponding linear form on T_xT^n induced by g_1. Then $\omega \wedge \star\omega(x) = \| \omega(x) \|^2 \ (dvol)(x)$ where $dvol$ is the volume form associated with g_1.

Let $\{v_1, \cdots, v_n\}$ be a lattice of $\mathbb{R}^n$ associated with (T^n, g_1). We denote by X_i the constant vector field defined by v_i, and by ω_i the dual 1-form of X_i. Now, as (T^n, g_1) is flat, we have $|v_i|_s = \| v_i \|_2$. Thus

$$|v_i|_s^2 = \| v_i \|_2^2 = \int_{T^n} \omega_i \wedge \star\omega_i = \int_{T^n} \| \omega_i(x) \|^2 \ (dvol)(x) = \| v_i \|_2,$$

for all $i = 1, \ldots, n$.

Since the same is valid for (T^n, g_2), and as the stable norms are equal, we conclude that g_1 and g_2 are isometric. □

Motivated by these results, it seems interesting to solve the next question:

Question: Suppose that a Riemannian metric g on the n-torus T^n, $n \geq 3$, has the same stable norm as a flat metric g_0 on T^n. Can we conclude that (T^n, g) and (T^n, g_0) are isometric by an isometry homotopic to the identity?

The case $n = 2$ was essentially proved by Bangert in [1].

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