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Base points in homotopy theory and the Fundamental Theorem of Algebra *

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Abstract

The importance of the base point in homotopy theory is emphasized, and illustrated with several examples. In particular, one common application of elementary homotopy theory is to prove the Fundamental Theorem of Algebra using the fundamental group, and the rôle of the base point in this proof is analyzed. Other topological methods for proving this theorem are discussed, as well as its analogue for quaternions and octonions. A brief survey of some proofs that are primarily non-topological in nature is also made.

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1 Introduction

There are many topology texbooks, including a fair number of recent ones, that introduce basic homotopy theory and the fundamental group. Almost all of these texts emphasize the importance of the base point in the definitions. This was not always the case. Indeed the classic 1934 text [22] (§42) of Seifert and Threlfall warns the reader that given a continuous map $f : X \to Y$ of path-connected spaces, the induced map f_* of the fundamental groups is not well defined, it is only determined up to an inner automorphism of the fundamental group of Y. This difficulty is overcome by assigning each topological space a distinguished point,

*Invited article.

i.e. the aforementioned base point. One then considers the fundamental group (as well as the higher homotopy groups) as defined on the *category* of pointed spaces and pointed maps. In this case f_* is well defined and the homotopy groups become functors.

In this note we explore the importance of the base point in homotopy theory by means of several illustrative examples, showing that neglect of the base point can in fact lead to absurd results and incomplete proofs. As a final example we consider the Fundamental Theorem of Algebra (henceforth FTA, see §3 for statement). Of course, the FTA has been proved by many authors using diverse techniques, indeed an entire book [9] and an entire chapter of [7] are devoted to this subject. Perhaps the first to offer a proof was d'Alembert [4]. The first rigorous proof was arguably Lagrange's (cf.[25], and also [7], Ch. 4, for the early history of the theorem). Gauss proved it in his doctoral dissertation and eventually gave three other proofs during the course of his life, partly because he was not completely satisfied with the rigour of his first proof.

There seem to be two main reasons for the difficulties in establishing a rigorous proof in the early period. The first was logical, in that some of the early proofs started by assuming that a root existed "somewhere or other" ([7], p.109), and then tried to demonstrate that this root was actually a complex number. The algebraic concept of a splitting field makes sense of this idea, but this concept was only formulated in the latter part of the nineteenth century. The second reason was topological, since ideas such as a continuous function necessarily assuming its maximum and minimum values on a closed bounded set were again not properly formulated and understood until the latter part of the nineteenth century.

Good summaries of the history and of the numerous proofs found today can also be found at the websites [10], [11], or in [9]. The methods of proof include complex analysis, real analysis with some complex geometry, topology, Galois theory, linear algebra, etc. Even within the discipline of topology there are various methods of proof. We shall mainly consider the rôle of the base point in versions using the fundamental group.

The remainder of this Introduction will establish some basic definitions, terminology, and state some basic facts of homotopy theory as well as introduce a couple of lemmas that will be quite useful in the sequel. All of this material (except possibly the two lemmas) is standard in any topology text that introduces homotopy theory. The lemmas are given as exercises in e.g. [14], p.38 and [17], p.235. In Section 2, using a few examples, we illustrate the importance of the base point in the definition of homotopy groups, and examine the relation between free and based homotopy. This is continued in Section 3, where we give a topological proof of FTA using the fundamental group, and point out (in the light of Section 2) where a gap in this proof can occur if the base points are forgotten. We also show that it is easily repaired, and give a couple of methods to do this. In Section 4 other topological proofs are mentioned, including the homology proof of [8] that has some major advantages, e.g. it can be readily generalized to a higher dimensional version of FTA for quaternions and octonions. Other methods of proof for FTA are also discussed and compared in Section 4.

1.1 Basic definitions

By a map we shall always mean a continuous function from one topological space X to another, Y. In case $X = I$, the closed unit interval $[0,1] \subset \mathbb{R}$, a map $f : I \to Y$ is called a path in Y with initial point $f(0)$ and terminal point $f(1)$. If also $f(0) = f(1)$, then the path f is called a loop based at $f(0)$. All our spaces will be based, i.e. each space X has a base point which we write x_0 (in particular X is non-empty). A map $f : X \to Y$ is called *based* if $f(x_0) = y_0$. For two paths f, g in X satisfying $f(1) = g(0)$, their product path $f \cdot g : I \to X$ is defined in the usual way (f followed by g), also simply written fg, and the reverse path to a path f is denoted $\overline{f}$. Strictly speaking the product of paths is non-associative, however it is associative up to homotopy (defined in the following paragraph) and it is customary to make a small abuse of notation and write $f \cdot g \cdot h$, or simply fgh, when this product of paths is being considered only up to homotopy.

A *free homotopy* between maps $f, g : X \to Y$ is a map $F : I \times X \to Y$ such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$ for any $x \in X$. A *based homotopy* between based maps $f, g : (X, x_0) \to (Y, y_0)$ is a free homotopy $F : I \times X \to Y$ such that $F(I \times \{x_0\}) = \{y_0\}$. The homotopy class $[f]_*$ (resp. $[f]$) of a map $f : X \to Y$ is defined as the set of all maps $g : X \to Y$ that are based (resp. free) homotopic to f. The set of based (resp. free) homotopy classes of based maps from X to Y is denoted $[X,Y]_*$ (resp. $[X,Y]$). Since based homotopic maps are trivially free homotopic there is a canonical map

$$\phi : [X,Y]_* \to [X,Y] \quad \text{where} \quad \phi[f]_* = [f].$$

In general ϕ is neither injective nor surjective, and this will be a key point in the discussion to follow.

If need arises, we shall be more specific about our choices of base points. For $X = S^n$, the unit sphere in $\mathbb{R}^{n+1}$, the base point is taken to be $s_0 = (1, 0, ..., 0)$. For any (as always pointed) space Y, the set $[S^n, Y]_*$ has a natural group structure for $n \geq 1$ which is abelian for $n \geq 2$, and is denoted $\pi_n(Y, y_0)$. The fundamental group of Y based at y_0 is $\pi_1(Y, y_0)$, and in general is not abelian.

We close this Introduction with the two aforementioned lemmas.

Lemma 1.1.1 *In a path-connected space X, let w be any path from a point x_0 to a point x_1. Then a loop γ, based at x_0, is freely homotopic to the loop $\delta = \overline{w} \cdot \gamma \cdot w$, based at x_1.*

Lemma 1.1.2 *Two loops γ, δ with base point x_0 are freely homotopic in a path-connected space X (i.e. $\phi[\gamma]_* = \phi[\delta]_*$) if and only if $[\gamma]_*$, $[\delta]_*$ are conjugate in $\pi_1(X, x_0)$.*

Other terms and results from basic algebraic topology that are fairly standard and found in all texts on the subject will be used in the sequel, without comment or definition.

2 The importance of the base point in homotopy theory

Example 2.1.3 *The Antipodal Map.*

Our first example, due to D. Handel, involves the higher homotopy groups. It shows how neglect of the base point can quickly lead to a "paradox." Let $a : S^n \to S^n$ denote the antipodal map $a(x) = -x$ for all $x \in S^n$, and real projective space $\mathbb{R}P^n$ be the usual identification space of S^n obtained by identifying each x with $a(x)$. Let $\kappa : S^n \twoheadrightarrow \mathbb{R}P^n$ denote the canonical projection map: $\kappa(x) = [x] = \{x, -x\}$. Since $\kappa(ax) = \kappa(x)$, one has a commutative diagram

$$\begin{tikzcd} S^n \arrow[rr, "a"] \arrow[dr, "\kappa"'] & & S^n \arrow[dl, "\kappa"] \\ & \mathbb{R}P^n & \end{tikzcd} .$$

It is well known that $\deg(a) = (-1)^{n+1}$. Assume now that $n \geq 2$ is even, so $\deg(a) = -1$. Applying the functor π_n yields the diagram

$$\begin{tikzcd}
\mathbb{Z} \approx \pi_n(S^n) \arrow[rr, "a_*=-1"] \arrow[dr, "\kappa_*", "\approx"'] & & \pi_n(S^n) \approx \mathbb{Z} \arrow[dl, "\kappa_*"', "\approx"] \\
& \pi_n(\mathbb{R}P^n) \approx \mathbb{Z} &
\end{tikzcd}$$

which, since $-1 \neq 1 \in \mathbb{Z}$, is *non-commutative*, apparently a blatant contradiction. The explanation is that π_n is not a functor on the category Top, rather it is a functor on the category Top_* of *pointed* spaces, maps, and homotopies, and the antipodal map a is not a based map.

We remark that, as with many "paradoxes," this example can also be put to constructive use. Indeed, any great semi-circle in S^n from s_0 to $-s_0$ induces a loop λ when projected into $\mathbb{R}P^n$. We take $p_0 = \kappa(s_0)$ to be the basepoint of $\mathbb{R}P^n$. What the above shows, with suitable interpretation, is that for $n \geq 2$ even, $[\lambda]_* \in \pi_1(\mathbb{R}P^n, p_0) \approx \mathbb{Z}_2$ acts on $\pi_n(\mathbb{R}P^n, p_0) \approx \mathbb{Z}$ by the non-trivial automorphism $m \mapsto -m$, $m \in \mathbb{Z}$. In other words, the fundamental group $\pi_1(\mathbb{R}P^n, p_0)$ acts non-trivially on $\pi_n(\mathbb{R}P^n, p_0)$ for $n \geq 2$ even. This also gives a nice example of a space $\mathbb{R}P^n$ with nilpotent fundamental group $\mathbb{Z}_2$, $n \geq 2$, but nilpotent as a space if and only if n is odd (cf. [26] p.46).

Example 2.1.4 *The Harmonic Comb.*

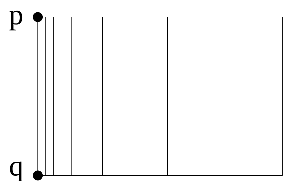

Figure 1: The Harmonic Comb

In this example we shall see that $\phi : [X, Y]_* \to [X, Y]$ need not be injective. The harmonic comb, a space studied in many topology texts, is defined as the set

$$H = \left(\left\{ 0, \ \frac{1}{n} \ \middle| \ n \in \mathbb{N} \right\} \times I \right) \cup (I \times \{0\}) \subset \mathbb{R}^2,$$

endowed with the topology inherited from that of the Euclidean plane. Consider the two points $p = (0,1)$, $q = (0,0)$ as possible base points of H. It is easy to see that H with base point q is contractible (with q remaining fixed), in which case the free homotopy classes $[H,H]$ must be a singleton. However, a slightly more delicate argument (cf. [5], p.48 for an elegant version) shows that with p as base point H is not contractible (again with p fixed during the homotopy). It follows that with p as base point $[1_H]_* \neq [p]_*$, i.e. the identity map of H is not based homotopic to the constant map p. Thus $[H,H]_*$ contains at least two elements, and ϕ cannot be injective.

This example also illustrates that "not all base points are created equal." More precisely, one should restrict attention to nondegenerate base points, a property that q has but p fails to have. Spanier [23] defines a base point x_0 to be nondegenerate if the inclusion map $\{x_0\} \hookrightarrow X$ is a cofibration. For compactly generated spaces this is equivalent to Whitehead's condition [27] that (X, x_0) be an NDR-pair of spaces. Furthermore, following [27], p.98, the fundamental group $\pi_1(X, x_0)$ is defined only when x_0 is a nondegenerate base point. Fortunately, the topological spaces most frequently used in algebraic topology, e.g. CW-complexes or simplicial complexes, have the property that every choice of base point is nondegenerate, and these spaces are also compactly generated.

Example 2.1.5 *The Figure Eight*

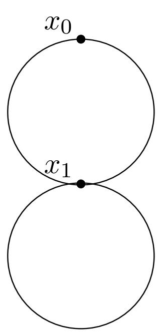

FIGURE 2. The Figure Eight

In this example we shall see that $\phi : [W,X]_* \to [W,X]$ need not be injective when $W = S^1$. Let X be the figure eight, a wedge of two circles. For convenience, we take this to be the union of two circles of radius one centred at $(0,1)$ and $(0,-1)$, respectively, meeting at the origin $x_1 := (0,0)$. Suppose that $x_0 := (0,2)$ is the basepoint. Let γ_l be the path which traverses the left hand portion of the upper circle from x_0 to x_1 once, and let γ_r be the path traversing the right hand semicircle between x_0 and x_1 once. Also, let τ be the loop based at x_1 which traverses the bottom circle once counterclockwise. Then $\gamma_l \cdot \tau \cdot \overline{\gamma}_l$ and $\gamma_r \cdot \tau \cdot \overline{\gamma}_r$ are loops based at x_0. We now show that these two loops can not be based homotopic. The fundamental group $\pi_1(X, x_0)$ is the free (non-abelian) group on two generators. It is not difficult to see that these two generators can be chosen to be $a := [\gamma_l \tau \overline{\gamma}_l]_*$ and $b := [\gamma_l \overline{\gamma}_r]_*$; this is accomplished by using the usual generators of $\pi_1(X, x_1)$ and the isomorphism which moves the basepoint of X from x_1 to x_0 via the path γ_l. Also define $c := [\gamma_r \tau \overline{\gamma}_r]_*$.

If loops $\gamma_l \tau \overline{\gamma}_l$ and $\gamma_r \tau \overline{\gamma}_r$ are based homotopic, then

$$\begin{aligned} a^{-1}ba &= [\gamma_l \overline{\tau \overline{\gamma}_l} \gamma_l \overline{\gamma}_r \gamma_l \tau \overline{\gamma}_l]_* = [\gamma_l \overline{\tau \overline{\gamma}_r} \gamma_l \tau \overline{\gamma}_l]_* \\ &= [\gamma_l \overline{\tau \overline{\gamma}_r} \gamma_r \tau \overline{\gamma}_r]_* = [\gamma_l \overline{\gamma}_r]_* = b. \end{aligned}$$

We thus conclude that $a^{-1}ba = b$, which implies that $\pi_1(X, x_0)$ is an abelian group, a glaring contradiction. However, by Lemma 1.1.2, the two loops are freely homotopic since their based homotopy classes a, c are conjugate in $\pi_1(X, x_0)$, as shown by the calculation

$$a = [\gamma_l \tau \overline{\gamma}_l]_* = [\gamma_l \overline{\gamma}_r \gamma_r \tau \overline{\gamma}_r \gamma_r \gamma_l]_* = [\gamma_l \overline{\gamma}_r]_* [\gamma_r \tau \overline{\gamma}_r]_* [\gamma_r \gamma_l]_* = bcb^{-1}.$$

This example can easily be extended to other popular topological spaces, such as the countable wedge of circles or the Hawaiian earring (see, e.g., [2] for examples and references).

Our last example, illustrating non-surjectivity of ϕ, is rather trivial.

Example 2.1.6 *Target Space not Path Connected*

Let $Y = \{a, b\}$ be the space with two points and the discrete topology, with base point $y_0 = a$. Then for any path connected space X, $[X, Y]_*$ has just a single element (the constant map taking X to $\{a\}$) whereas $[X, Y]$ has two elements, so ϕ is not surjective.

These examples demonstrate the importance of the base point in homotopy theory, in particular, knowing whether or not $\phi : \pi_1(X, x_0) = [S^1, X]_* \to [S^1, X]$ is a bijection. This is easily answered, using the two lemmas in Section 1. Indeed Lemma 1.1.1 shows that ϕ is surjective whenever X is path connected (the converse is also obviously true).

And Lemma 1.1.2 shows that ϕ is injective if and only if $\pi_1(X, x_0)$ is abelian. We formulate this as follows.

Proposition 2.1.7 *Let X be path connected. Then $\phi : \pi_1(X, x_0) = [S^1, X]_* \to [S^1, X]$ is bijective if and only if $\pi_1(X, x_0)$ is abelian.*

This proposition is covered in the exercises of the two books [14], [17], as already mentioned, and also fully covered in Stöcker-Zieschang [24].

3 Usual topological proof of FTA

The FTA states that every polynomial $p(z)$ of degree $n \geq 1$, with real or complex coefficients, has exactly n zeros (roots), when the zeros are counted with multiplicities, in the complex numbers $\mathbb{C}$. The proof of the general statement follows, by an obvious induction and the remainder theorem, from the statement that every polynomial of degree $n \geq 1$ has at least one zero in $\mathbb{C}$. This latter statement is also called the FTA, and it is the proof of this with which we are concerned. Since we are using the complex numbers, it will be convenient to write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and also to take $S^1 \subset \mathbb{C}^*$ as the unit complex numbers, with basepoint $s_0 = 1$. Of course S^1 sits in $\mathbb{C}^*$ as a strong deformation retract, the map $z \mapsto z/|z|$ providing the retraction of $\mathbb{C}^*$ onto S^1. As usual, we write $S^1 = \{e^{2\pi i s} \mid 0 \leq s \leq 1\}$.

Let us now outline a common topological proof. We shall use the fundamental group $\pi_1(S^1, s_0) \approx \mathbb{Z}$ of the circle, an infinite cyclic group with generator given by the homotopy class of the identity map, i.e. the loop which goes once around the circle counterclockwise. We shall also be "careless" and not worry, at least for the time being, about base points. The proof proceeds by supposing that the polynomial $p(z)$ has no zeros in $\mathbb{C}$, then by using $p(z)$ to produce two elements of $\pi_1(S^1, s_0)$, which should be equal but are not, thus giving a contradiction.

Indeed, suppose without loss of generality that $p(z)$ is monic, so

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \ , \quad a_i \in \mathbb{C}.$$

Choose any $R > 0$, and fix any r with $0 \leq r \leq R$. Note that restricting p to the circle $|z| = r$ produces a loop λ_r, where $\lambda_r(s) := p(re^{2\pi i s})$, $0 \leq s \leq 1$, in $\mathbb{C}$. Furthermore, the assumption that p has no zeros implies that this loop λ_r lies in $\mathbb{C}^*$. Letting r vary from 0 to R produces a homotopy in $\mathbb{C}^*$ which shows λ_R is contractible in $\mathbb{C}^*$ to the constant loop λ_0. Since p has no zeros we are free to define $g_r(s) := p(re^{2\pi is})/|p(re^{2\pi is})|$, $0 \leq r \leq R$. This is now a loop in S^1, so its homotopy class represents an element of $\pi_1(S^1)$. The new loop g_R is again contractible; just as for λ_R one produces a homotopy to the constant map g_0 by letting r vary from 0 to R. Thus $[g_R] = 0 \in \pi_1(S^1)$.

If we now choose R sufficiently large, so that $R > \max\{|a_1| + \cdots + |a_n|, 1\}$, then the polynomials

$$p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n), \quad 0 \leq t \leq 1,$$

are guaranteed not to have any zeros on the circle $|z| = R$, by elementary estimations of $|p_t(z)/z^n|$ (which clearly approaches 1 as $|z| \to \infty$). As t varies from 0 to 1, the polynomials p_t thus provide a homotopy f_t in $\mathbb{C}^*$ between $g_R(s)$ and the well known n'th power function $w_n(z) = z^n$. When $n > 0$, the latter is known to represent the non-trivial element $n \in \pi_1(\mathbb{C}^*) \approx \pi_1(S^1) \approx \mathbb{Z}$, contradicting the final assertion of the previous paragraph.

The problem is that the above "argument" uses the fundamental group but takes place in the free homotopy classes $[S^1, S^1]$ (or its equivalent $[S^1, \mathbb{C}^*]$), and as we have seen in Section 2, these are not in general the same. Indeed, while $w_n(1) = 1^n = 1$ implies w_n is a loop based at $s_0 = 1$, the loop g_r is based at

$$g_r(1) = \frac{1 + a_1 + ... + a_n}{|1 + a_1 + ... + a_n|},$$

which in general is not equal to 1. A similar objection applies to the homotopy f_t. Thus the proof as presented above has a gap in it, and this does occur in a few texts on the subject.

However, the gap is easily fixed. We present three distinct methods to do so. The first "repair method" is to simply redefine all the above maps so as to be based. Indeed, given any map whatsoever $\alpha : X \to \mathbb{C}^*$, for any space X with base point x_0, changing α to the new map $\widetilde{\alpha}$ given by $\widetilde{\alpha}(x) = \alpha(x)/\alpha(x_0)$ does the trick. For example, in the above proof, we use

$$\widetilde{g}_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|},$$

and similarly for $\widetilde{f}_t$.

The second "repair method" is equally simple, just apply Proposition 2.1.7 to either of the spaces S^1 or $\mathbb{C}^*$. Since the fundamental group of either space, $\mathbb{Z}$, is abelian, the proposition will apply. One then sees that ϕ is a bijection in this case. Thus, the proof can legitimately be carried out using free homotopies, as long as the bijectivity of ϕ in this situation is *first* established.

The third "repair method" is to use the idea of a nullhomotopic map (which does not involve any base points), first proving a lemma that asserts that for any map $f : S^1 \to X$, where the base point of X is taken to be $x_0 = f(1)$, the induced homomorphism $f_* : \pi_1(S^1, 1) \to \pi_1(X, x_0)$ is trivial if and only if f is nullhomotopic. The remainder of the proof proceeds along lines similar to those outlined above.

Without making any attempt to list all the texts giving topological proofs of FTA similar to this one, we mention McCarty [18] as an early text and Deo [5], Hatcher [14] as recent texts giving a complete proof, using the first repair method (see also [9], Proof 11). The second repair method is implicit in the exercises given in [14], [17], and is given a full, careful treatment in Stöcker-Zieschang [24]. The third repair method is found in the popular text of Munkres [20], among others.

4 Other topological proofs of FTA and its extension to quaternions and octonions

Variations of the topological proofs of FTA given in Section 3 can be found in various texts. Without attempting to list all of them, we mention Bredon [1], where the homotopies are treated in an elegant way, and Lawson [17] where the idea of the degree of an arbitrary map $S^1 \to S^1$ is defined (independent of any choice of base point) and used to complete the proof. Similarly, the notion of winding number is used in the elementary text of Chinn-Steenrod [3].

These methods are actually very close to the proof using homology, as given by Eilenberg-Steenrod [8], which will be discussed in the next paragraph. It is also possible to give proofs of FTA using differential topology (usually involving the notion of degree), again with no attempt at completeness we mention three essentially different proofs to be found in Guillemin-Pollack [13], Hirsch [15], and Milnor [19].

The proof given in Section 3, using the fundamental group $\pi_1(X, x_0)$, can easily be mimicked using the first homology group $H_1(X)$. There is, however, the advantage that homology does not depend on base points, thereby removing any need to keep track of base points and simplifying the proof. This is the approach taken in [8], Ch. 11, also see [9], Proof

6, and has the further advantage that it easily generalizes to a version of FTA for the quaternions and octonions which we shall explain below. The only disadvantage of this approach may be pedagogical, since it first requires the development of homology theory, which is substantially more involved than elementary homotopy theory.

To state the Eilenberg-Steenrod generalization, one must first carefully define what a polynomial over the quaternions $\mathbb{H}$ or octonions $\mathbb{O}$ is, since $\mathbb{H}$ is non-commutative and $\mathbb{O}$ is both non-commutative and non-associative. This involves first defining a general monomial, then a polynomial is simply a sum of monomials. Rather than giving the precise inductive definition as in [8], we will simply illustrate this by giving the example $q_1 z q_2 z q_3$, $q_i \in \mathbb{H}$, as the most general quaternionic monomial of degree 2 in the variable z. For octonions one also must allow all possible insertions of parentheses that give a well formed product, creating even more monomials. Clearly the degree of a monomial is well defined, any polynomial is a sum of distinct monomials, and the degree of a polynomial is then simply the maximum degree of any of these monomials. We will call a polynomial having the usual form $p(z) = a_n z^n + ... + a_1 z + a_0$ a *standard polynomial* (since any two-dimensional subalgebra of $\mathbb{O}$ is associative, this is well defined even for $\mathbb{O}$).

Theorem 4.1.8 *(Eilenberg-Steenrod). Let $p(z)$ be any complex, quaternionic, or octonionic polynomial of degree $n > 0$ and assume further that it has only a single monomial term of degree n. Then it has a zero in respectively $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Corollary 4.1.9 *Any standard polynomial $p(z)$ over $\mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ of degree $n > 0$ has a zero in respectively $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

We remark that Eilenberg and Steenrod actually state the theorem in even greater generality, namely for normed division algebras over the reals of dimension at least two. But at the time their book was written it was not yet known that the only normed division algebras over the reals are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. The necessity of the condition of just one monomial term of highest degree is not addressed by them, however the degree 1 quaternionic polynomial $p(z) = iz - zi + 1$ can easily be seen to have no zeros in $\mathbb{H}$, thereby showing the necessity of this hypothesis. Thus FTA, in its most general form, is false for $\mathbb{H}$, $\mathbb{O}$.

Returning to the usual FTA for $\mathbb{C}$, note that all the topological proofs thus far discussed use concepts from algebraic topology. The proof given in the following paragraph uses only a little general topology and the basic fact from complex analysis that a non-constant holomorphic function is an open map, cf. [16], Chapter 2. This proof, which is roughly outlined in Fulton's text [12], p.49, will be given in full here, due to its brevity and intrinsic interest.

Letting $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$, $n > 0$, we have already noted in §3 that

$$\lim_{|z|\to\infty} \frac{p(z)}{z^n} = 1,$$

which implies in particular that $p(z) \neq 0$, $|z| \geq R$, for some suitably large $R > 0$. It follows that on the Riemann sphere $S^2 \cong \mathbb{C} \sqcup \{\infty\}$, the one-point compactification of $\mathbb{C}$, p extends to a continuous function that we shall call P, with $P(\infty) = \infty$. Since S^2 is compact, $P(S^2)$ is closed in S^2. We have already mentioned that P is an open mapping on $\mathbb{C}$. Assuming for the moment that it is also open at ∞, then its image is a non-empty open and closed subset of S^2, hence all of S^2 by connectedness. To see that P is also open at ∞, consider the involution $\tau(z) = z^{-1}$ of the Riemann sphere. We shall use the obvious fact that τ is a homeomorphism, in fact it is even a biholomorphic equivalence. Since $P(z) \neq 0$ on some open neighbourhood $|z| > R$ of ∞, it follows that

$$\tau P \tau(z) = \frac{1}{P(z^{-1})} = \frac{z^n}{1 + a_{n-1}z + ... + a_0 z^n}$$

is a well defined non-constant holomorphic function on the disc $|z| < 1/R$ with $P\tau(0) = 0$. Thus $\tau P \tau$ takes some open neighbourhood U of 0 onto an open neighbourhood V of 0, whence P takes the open neighbourhood $\tau(U)$ of ∞ onto the open neighbourhood $\tau(V)$ of ∞, concluding the proof that P is surjective. In particular, $0 \in \mathrm{Im}(P)$.

We finish with a brief discussion of the many "non-topological" proofs of FTA, where by non-topological we mean proofs where the *primary* method used is outside topology, e.g. complex geometry, field theory, linear algebra, or complex analysis. All proofs must use some topological ideas in one form or another. In particular the non-topological proofs all use compactness (e.g. the max-min theorem for continuous real valued functions on a compact domain), or the intermediate value theorem (usually in the form that a real polynomial of odd degree must have a real zero), at some stage. Of course many such proofs of FTA have been given, indeed an article [21] reviewing nearly 100 proofs of the theorem was written in 1907 by Netto and Le Vavasseur, when the concepts of algebraic and differential topology (which as we have seen led to many more proofs) were still in their infancy. Finally, it is worth mentioning that the proofs we have been talking about have all been existential. In the light of modern day computer science, as well as logic, it is important to note that constructive proofs of FTA have also been given, starting with attempts by Weierstrass in 1859 and 1891, a solution by H. Kneser in 1940, as well as Hirsch and Smale in 1979 and M. Kneser in 1981. These proofs, of course, give some algorithm for finding successive z_n that converge to a zero. For further discussion of this small but important class of proofs cf. [7], p. 114–115.

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References

[1] Bredon G. E., Topology and Geometry (Grad. Texts in Math.**139**), Springer-Verlag, New York, 1993.

[2] Cannon J. W.; Conner J. R., *The Combinatorial Structure of the Hawaiian Earring Group*, Topology and its Applications **106** (2000), 225–271.

[3] Chinn W. G.; Steenrod N. E., First Concepts of Topology, Random House, New York, 1966.

[4] D'Alembert J., *Recherches sur le Calcul Intégral*, Histoire de l'Académie Royale des Sciences et Belles Lettres, Berlin (1748), 182–224.

[5] Deo S., Algebraic Topology – A Primer, Hindustan Book Agency, New Delhi, 2nd printing 2006.

[6] Derksen H., *The Fundamental Theorem of Algebra and linear algebra*, Amer. Math. Monthly **110** (2003), 620–623.

[7] Ebbinghaus H. D.; Hermes H.; Hirzebruch F.; Koecher M.; Mainzer K.; Neukirch J.; Prestel A.; Remmert R., Numbers (Grad. Texts in Math. **123**), Springer-Verlag, New York, 1991.

[8] Eilenberg S.; Steenrod N., Foundations of Algebraic Topology, Princeton Univ. Press, Princeton, 1952.

[9] Fine B.; Rosenberger G., The Fundamental Theorem of Algebra, Springer-Verlag, N.Y., Berlin, Heidelberg, 1997.

[10] http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Fund_theorem_of_algebra.html

[11] http://math.fullerton.edu/mathews/c2003/FunTheoremAlgebraBib/Links/FunTheoremAlgebraBib_lnk_2.html

[12] Fulton W., Algebraic Topology, a First Course (Grad. Texts in Math **153**), Springer-Verlag, New York, 1991.

[13] Guillemin V.; Pollack A., Differential Topology, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.

[14] Hatcher A., Algebraic Topology, Cambridge University Press, Cambridge, 2002.

[15] Hirsch M. W., Differential Topology, Springer-Verlag, New York, 1976.

[16] Lang S., Complex Analysis, Fourth Edition, Springer-Verlag, New York, 2003.

[17] Lawson T., Topology : a Geometric Approach (Oxford Graduate Texts in Mathematics **9**), Oxford Science Publications, Oxford University Press, NYC, 2003.

[18] McCarty G., Topology, McGraw Hill, New York, 1967.

[19] Milnor J. W., Topology from the Differentiable Viewpoint, The Univ. Press of Virginia, Charlottesville, 1965.

[20] Munkres J. R., Topology, 2nd Ed., Prentice Hall, New Jersey, 2000.

[21] Netto E.; Le Vavasseur R., Enc. Sciences Math. Pures et Appl. I.2 (1907), 189–205.

[22] Seifert H.; Threlfall W., Lehrbuch der Topologie, Chelsea Pub. Co., New York, 1934.

[23] Spanier E., Algebraic Topology, McGraw-Hill Inc., 1966.

[24] Stöcker R.; Zieschang H., Algebraische Topologie, eine Einführing, Tuebner, Stuttgart, 1988.

[25] Suzuki J., *Lagrange's proof of the Fundamental Theorem of Algebra*, Amer. Math. Monthly **113** (2006), 705–714.

[26] Varadarajan K., The Finiteness Obstruction of C.T.C. Wall (Can. Math. Soc. Series of Monographs and Advanced Texts), J. Wiley and Sons, N.Y., 1989.

[27] Whitehead G., Elements of Homotopy Theory (GTM 61), Springer-Verlag, New York, 1978.