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Geometric dimension of stable vector bundles over spheres

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Abstract

We present a new method to determine the geometric dimension of stable vector bundles over spheres, using a constructive approach. The basic tools include K-theory and representation theory of Lie groups, and the use of spectral sequences is totally avoided.

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1 Introduction

Let X be a connected finite cell complex of dimension m and η a vector bundle over X. In particular, the k-dimensional trivial vector bundle is denoted by $k\epsilon$. Given η, one often seeks trivial sub-bundles of $\eta \oplus m\epsilon$ of least possible co-dimension q. This q is commonly called the geometric dimension of η, denoted $q = gd(\eta)$. The definition, as formulated, is in such a manner that

$$gd(\eta) = gd(\eta \oplus \epsilon) = gd(\eta \oplus 2\epsilon) = gd(\eta \oplus 3\epsilon) = \dots$$

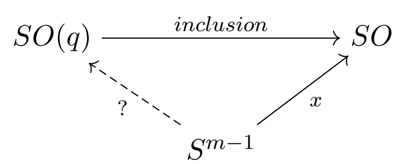
holds, so that one can also speak of the geometric dimension $gd(x)$ for an arbitrary element x in the Grothendieck cohomology group $KO(X)$, see [1].

Bott's periodicity theorem on the homotopy groups of the infinite orthogonal group SO tells us that for $X = S^m$ = the m-dimensional sphere where $m > 1$, one has, in reduced cohomology,

$$\widetilde{KO}(S^m) = [S^m, BSO] =$$

$$\pi_{m-1}(SO) = \begin{cases} 0 & m \equiv 3, 5, 6 \text{ or } 7 \pmod 8, \\ \mathbb{Z} & m \equiv 0, 4 \pmod 8, \\ \mathbb{Z}/2\mathbb{Z} & m \equiv 1, 2 \pmod 8. \end{cases}$$

An obvious question then is to determine $gd(x)$ for x a generator of any of the above nonzero groups. Here in fact one is trying to further pinpoint Bott's result by asking: if x is interpreted as an essential map in the diagram



what is the smallest q such that x can be homotopically compressed into the finite orthogonal group $SO(q)$? The complete answer, obtained through a period of 25 years, is due to Mark Mahowald and his co-workers [3, 5]. In these papers the principal tool is to use homotopy spectral sequences. The drawback of such an approach is that geometric features behind claimed results often become obscure, especially when calculation steps or analysis of differentials in spectral sequences are sometimes suppressed in favor of brevity. In this paper we aim at providing a non-spectral sequence approach to determine $gd(x)$ for all $x \in \widetilde{KO}(S^m)$, re-obtaining Theorem 1.1 of [5]:

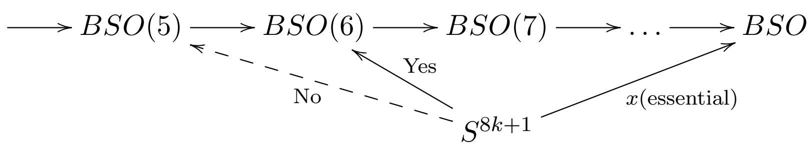
(I) For the generator x of $\widetilde{KO}(S^{8k+1})$ or $\widetilde{KO}(S^{8k+2})$ where $k \geqslant 1$, the geometric dimension $gd(x) = 6$.

(II) For any nonzero element y in $\widetilde{KO}(S^{4k})$, $k \geqslant 5$, the geometric dimention $gd(y) = 2k + 1$.

2 Spheres of dimension $8k + 1$ and $8k + 2$

Theorem 2.1. *For $k \geqslant 1$, the generator x in $\widetilde{KO}(S^{8k+1}) = \mathbb{Z}/2\mathbb{Z}$ has $gd(x) = 6$.*

This is the second part of Theorem 1.1 of Davis-Mahowald [5], summarisable by a "compression diagram" below:



Alternatively, one can say that Bott's generator in $\pi_{8k}(SO) = \mathbb{Z}/2\mathbb{Z}$ "originates" from the finite orthogonal group $SO(6)$, but not from $SO(5)$. Our proof shall be accomplished in a number of geometric steps.

Step A. For $k = 1$, we construct a 6-dimensional vector bundle η over S^9 that is stably non-trivial.

Let $X = S^9 \cup_8 e^{10}$ be the Moore space obtained from S^9 via attachment of a 10-dimensional cell using a self-map of S^9 of degree 8. Let $c : X \longrightarrow S^{10}$ be the collapse map which pinches S^9 to a point. In complex K-theory, the generator of $\widetilde{K}(S^{10}) = \mathbb{Z}$ can be represented by a $\mathbb{C}$-vector bundle ω on S^{10} of $\mathbb{C}$-dimension 5. It is well-known [4] that in $H^{10}(S^{10}; \mathbb{Z}) = \mathbb{Z}$ the 5th Chern class

$$c_5(w) = (5-1)! = 24$$

so that ω as a real vector bundle has Euler class 24. It follows that the pull-back vector bundle $c^!(\omega)$ over X has zero Euler class, and is thus sectionable.

As a result $c^!(\omega) = \zeta \oplus \epsilon_{\mathbb{C}}$ for some complex vector bundle ζ over X with $\dim_{\mathbb{C}} \zeta = 4$ while $\epsilon_{\mathbb{C}}$ is the trivial complex line bundle. Furthermore, since the first Chern class $c_1(\zeta) = 0$, ζ can be regarded to have structural group $SU(4)$ rather than $U(4)$.

Step B. In Lie group representation theory [6, Chapter 13], one has

$$\triangle_6^+ : \mathrm{Spin}(6) \xrightarrow{\approx} SU(4) \hookrightarrow U(4),$$

namely, the positive spinor representation $\triangle_6^+$ of Spin(6) into $U(4)$ sends Spin(6) isomorphically onto the subgroup $SU(4)$. It thus follows that there is a 6-dimensional spinor bundle η over X such that $\triangle_6^+(\eta) = \zeta$. i.e., such that ζ is the associated bundle of η via $\triangle_6^+$. Also, from construction one knows that $y = \zeta - 4\epsilon_{\mathbb{C}}$ is a generator of $\widetilde{K}(X) = \widetilde{K}(S^9 \cup_8 e^{10}) = \mathbb{Z}/8\mathbb{Z}$.

Step C. We now make the crucial claim that η restricted to S^9 is stably nontrivial, in other words $\eta|_{S^9}$ represents the generator of $\widetilde{KO}(S^9) = \mathbb{Z}/2\mathbb{Z}$. Our argument is inspired by the methods in [1]. Recall from representation theory that the double covering of $SO(6)$ by Spin(6) fits into a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spin}(6) & \xrightarrow{\triangle_6^+} & SU(4) & \hookrightarrow & U(4) \\ \downarrow & & & & \downarrow \lambda_2^{\mathbb{C}} \\ SO(6) & & \xhookrightarrow{\quad inclusion \quad} & & U(6) \end{array}$$

in which $\lambda_2^{\mathbb{C}}$ denotes second exterior power operation for complex vector spaces. Going around clockwise one sees that the complexification of η can be computed via

$$\begin{aligned} \eta \otimes \mathbb{C} = \lambda_2^{\mathbb{C}}(\triangle_6^+(\eta)) &= \lambda_2^{\mathbb{C}}(y + 4\epsilon_{\mathbb{C}}) \\ &= \lambda_2^{\mathbb{C}}(y) + \lambda_1^{\mathbb{C}}(y) \otimes_{\mathbb{C}} \lambda_1^{\mathbb{C}}(4\epsilon_{\mathbb{C}}) + \lambda_2^{\mathbb{C}}(4\epsilon_{\mathbb{C}}) \\ &= -16y + 4y + 6\epsilon_{\mathbb{C}} = 4y + 6\epsilon_{\mathbb{C}} \\ &\neq 0 \text{ in } \tilde{K}(X) \end{aligned}$$

Here the term $-16y$ is due to the fact that on $\tilde{K}(S^{10})$, $\lambda_2^{\mathbb{C}}$ operates as multiplication by $-2^{5-1} = -16$. If η were a stably trivial vector bundle on S^9, then $\eta \oplus 4\epsilon$ would be trivialisable on S^9, so that it can be regarded as a pullback, via c, of some 10-dimensional vector bundle defined over S^{10}. But the complexification morphism

$$\widetilde{KO}(S^{10}) \xrightarrow{\otimes\mathbb{C}} \widetilde{K}(S^{10})$$

is a trivial homomorphism $\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}$, so $\eta \otimes \mathbb{C} = (\eta \oplus 4\epsilon) \otimes \mathbb{C} - 4\epsilon_{\mathbb{C}}$ would be equal to $10\epsilon_{\mathbb{C}} - 4\epsilon_{\mathbb{C}} = 6\epsilon_{\mathbb{C}}$, contradicting the computation above.

Step D. On spheres S^{8k+1} with $k > 1$ we can obtain 6-dimensional vector bundle by pulling back η via Adams maps A between mod 8 Moore spaces [2], as in

$$S^{8k+1} \cup_8 e^{8k+2} \xrightarrow{A} S^{8k-7} \cup_8 e^{8k-6} \xrightarrow{A} \ldots \xrightarrow{A} S^{17} \cup_8 e^{18} \xrightarrow{A} S^9 \cup_8 e^{10} = X$$

These spaces are $8j$-fold suspensions of X with their $\widetilde{K}$ groups all equal to $\mathbb{Z}/8\mathbb{Z}$, mutually isomorphic [2] under the induced homomorphisms A^*. We go on to point out, moreover, that for all $k \geqslant 1$,

$\widetilde{KO}(S^{8k+1}) = \mathbb{Z}/2\mathbb{Z} = \widetilde{KO}(S^{8k+2})$, so that $\widetilde{KO}(S^{8k+1} \cup_8 e^{8k+2})$ equals $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, with generators u, v such that v restricts trivially to $\widetilde{KO}(S^{8k+1})$ while u doesn't. One needs to argue, further beyond Adams, that these $\widetilde{KO}$ groups are mutually isomorphic under A^* as well. To this end notice that the v generators are obtained from the generators of the $\mathbb{Z}/8\mathbb{Z}$ groups of $\widetilde{K}$ theory by realification, i.e. by forgetting complex structures. They thus do correspond under A^*. The pullback of the 6-dimensional vector bundle η on X to $S^{8k+1} \cup_8 e^{8k+2}$ retains the crucial property of η pointed out in Step C, namely that it stably complexifies into the element of order 2 in $\widetilde{K}(S^{8k+1} \cup_8 e^{8k+2})$. This entails, again as in Step C, that the pullback of η is stably non-trivial on S^{8k+1}. It must therefore represent an u generator. In particular the generator x of $\widetilde{KO}(S^{8k+1})$ now has a 6-dimensional vector bundle as representative, and $gd(x) \leqslant 6$ follows.

Step E. Finally we will rule out the possibility that $gd(x) \leqslant 5$, by establishing

Proposition 2.2. *Any 5-dimensional vector bundle ξ over S^{8k+1} must be stably trivial for all $k \geqslant 1$.*

For the proof let us consider

$$\triangle_5 : \mathrm{Spin}(5) \xrightarrow{\approx} Sp(2) \hookrightarrow U(4) \quad ,$$

the 5-dimensional spinor representation into $U(4)$ taking up isomorphic image $Sp(2)$ inside $U(4)$. From representation theory one knows that the (left) quaternionic 2-plane bundle $\triangle_5(\xi)$ associated to ξ satisfies

$$\mu_2(\triangle_5(\xi)) \approx \xi \oplus \epsilon$$

as real vector bundles, where μ_2 is the functorial operation described as in [8, §4].

Briefly, like the second symmetric power on real vector spaces, μ_2 is a functor which converts left vector spaces of dimension m over the quaternions into real vector spaces of dimension $m(2m-1)$. It enjoys the property

$$\mu_2(V \oplus W) = \mu_2(V) \oplus V \hat{\otimes} W \oplus \mu_2(W)$$

where $\hat{\otimes}$ means taking tensor product over the quaternions after converting V into a right quaternionic vector space in the usual manner.

Since $\widetilde{KSP}(S^{8k+1}) = 0$ one has an isomorphism of symplectic vector bundles

$$\triangle_5(\xi) \oplus n\epsilon_{\mathbb{H}} = (n+2)\epsilon_{\mathbb{H}}$$

for n sufficiently large. Taking n large as well as even and applying μ_2, one gets, in $KO(S^{8k+1})$,

$$\mu_2(\triangle_5(\xi)) \oplus \triangle_5(\xi)\hat{\otimes}_{\mathbb{H}}(n\epsilon_{\mathbb{H}}) \oplus \mu_2(n\epsilon_{\mathbb{H}}) = \mu_2\left((n+2)\epsilon_{\mathbb{H}}\right)$$

See [8]. The last two terms in this equation are, of course, trivial $\mathbb{R}$-vector bundles. Since n, when even, annihilates $\widetilde{KO}(S^{8k+1})$, the middle term on the left side is also $\mathbb{R}$-trivial. Hence the equation implies that $\mu_2(\triangle_5(\xi))$, and hence ξ itself, must be stably trivial. □

The case of S^{8k+2} follows easily, as in

Corollary 2.3. *For $k \geqslant 1$ the generator x' in $\widetilde{KO}(S^{8k+2}) = \mathbb{Z}/2\mathbb{Z}$ has $gd(x') = 6$.*

This is because the unique essential map

$$\eta:\, S^{8k+2} \longrightarrow S^{8k+1}$$

is well-known to satisfy $\eta^!(x) = x'$. Also, Proposition (2.2) on 5-dimensional vector bundles carries over from S^{8k+1} to S^{8k+2}, with no change whatsoever.

3 The 6-dimensional bundle η over S^9

When octonions are regarded as pairs of quaternions, their multiplication is well-known to be given by the formula

$$(a_1, a_2) \times (b_1, b_2) \longrightarrow (a_1 b_1 - \overline{b}_2 a_2, b_2 a_1 + a_2 \overline{b}_1).$$

This provides a bilinear multiplication $\mathbb{R}^8 \times \mathbb{R}^8 \longrightarrow \mathbb{R}^8$ free of zero divisors. One can use the same formula to define multiplication of "bi-octonions", by regarding a_1, a_2, b_1, b_2 themselves as octonions. This gives a bilinear "Cayley-Dickson" multiplication

$$\mu:\, \mathbb{R}^{16} \times \mathbb{R}^{16} \longrightarrow \mathbb{R}^{16}$$

which is long known to have zero divisors. However [7] some restrictions of μ to "partial multiplications" will be zero-divisior free. Two such restrictions are the μ_r and the μ_c below:

$$\begin{array}{cccc} \mu_r : & \mathbb{R}^9 \times \mathbb{R}^{16} & \longrightarrow & \mathbb{R}^{16} \\ & & & \| \\ \mu_c : & \mathbb{R}^{10} \times \mathbb{R}^{10} & \longrightarrow & \mathbb{R}^{16} \end{array}$$

where for μ_r we take (a_1, a_2) only with $a_1 \in \mathbb{R}$, and for μ_c we take (a_1, a_2), (b_1, b_2) with $a_1 \in \mathbb{C}$, $b_1 \in \mathbb{C}$. These two zero-divisor free multiplications are "compatible" in that they coincide over the mutual subdomain $\mathbb{R}^9 \times \mathbb{R}^{10}$. Both multiplications are in fact "orthogonal", in the sense that the norm of the product equals the product of the norms of the factors.

For a line λ through the origin in $\mathbb{R}^n$ write $[\lambda]$ for the corresponding point of the real projective space $\mathbb{R}P^{n-1}$. Let η_0 be the 6-dimensional vector bundle over $\mathbb{R}P^9$ obtained by setting up over $[\lambda]$ the fiber $\eta_0^{[\lambda]}$, given by the short exact sequence

$$0 \longrightarrow \lambda \otimes \left(\lambda \otimes \mathbb{R}^{10}\right) \xrightarrow{id_\lambda \otimes \hat{\mu}_c} \lambda \otimes \mathbb{R}^{16} \longrightarrow \eta_0^{[\lambda]} \longrightarrow 0$$

where $\hat{\mu}_c$ sends $\lambda \otimes \vec{v} \subset \lambda \otimes \mathbb{R}^{10}$ to $\mu_c(\lambda \times \overrightarrow{v})$. To see the geometric meaning of this sequence, note that $\lambda \otimes \left(\lambda \otimes \mathbb{R}^{10}\right)$ is canonically isomorphic to $\mathbb{R}^{10}$. Exactness means that, over $\mathbb{R}P^9$, the Whitney sum of 16 copies of the Hopf line bundle has 10 independent sections, with η_0 as direct sum complement. Note that η_0 isn't stably trivial, as $\widetilde{KO}\left(\mathbb{R}P^9\right) = \mathbb{Z}/32\mathbb{Z}$, with $\lambda - \epsilon$ as generator. When $[\lambda]$ happens to be in $\mathbb{R}P^8$, this exact sequence fits into a commutative diagram

$$\begin{array}{ccccc} \lambda \otimes \left(\lambda \otimes \mathbb{R}^{10}\right) & \xrightarrow{id_\lambda \otimes \hat{\mu}_c} & \lambda \otimes \mathbb{R}^{16} & \longrightarrow & \eta_0^{[\lambda]} \\ \big\downarrow & \overset{\approx}{\nearrow}{\scriptstyle id_\lambda \otimes \mu_r} & & & \\ \lambda \otimes \left(\lambda \otimes \mathbb{R}^{16}\right) & & & & \end{array}$$

This is to say that over $\mathbb{R}P^8$, η_0 is isomorphic to the 6-dimensional trivial bundle whose "constant" fiber is the normal bundle of $\lambda \otimes (\lambda \otimes \mathbb{R}^{10})$ inside $\lambda \otimes \left(\lambda \otimes \mathbb{R}^{16}\right)$, which is framed by the standard normal frame of $\mathbb{R}^{10}$ inside $\mathbb{R}^{16}$. By pinching $\mathbb{R}P^8$ to a point and identifying all

$$\left\{\eta_0^{[\lambda]}\right\}_{[\lambda] \in \mathbb{R}P^8}$$

into one vector space according to such framing, one obtains a 6-dimensional vector bundle over S^9, denoted again by η_0, which represents the generator of $\widetilde{KO}(S^9)$. Indeed it is not hard to show that this η_0 further extends to a vector bundle $\widetilde{\eta}_0$ over $S^9 \cup_8 e^{10}$, but we'll omit the details. Such $\widetilde{\eta}_0$ can provide a concrete alternative to the vector bundle η in §2.

4 Spheres of dimension $4k$, $k \geqslant 5$

Theorem 4.1. *For $k \geqslant 5$ any nonzero element x in $\widetilde{KO}(S^{4k}) = \mathbb{Z}$ has $gd(x) = 2k+1$.*

This is the first part of Theorem (1.1) of Davis-Mahowald [5]. It was actually established much earlier in [3]. Very briefly, the complexification map $\widetilde{KO}(S^{4k}) \xrightarrow{\otimes\mathbb{C}} \widetilde{K}(S^{4k})$ is a monomorphism. If x is represented by a vector bundle ζ of $\mathbb{R}$-dimension m then $x \neq 0 \Rightarrow x \otimes \mathbb{C} \neq 0$. Thus $x \otimes \mathbb{C}$ has nontrivial Chern character, and so $\zeta \otimes \mathbb{C}$ must have a nonzero $2k^{th}$ Chern class. This already forces $m \geqslant 2k$. If $m = 2k$, then $\zeta \otimes \mathbb{C} \approx \zeta \oplus \zeta$ as real vector bundles and in terms of Euler classes χ

$$0 \neq c_{2k}\,(\zeta \otimes \mathbb{C}) = \chi_{4k}\,(\zeta \oplus \zeta) = \chi_{2k}(\zeta) \smile \chi_{2k}(\zeta) = 0$$

because cup product is taken in the ring $\widetilde{H^*}(S^{4k})$. This contradiction forces $m \geqslant 2k+1$.

Finally, the fact that there exists indeed a $(2k+1)$-dimensional vector bundle which represents the generator x of $\widetilde{KO}(S^{4k})$ has been established quite early in [3]. This is again established in [5], through a comparison of the Postnikov tower of BSO with its connective covers [9]. We are not aware of any other existence proofs in the literature. In any event, with the above theorems, the geometric dimension of nonzero elements in the $\widetilde{KO}$ theory of an arbitrary sphere of dimensions different from 1,2,4,8,12 and 16 is now completely determined. There is little difficulty in handling these exceptional dimensions case by case. Indeed the answer to all nontrivial cases are already tabulated at the end of [5].

Postscript: The content of this article was presented by the first author at the 2008 meeting of the Sociedad Matematica Mexicana in Guanajuato, in a special session on algebraic topology dedicated to Professor Sam Gitler. At that time the authors were unaware of the paper by M. Cadek and M. Crabb entitled "$G-$structures on spheres", published in Proc. London Math. Soc. (3)93(2006), 791-816. In an appendix to their paper, Cadek and Crabb included a self-contained proof of the

Davis-Mahowald Theorem 1.1 in [5]. Their proof and ours, while independently arrived at, overlap to some extent. In our treatment more emphasis was placed on explicit constructions, such as the algebraic description of the 6−dimensional vector bundle η over S^9. Also, as a conclusive step, we provided an original proof that any 5−dimensional vector bundle over a sphere of dimension at least 9 must be stably trivial. Our approach is elementary throughout, using standard $K-$theory. Such an approach, we believe, is much in line with Sam Gitler's mathematical style.

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