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Topological complexity and related invariants*

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Abstract

Early this century, Michael Farber introduced and developed the notion of topological complexity, applying it to robotics (in greater detail, to robot motion planning). This is a numerical invariant of Lusternik–Schnirelmann type. We survey recent progress in the area.

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1 Motion planning problem

Let X be a topological space that we can regard as the configuration of a mechanical system. Points of X are the states the system, and a continuous motion can be regarded as a continuous path $\alpha : I \to X$ where $I = [0, 1]$. Here $\alpha(0)$ is the initial point and $\alpha(1)$ is the final point.

We denote by X^I the space of continuous paths $I \to X$ equipped with the compact-open topology.

We assume that X is path-connected, and so we can move any given point of X to another given point.

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A *motion planning algorithm* is a rule that assigns to each pair $(x, y) \in X$ a path $\alpha : I \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$.

For more on motion planning see [La91, LV06].

To say it more formally, consider the fibration

$$\zeta_X = \{\pi : X^I \to X \times X, \quad \pi(\alpha) = (\alpha(0), \alpha(1)).$$

Now the motion planning algorithm turns out to be a map (not necessarily continuous) $s : X \times X \to X^I$ such that $\pi(s(x, y)) = (x, y)$ for all $(x, y) \in X \times X$. In other words, $\pi \circ s = 1_{X \times X}$, or we can say that $s : X \times X \to X^I$ is a section of ζ_X. Now we (can) interpret a *motion planning algorithm* as a section of the fibration ζ_X.

It is easy to see that a *continuous* motion planning algorithm exists if and only if X is contractible, see [Fa08, Lemma 4.2]

However, usually people do not like discontinuity or, at least, want to control this. Now we describe a mathematical apparatus that helps us to manage this situation.

2 Sectional category: preliminaries

Below "fibration" denotes "Hurewicz fibration" and the base B of any fibration is assumed to be path-connected CW space of finite type. All maps are assumed to be continuous unless something other is said explicitly. All functional spaces of the form Y^X are assumed to be equipped with compact-open topology.

Definition 2.1 (A. Schwarz [S66]). Given a fibration $\xi = \{p : E \to B\}$, the *sectional category* or *Schwarz genus* of ξ is the least number k such that there exists an open covering $A_0, A_1, \dots A_k$ of B and, for each A_i, a map $s_i : A_i \to E$ having $p \circ s_i = 1_{A_i}$. In other words, each s_i is a (continuous) local section of p. We also agree that the sectional category of ξ is equal to -1 if $E = \varnothing$.

We use the notation $\operatorname{secat} \xi$ or $\operatorname{secat} p$ for the sectional category of ξ. Clearly, $\operatorname{secat} \xi = 0$ iff p has a section.

Proposition 2.2 (Serre). *Given a fibration $\xi = \{p : E \to B\}$, for any two points $b, b' \in B$ the fibers $p^{-1}(b)$ and $p^{-1}(b')$ are homotopy equivalent.*

Definition 2.3. The *homotopy fiber* of a fibration ξ is defined to be the homotopy equivalence class of $p^{-1}(b), b \in B$. The notion is well-defined because of Proposition 2.2.

For ξ as above, we frequently speak about the fibration $F \to E \to B$ meaning that F is a homotopy fiber of ξ.

Definition 2.4. Let X be a path-connected space, let $x_0 \in X$, and let $P(X) = P(X, x_0)$ be the space of paths that start at x_0. So, $PX = \{\alpha \in X^I \bigm| \alpha(0) = x_0\}$. Define the *Serre path fibration* $\eta_X = \{p_X : PX \to X\}$ by setting $p(\alpha) = \alpha(1)$.

Example 2.5. Consider the Serre path fibration $\eta_X = \{p = p_X : P(X, x_0) \to X\}$. It is clear that $p_X^{-1}(x_0)$ is the loop space $\Omega(X, x_0)$. So, the homotopy fiber of η_X is (the homotopy class of) $\Omega(X, x_0)$.

It is worthy to note that, generally, $p_X^{-1}(x_1)$ for $x_1 \neq x_0$ is homeomorphic neither to $\Omega(X, x_0)$ nor to $\Omega(X, x_1)$.

In next three sections we give three main examples of sectional category.

3 Lusternik–Schnirelmann category

Definition 3.1. *The Lusternik–Schnirelmann category* of a space X (denoted by $\operatorname{cat} X$) is the least number k such that there exists an open covering $A_0, A_1, \dots A_k$ of X where each A_i is contractible in X.

Now, assume that X is path-connected. Then the space PX is contractible. Hence, a local section $s : A \to PX$ of p exists if and only if the subspace A of X is contractible in X. So, Lusternik–Schnirelmann category is equal to the sectional category of η_X. So, $\operatorname{cat} X := \operatorname{secat} \eta_X$.

It is worth noting one of the main applications of the Lusternik–Schnirelmann theory: Given a smooth function $f : M \to \mathbb{R}$ on a closed smooth manifold M, the number of critical points of f is at least $1 + \operatorname{cat} M$. This result turned out to be the starting point of LS theory, [LS29, LS34]. Currently, the LS theory is a wide area of intensive topology research.

More information on Lusternik–Schnirelmann theory can be found in [CLOT03].

4 Topological complexity

Most results of this subsection are due by Farber and his collaborators, [Fa03, Fa04, Fa06, Fa08, FG08].

Definition 4.1 (Farber [Fa08])**.** Let X be a path-connected CW space of finite type. A *topological complexity* of a space X (denoted by $\mathrm{TC}(X)$) is the sectional category of ζ_X. So, $\mathrm{TC}(X) := \operatorname{secat} \zeta$.

How is it related to motion planning problem? We already noticed that a continuous motion planning algorithm exists for contractible X only. So, as a first step, it makes sense to consider subsets $\{A_i\}$ with $\cup A_i = X \times X$ and such that each A_i admits a section of ζ_X over A_i. Note that if $\mathrm{TC}\,(X) = k$ then $X \times X$ admits such a family $\{A_i\}_{i=0}^k$ (and even with open $A_i, i = 0, \ldots, k$).

However, this is not enough for our goals. In fact, the local sections s_i can overlap since, in general, we have $A_i \cap A_i \neq \emptyset$. So, here we will not get a well-defined motion planning algorithm.

To cope with this inconvenience, it makes sense to enlarge the class of considered domains of continuity (by using not only open subsets but something more), while to keep good properties of $\{A_i\}$'s. This needs some expenses, such as restrictions on the configuration space X, but this is enough for most applications. This program was successfully realized by Farber, who used Euclidean Neighborhood Retracts (ENRs). See [Do95] concerning ENRs. From our point of view, the advantage of ENR is the property that, given two open subsets U and V of an ENR X, the $U \setminus V$ is also an ENR.

Theorem 4.2 (Farber [Fa08])**.** *Assume X is a polyhedron in $\mathbb{R}^N$ with $\mathrm{TC}(X) = k$. There exist a motion planning algorithm $s : X \times X \to X^I$ and a partition $X = F_0 \cup F_1 \cup \cdots \cup F_k$ such that*

- *each F_i is an Euclidean Neighborhood Retract (ENR);*
- *for each i the restriction $s_{|F_i} : F_i \to X \times X$ is continuous;*
- *$F_i \cap F_j = \emptyset$ for $i \neq j$.*

Thus, if $\mathrm{TC}(X) = k$ then there exists a motion planning algorithm $s : X \times X \to X^I$ that has $k + 1$ domains of continuity of s, and each domain of contnuity is an ENR.

5 Higher topological complexity

Let J_n denote the wedge of n copies of the closed interval [0, 1], in all of which $0 \in [0, 1]$ is the base point. Given a space X, every element $\alpha \in X^{J_n}$ can be regarded as an n-tuple (multipath) $(\alpha_1, \ldots, \alpha_n)$ of paths in X all of which start at a common point.

Consider a fibration $\zeta_n = \zeta_{n,X} = \{e_n = e_n^X : X^{J_n} \to X^n\}$, $e_n(\alpha) = (\alpha_1(1), \ldots, \alpha_n(1))$ where X is a path-connected CW space of finite type.

Definition 5.1 (Rudyak [Ru10])**.** A *higher topological complexity* (of order n) of a space X (denoted by $\mathrm{TC}_n(X)$) is the sectional category of ζ_n. So, $\mathrm{TC}_n(X) := \operatorname{secat} \zeta_n$.

Clearly, $\mathrm{TC}(X) = \mathrm{TC}_2(X)$.

It is worthy to note that, given n, the equality $\mathrm{TC}_n(X) = 0$ holds if and only if X is contractible.

Proposition 5.2. *If A is a retract of X then* $\operatorname{cat} A \leq \operatorname{cat} X$ *and* $\mathrm{TC}_n(A) \leq \mathrm{TC}_n(X)$.

Proof. Obvious. □

Remark 5.3. In Sections 3, 4, 5, assume that X is a polyhedron. Then the values $\operatorname{cat} X$ and $\mathrm{TC}_n(X)$ do not change if, in the definitions, we assume that each A_i is an euclidean neighborhood retract, not necessary an open subset of the base. This is proved for $n = 2$ in Theorem 4.2, and the general case can be proved similarly.

There is another interpretation of TC_n. Consider a fibration

$$\upsilon_n = \{u_n : X^I \to X^n\},$$
$$\upsilon_n(\alpha) = \left(\alpha(0), \alpha\left(\frac{1}{n-1}\right), \ldots, \alpha\left(\frac{n-2}{n-1}\right), \alpha(1)\right).$$

It is easy to check (and we will see it below) that ζ_n and υ_n have equal sectional categories.

Now you can see how TC_n is related to motion planning theory. Indeed, $\mathrm{TC}(X)$ is related to motion planning algorithm when a robot moves from a point to another point, while $\mathrm{TC}_n(X)$ is related to motion planning problem whose input is not only an initial and final point but also $n - 2$ intermediate additional points.

See [BGRT14, GLO13, GLO15b, KL12] for more information on TC_n.

6 More on sectional category

Given two fibrations $\xi = \{p : E \to B\}$ and $\xi' = \{p' : E' \to B'\}$, consider their product

$$\xi \times \xi' = \{p \times p' : E \times E' \to B \times B'\}.$$

Theorem 6.1 (Schwarz)**.** *We have*

$$\operatorname{secat}(\xi \times \xi') \le \operatorname{secat} \xi + \operatorname{secat} \xi'.$$

In particular,

$$\operatorname{cat}(X \times Y) \le \operatorname{cat} X + \operatorname{cat} Y \text{ and } \operatorname{TC}_n(X \times Y) \le \operatorname{TC}_n(X) + \operatorname{TC}_n(Y).$$

Proof. For the proof, see [S66, Prop. 21]. □

This theorem dates back to Bassi [B37], who proved the similar inequality for Lusternik-Schnirelmann category.

Let $\xi = \{p : E \to B\}$ be a fibration and $f : X \to B$ be a map. Consider the induced fibration $f^*\xi$ over X.

Proposition 6.2. *We have* $\operatorname{secat} f^*\xi \le \operatorname{secat} \xi$.

Proof. Indeed, if ξ has a local section over a subspace A of B then $f^*\xi$ has a local section over the subspace $f^{-1}(A)$ of X. □

Now we settle homotopy invariance of sectional category.

Consider two fibrations $\xi = \{p : E \to B\}$ and $\overline{\xi} = \{p' : E' \to B\}$ over the same base B, and a commutative diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, equal] & B. \end{tikzcd}$$

Proposition 6.3. *We have* $\operatorname{secat} \overline{\xi} \le \operatorname{secat} \xi$. *Furthermore, if f is a fiber homotopy equivalence over B then* $\operatorname{secat} \overline{\xi} = \operatorname{secat} \xi$.

Proof. Indeed, if $s : A \to E$ is a local section of ξ over A then fs is a local section of $\overline{\xi}$ over the same A. Hence, $\operatorname{secat} \xi \ge \operatorname{secat} \overline{\xi}$. Furthermore, if f is a fiber homotopy equivalence over B then there exists a homotopy inverse $h : E' \to E$ over B to f, and hence $\operatorname{secat} \xi \le \operatorname{secat} \overline{\xi}$. Thus, $\operatorname{secat} \xi = \operatorname{secat} \overline{\xi}$. □

Consider two fibrations $\xi = \{p : E \to B\}$ and $\xi' = \{p' : E' \to B'\}$ and a commutative diagram

$$\begin{CD} E @>f>> E' \\ @VpVV @VVp'V \\ B @>g>> B'. \end{CD}$$

Theorem 6.4. *If f is a fiber homotopy equivalence and g is a homotopy equivalence then* $\operatorname{secat}\xi = \operatorname{secat}\xi'$.

Proof. The bundle map $\xi \to \xi'$ can be decomposed as

$$\xi \to g^*\xi' \to \xi'$$

where the correcting map $\xi \to g^*\xi'$ yields the identity map 1_B on bases. Now, $\operatorname{secat}\xi = \operatorname{secat} g^*\xi'$ by Proposition 6.3, while $\operatorname{secat} g^*\xi' \le \operatorname{secat}\xi'$ by Proposition 6.2. Hence $\operatorname{secat}\xi \le \operatorname{secat}\xi'$. Since f is fiber homotopy equivalence, we can find a fiber homotopy equivalence $h : E' \to E$ that is fiber homotopy inverse to f and prove that $\operatorname{secat}\xi' \le \operatorname{secat}\xi$. $\square$

Corollary 6.5. *The invariant* $\operatorname{cat} X$*, as well as* $\mathrm{TC}_n(X)$*, is a homotopy invariant.*

Remark 6.6. Now you see the above-mentioned equality $\operatorname{secat} e_n = \operatorname{secat} u_n$. Indeed, both maps $e_n : X^{J_n} \to X^n$ and $\upsilon_n : X^i \to X^n$ are homotopy equivalent to the diagonal $d_n : X \to X^n$, and so the fibration e_n and υ_n are fiber homotopy equivalent, like in Theorem 6.4. Thus, $\operatorname{secat} e_n = \operatorname{secat} u_n$.

7 Several inequalities

Proposition 7.1. *For any fibration $\xi = \{p : E \to B\}$ we have the inequality* $\operatorname{secat}\xi \le \operatorname{cat} B$.

Proof. This holds because, for any subset A of B that is contractible in B, the fibration ξ admits a local section over A. $\square$

Theorem 7.2. *For every n we have* $\operatorname{cat} X^{n-1} \le \mathrm{TC}_n(X) \le \operatorname{cat} X^n \le \mathrm{TC}_{n+1}(X)$.

Proof. The second inequality follows from the Proposition 7.1. For the inequality $\operatorname{cat} X^{n-1} \le \mathrm{TC}_n(X)$, see [BGRT14, Proposition 3.1]. (Note that Farber [Fa08] considered the case $n = 2$.) $\square$

Corollary 7.3. $\mathrm{TC}_n(X) \le \mathrm{TC}_{n+1}(X)$.

Open Problem 7.4. *Do there exist a non-contractible space X and a natural number n such that $\mathrm{TC}_n(X) = \mathrm{TC}_{n+1}(X)$?*

Proposition 7.5. *If X is not contractible then $\mathrm{TC}_n(X) \ge n - 1$.*

Proof. This is proved in [Ru10, Proposition 3.5]. We present one more proof. Ganea and Hilton [GH59] proved that $\operatorname{cat} X^n \ge n$ for X non-contractible. Now the proposition follows from the inequality $\mathrm{TC}_n(X) \ge \operatorname{cat} X^{n-1}$. □

Theorem 7.6. *If G is a path-connected H-space (e.g. a topological group) then $\mathrm{TC}_n(G) = \operatorname{cat} G^{n-1}$.*

Proof. For a topological group and $n = 2$ this is proved in [Fa04], for $n > 2$ see [BGRT14]. For arbitrary H-spaces see [LuSh13]. □

Note also the following difference between cat and TC. We know that $\operatorname{cat}(X \vee Y) = \max\{\operatorname{cat} X, \operatorname{cat} Y\}$. This is not true for TC. Namely, $\mathrm{TC}(S^1) = 1$ while $\mathrm{TC}(S^1 \vee S^1) = 2$.

Theorem 7.7 (Dranishnikov [Dr14])**.** Assume X, Y are two absolute neighborhood retracts. Then

$$\max\{\mathrm{TC}(X), \mathrm{TC}(Y), \operatorname{cat}(X \times Y) \le \mathrm{TC}(X \vee Y) \le \mathrm{TC}(X) + \mathrm{TC}(Y) + 3.$$

We know that if $\widetilde{X} \to X$ is a cover map then $\operatorname{cat} \widetilde{X} \le \operatorname{cat} X$. This is not true for TC.

Example 7.8 (Dranishnikov [Dr14])**.** Let $X = S^3 \times S^3 \vee S^1$, and let $\widetilde{X}$ be the universal cover of X. Then $\mathrm{TC}(X) \le 3$ while $\mathrm{TC}(\widetilde{X}) \ge 4$.

8 Topological complexity of discrete groups

Let π be a discrete group. Define $\mathrm{TC}(\pi) := \mathrm{TC}(B\pi)$ where $B\pi$ denotes the classifying space of π. Since the classifying space (assumed to be CW) is defined uniquely up to homotopy equivalence, and because of the homotopy invariance of TC, the invariant $\mathrm{TC}(\pi)$ is well-defined.

Note that the invariant $\operatorname{cat} \pi := \operatorname{cat}(B\pi)$ has a known purely group-theoretical description. In fact, $\operatorname{cat} \pi$ is equal to the cohomological dimension $\operatorname{cd}(\pi)$ of π, see [EG57] for $\operatorname{cat} \pi \ne 2$ and [St68, Sw69] for $\operatorname{cat} \pi = 2$.

The situation for TC looks more complicated. We know that $\operatorname{cat} X \leq \mathrm{TC}(X) \leq \operatorname{cat}(X \times X)$ for all X. The following proposition tells us that, in the class of $K(\pi, 1)$-spaces, the above-mentioned inequality gets no new bounds. In other words, we have examples of two group π, π' such that $\operatorname{cat} \pi = \operatorname{cat} \pi'$ while $\mathrm{TC}(\pi) \neq \mathrm{TC}(\pi')$.

Proposition 8.1 ([Ru16])**.** *For every natural k and every natural l with $k \leq l \leq 2k$ there exists a discrete group π such that $\operatorname{cat} \pi = k$ and $\mathrm{TC}(\pi) = l$. In fact, we can put $\pi = \mathbb{Z}^k * \mathbb{Z}^{l-k}$.*

Because of the proposition, the following problem turns out to be essential.

Open Problem 8.2 (Farber)**.** *Describe* $\mathrm{TC}(\pi)$ *in purely group-theoretical terms.*

9 A homotopy-theoretical description of sectional category

Recall that the *join* $X * Y$ of two CW spaces X and Y is defined to be a quotient space $(X \times I \times Y)/R$, where R is the equivalence generated by the equivalences $(x, 0, y_1) \sim (x, 0, y_2)$ for all $x \in X$, $y_1, y_2 \in Y$ and $(x_1, 1, y) \sim (x_2, 1, y)$ for all $x_1, x_2 \in X$, $y \in Y$.

Note also that $X * Y$ is the double mapping cylinder of the diagram

$$X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y.$$

More generally, given two maps $f : X \to Z$ and $g : Y \to Z$, the *fiberwise join* of f and g is defined to be the map $f * g : X *_Z Y \to Z$ where

$$X *_Z Y = \{[x, t, y] \in X * Y \bigm| f(x) = g(y)\}$$

and $(f * g)(x, t, y) = f(x)$.

Note that $X *_Z Y$ turns into $X * Y$ if Z is the point.

We can iterate the join construction. In particular, given a fibration $\xi = \{p : E \to B\}$ we can form the fibration

$$\xi^{*k} := \{p * p * \cdots * p : \underbrace{E *_B \cdots *_B E}_{k \text{ times}} \to B\}.$$

If we denote the homotopy fiber of ξ by F then the homotopy fiber of ξ^{*k} is F^{*k}.

Theorem 9.1 (Schwarz[S66]). *The fibration ξ^{*k} has a section if and only if* $\operatorname{secat}\xi < k$.

In other words, $\operatorname{secat}\xi$ is the least value m such that $\xi^{*(m+1)}$ admits a section.

For example, let $\eta_X = \{p_X : PX \to X\}$ be the Serre path fibration. So, we have the iterated fiber join

$$(\eta_X)^{*m} = \{P_m(X) := PX * \cdots * PX \to X\}$$

over X. This is a fibration with homotopy fiber $(\Omega X)^{*k}$.

Corollary 9.2. *We have:* $\operatorname{cat} X < m$ *iff* $(\eta_X)^{*m}$ *has a section.*

Similarly, we can apply Theorem 9.1 and get the following.

Corollary 9.3. *We have* $\operatorname{TC}_n(X) < m$ *iff* $(\zeta_{n,X})^{*m}$ *has a section.*

There is another description of $(\eta_X)^{*m}$, the so-called fiber-cofiber construction. We construct a certain fibration $q_n\colon G_n(X) \to X$ by induction on n. Put $G_0(X) = PX$, $F_0(X) = \Omega X$ and write η_X as the fibration $F_0(X) \to G_0(X) \to X$. Assume that we have a fibration

$$F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{q_n} X$$

and consider the mapping cone $C(i_n) = C(F_n(X)) \cup G_n(X)$. The map q_n extends to a map $r : C(i_n) \to X$ so that r is a constant map on $C(F_n)$. Define $q_{n+1} : G_{n+1}(X) \to X$ to be the Serre fibration substitute of r. We denote by $F_{n+1}(X)$ the (homotopy) fiber of q_{n+1} and get the fibration

$$F_{n+1}(X) \xrightarrow{i_{n+1}} G_{n+1}(X) \xrightarrow{q_{n+1}} X.$$

It turns out to be that $G_n(X) \to X$ and $P_{n+1}(X) \to X$ are homotopy equivalent fibrations over X.

10 Dimension–connectivity relations

Theorem 10.1 (Schwarz[S66]). *Given a fibration $\xi = \{p : E \to B\}$, take a point $b \in B$ and put $F = p^{-1}(b)$. Put $\dim B = d$ and assume that $\pi_k(F) = 0$ for $k < s$ (i.e., the space F is $(s-1)$-connected). Then*

$$\operatorname{secat}\xi < \frac{d+1}{s+1}.$$

Remark 10.2. By Proposition 2.2, the condition $\pi_k(F) = 0$ does not depend on choise of b, and we can express the condition as follows: the homotopy fiber of ξ is $(s-1)$-connected.

Example 10.3. Let X be an $(s-1)$-connected space with $s > 0$ (that tells us that X is path-connected). Put $d = \dim X$. Let us estimate $\operatorname{cat} X$. Recall the fibration $\eta_X = \{p : PX \to X\}$. The homotopy fiber of η_X has the homotopy type of ΩX (the loop space of X). Note that ΩX is $(s-2)$-connected. Thus, because of Theorem 10.1, we have

$$\operatorname{cat} X = \operatorname{secat}(\eta_X) < \frac{d+1}{s}, \quad \text{or } \operatorname{cat} X \leq \frac{d}{s}.$$

Similarly, $\mathrm{TC}(X) \leq 2d/s$ and $\mathrm{TC}_n(X) \leq dn/s$.

Example 10.4. Let $T^n, \mathbb{R}P^n$, and $\mathbb{C}P^n$ denote the torus, the real projective space, and the complex projective space, respectively. It follows from Theorem 10.1 $\operatorname{cat} T^n \leq n$ and $\operatorname{cat} \mathbb{R}P^n \leq n$. Furthermore, $\operatorname{cat} \mathbb{C}P^n \leq n$ since $\mathbb{C}P^n$ is simply connected ($\pi_i(\mathbb{C}P^n) = 0$ for $i < 2$).

11 Cohomological tools: cup-length

We start with a special case: Lusternik–Schnirelmann category.

Definition 11.1. Given a path-connected space X and a commutative ring R, define the *cup-length of X with coefficients in R* (denoted by $\mathrm{cl}_R(X)$) to be the maximal number k such that there exist $u_1, \cdots, u_k \in \widetilde{H}^*(X; R)$ with $u_1 \smile \cdots \smile u_k \neq 0$.

Theorem 11.2. *We have the following estimate:* $\mathrm{cl}_R(X) \leq \operatorname{cat} X$.

Proof. The idea of the proof is quite simple. Let $\operatorname{cat} X = n$. Take a covering $\{A_0, A_1, \dots, A_n\}$ by open and contractible in X sets. Suppose that $\mathrm{cl}_R(X) = k > n$ and take $u_1, \dots u_k$ with $u_1 \smile \cdots \smile u_k \neq 0$. Now,

$$u_{1|A_1} = 0, \dots, u_{n|A_n} = 0, u_{n+1|A_0} = 0.$$

Therefore, the classes u_i come from corresponding relative classes w_i in $H^*(X, A_i; R)$ (where $A_{n+1} := A_0$). In particular, the non-zero product $u_1 \smile \cdots \smile u_{n+1}$ comes from the relative product $w_1 \smile \cdots \smile w_{n+1}$, which lies in the trivial group $H^*(X, X; R)$.

This is a contradiction. $\square$

Examples 11.3. 1. It is easy to see that $\mathrm{cl}_{\mathbb{Z}}(T^n) = n$. Hence, $\mathrm{cat}\, T^n \geq n$. Together with the inequality $\mathrm{cat}\, T^n \leq \dim T^n = n$ we conclude that $\mathrm{cat}\, T^n = n$.

2. We have $H^*(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2[u]/u^{n+1}$, with $\dim u = 1$. Hence, $\mathrm{cl}_{\mathbb{Z}/2} = n$. So, $\mathrm{cat}(\mathbb{R}P^n) \geq n$, and we have $\mathrm{cat}(\mathbb{R}P^n) = n$ because $\mathrm{cat}(\mathbb{R}P^n) \leq \dim \mathbb{R}P^n) = n$.

3. We have $H^*(\mathbb{C}P^m) = \mathbb{Z}/[u]/(u^{n+1})$, with $\dim u = 2$. Hence $\mathrm{cl}_{\mathbb{Z}}(\mathbb{C}P^n) = n$, so $\mathrm{cat}(\mathbb{C}P^n) \geq n$. Further, $\mathrm{cat}(\mathbb{C}P^n) \leq \dim(\mathbb{C}P^n)/2 = 2n/2 = n$; the denominator 2 appears because $\mathbb{C}P^n$ is simply connected. Thus, $\mathrm{cat}(\mathbb{C}P^n) = n$.

4. For completeness, note that $\mathrm{cat}\, S^m = 1$ for all $m > 0$. Indeed, $\mathrm{cat}\, S^m > 0$ because S^m is not contractible, while S^m can be covered by two contractible spaces (discs).

We leave it to the reader to check that $\mathrm{cat}\, S = 2$ for all closed surfaces except S^2.

Now we pass to the general situation. Consider a fibration $\xi = \{p : E \to B\}$.

Definition 11.4. Define the *cup-length of ξ with coefficients in R* (denoted by $\mathrm{cl}_R(\xi)$) to be the maximal number k such that there exist elements $u_1, \cdots, u_k \in \mathrm{Ker}\{\widetilde{H}^*(B; R) \to \widetilde{H}^*(E; R)\}$ with

$$u_1 \smile \cdots \smile u_k \neq 0.$$

Theorem 11.5 (Schwarz)**.** *We have the following estimate:* $\mathrm{cl}_R(\xi) \leq \mathrm{secat}\, \xi$*.*

Remarks 11.6. 1. In a special case of the Serre fibration $\eta_X = \{p : P_X \to X\}$ the space PX is contractible. Therefore $\mathrm{cl}(\eta_X) = \mathrm{cl}(X)$.

2. In the definition and application of cup-length, we can consider more general situation: to consider $u_i \in H^*(B; A_i)$ for arbitrary coefficient groups (and even local coefficient systems) A_i with $u_1 \smile \cdots \smile u_k \in H^*(B; A_1 \otimes \cdots \otimes A_k)$.

12 Zero-divisors, TC and higher TC of spheres

Consider the fibration $\zeta_n = \{e_n : X^{J_n} \to X^n\}$ and the homotopy commutative diagram

$$\begin{array}{ccc} X^{J_n} & \longrightarrow & X \\ {\scriptstyle e_n}\big\downarrow & & \big\downarrow{\scriptstyle d_n} \\ X^n & \xrightarrow{1} & X^n \end{array}$$

where d_n is the iterated diagonal map, $d_n(x) = (x, \dots, x)$ and the top map has the form $(\alpha_1, \dots, \alpha_n) \mapsto \alpha_1(0)$.

Definition 12.1 (Farber [Fa08])**.** A cohomology class $u \in H^*(X^n; R)$ is called an *zero-divizor* if $d_n^*(u) = 0$.

So, the cup-length of ζ_n can be reformulated as the maximal number k such that there exist zero-divizors $u_1, \cdots, u_k$ with $u_1 \smile \cdots \smile u_k \neq 0$.

Theorem 12.2 (Farber [Fa08])**.** *We have*

$$\mathrm{TC}(S^{2k+1}) = 1 \quad \textit{and} \quad \mathrm{TC}(S^{2k}) = 2.$$

So, we have a remarkable contrast with the claim $\mathrm{cat}(S^m) = 1$.

Proof. Proof of $\mathrm{TC}(S^{2k-1}) = 1$. We must construct two continuous sections $s_i : A_i \to (S^{2k+1})^I$ where $A_0 \cup A_1 = S^{2k-1} \times S^{2k-1}$. In other words, cover $S^{2k-1} \times S^{2k-1}$ by subsets A_0, A_1 such that every two points $(x, y) \in A_i, i = 1, 2$ can be joined by an arc in S^{2k-1}, and the arc depends on x, y continuously in each A_i.

Put $A_0 = \{(x, y) \mid x, y \in S^{2k+1} \text{ with } x \neq -y\}$, and join x to y by the shortest geodesic. Put $A_1 = \{(x, y) \mid x = -y\}$. To construct s_1, recall that S^{2k-1} possesses a non-vanishing continuous tangent vector field $\mathbf{v}$. Now, given $x \in S^{2k-1}$, join x to $y = -x$ by the geodesic whose tangent vector at x is equal to $\mathbf{v}(x)$.

Proof of $\mathrm{TC}(S^{2k}) = 2$. Take a generator $u \in H^{2k}(S^{2k})$ and consider the element $v := u \otimes 1 - 1 \otimes u$ which lies in

$$H^{2k}(S^{2k} \times S^{2k}) = H^{2k}(S^{2k}) \otimes H^0(S^{2k}) \oplus H^0(S^{2k}) \otimes H^{2k}(S^{2k})$$

Note that v is a zero-divisor. Indeed, $d_2^*(u \otimes 1) = u = d_2^*(1 \otimes u)$, and so $d_2^*(v) = 0$. Furthermure, since $\dim u = 2k$ is even, we have $v \smile v = -2u \otimes u \neq 0$. Indeed

$$\begin{aligned} v \smile v &= ((u \otimes 1) - (1 \otimes u)) \smile ((u \otimes 1) - (1 \otimes u)) \\ &= -(u \otimes 1) \smile (1 \otimes u) - (1 \otimes u) \smile (u \otimes 1) \\ &= -2u \otimes u, \text{ since } \dim(1 \otimes u) = \dim(u \otimes 1) \text{ is even.} \end{aligned}$$

So, $v \smile v \neq 0$, and hence $\mathrm{cl}(\zeta_{S^{2k}}) \geq 2$. So, $\mathrm{TC}(S^{2k}) \geq 2$. Furthermore, $\mathrm{TC}(S^{2k}) \leq 2$ because of the dimension-connectivity relation, and thus $\mathrm{TC}(S^{2k}) = 2$. $\square$

Theorem 12.3 (Rudyak [Ru10]). *We have*

$$\mathrm{TC}_n(S^{2k+1}) = n-1 \quad and \quad \mathrm{TC}(S^{2k}) = 2n.$$

Proof. First, we prove that $\mathrm{TC}_n(S^{2k+1}) = n-1$. Consider a unit tangent vector field $\mathbf{v}$ on S^{2k+1}. Given $x, y \in S^{2k+1}, y = -x$, denote by $[x, y]$ the path determined by the geodesic semicircle joining x to y and such that the $\mathbf{v}(x)$ is the direction of the semicircle at x. If $x \neq -y$, denote by $[x, y]$ the path determined by the shortest geodesic from x to y.

Determine a (non-continuous) function $\varphi : (S^{2k+1})^n \to (S^{2k+1})^{J_n}$, by $\varphi(x_1, \ldots, x_n) = \{[x_1, x_1], \ldots, [x_1, x_n]\}$

For each $j = 0, \ldots, n-1$ consider the submanifold (with boundary) U_j in S^{2k+1} such that each n-tuple $(x_1, \ldots, x_n)$ in U_j has exactly j antipodes to x_1. Then $\varphi_{|U_j} : U_j \to (S^{2k+1})^{J_n}$ is a continuous section of $\zeta_{n,S^{2k+1}}$. Hence, $\mathrm{TC}_n(S^{2k+1}) \leq n-1$, and thus $\mathrm{TC}_n(S^{2k+1}) = n-1$.

Now we prove that $\mathrm{TC}_n(S^{2k}) = n$. Take a generator $u \in H^{2k}(S^{2k})$ and consider the element

$$w = \left(\sum_{i=1}^{n-1} 1 \otimes \cdots 1 \otimes u(i\text{th place }) \otimes 1 \ldots \otimes 1 \right) - 1 \otimes \cdots 1 \otimes (n-1)u$$

Note that w is a zero-divisor class. Furthermore, $w^{\smile n} = (1-n)n!(u \otimes \cdots \otimes u)$ (since $\dim S^{2k}$ is even). Hence $\mathrm{TC}_n(S^{2k}) \leq n$ by the cup-length argument, and thus $\mathrm{TC}_n(S^{2k}) = n$ by the dimension-connectivity argument. $\square$

Note also the following fact.

Theorem 12.4 (Grant-Lupton-Oprea[GLO13]). *If* $\mathrm{TC}(X) = 1$ *then* $X \cong S^{2n+1}$.

Generally, for $n > 2$ we do not know if the equality $\mathrm{TC}_n(X) = n-1$ implies that $X \cong S^{2k+1}$. This is true for many cases (for example, if X is a simply connected space), but it is an open question in general.

Open Problem 12.5. *Does the equaility* $\mathrm{TC}_n(X) = n-1$ *imply the homotopy equivalence* $X \cong S^{2k+1}$*?*

13 Surfaces

In this section, for brevity we write xy for $x \smile y$ for $x, y \in H^*(X)$.

For orientable closed surface, we have the following facts:

- $\mathrm{TC}(S^2) = 2$
- $\mathrm{TC}(T^2) = \operatorname{cat} T^2 = 2$, since T^2 is a group.
- $\mathrm{TC}(S_g) = 4$ provided S_g is a closed orientable surface of genus $g > 1$ [Fa08]. Indeed, take $a_1, a_2, b_1, b_2 \in H^1(S_g)$ such that

  $$a_1a_2 = b_1b_2 = a_1b_2 = a_2b_1 = a_1^2 = a_2^2 = b_1^2 = b_2^2 = 0$$

  and that $a_1b_1 = a_2b_2 \in H^2(S_g) = \mathbb{Z}$ is a non-zero element. Now, we can see the non-zero product of zero-divisors

  $$\prod_{i=1}^{2}(a_i \otimes 1 - 1 \otimes a_i)(b_i \otimes 1 - 1 \otimes b_i).$$

  Hence, $\mathrm{TC}(S_g) \geq 4$, and we get the equality $\mathrm{TC}(S_g) \leq 4$ by the dimension-connectivity relation.

Now consider non-orientable surfaces N_g ($N_1 = \mathbb{R}P_2$, N_2 is the Klein bottle).

First, $\mathrm{TC}(\mathbb{R}P^2) = 3$. Indeed, we have $H^2(\mathbb{R}P^2) = \mathbb{Z}/2[u]/u^3 = 0$, and $u \otimes 1 + 1 \otimes u$ is a zero-divisor in $H^*(\mathbb{R}P^2 \times RP^2; \mathbb{Z}/2)$. Now

$$(u \otimes 1 + 1 \otimes u)^3 = u^2 \otimes u + u \otimes u^2 \neq 0.$$

So, $\mathrm{TC}(\mathbb{R}P^2) \geq 3$. Now we can see the inequality $\mathrm{TC}(\mathbb{R}P^2) \leq 3$ geometrically, [Fa08].

We do not know $\mathrm{TC}(N_g)$ for $g = 2, 3$. (It is clear, however, that $3 \leq \mathrm{TC}(N_g) \leq 4$). Recently it was proved (Dranishnikov [Dr15]) that $\mathrm{TC}(N_g) = 4$ for $g > 3$.

Concerning TC_n.

We have $\mathrm{TC}_n(S^2) = n$

We have $\mathrm{TC}_n(T^2) = \operatorname{cat}((T^2)^{n-1}) = 2n - 2$, since T^2 is a group.

We note the following surprising fact:

Theorem 13.1 ([GGGL15])**.** *If $n > 2$ then $\mathrm{TC}_n(F) = 2n$ for all other surfaces F, no matter whether F is orientable or not.*

Why surprising? Two exciting moments. First, $\mathrm{TC}(\mathbb{R}P^2) = 3$ while $\mathrm{TC}_n(\mathbb{R}P^2) = 2n$ for $n > 2$. Second, for the Klein bottle K (and $K\#\mathbb{R}P_2$), we do not know the value of $\mathrm{TC}(K)$ while we know that $\mathrm{TC}_n(K) = 2n$ for $n > 2$.

14 Some high-dimensional examples

Theorem 14.1 ([BGRT14]). *For any path-connected space X and positive integers n and k we have $\mathrm{cl}(\zeta_{(n,X\times S^k)} \geq \mathrm{cl}(\zeta_{n,X}) + n - 1$. This inequality can be improved to $\mathrm{cl}(\zeta_{(n,X\times S^k)} \geq \mathrm{cl}(\zeta_{n,X}) + n$ provided k is even and $H^*(X)$ is torsion-free.*

Corollary 14.2. $\mathrm{TC}_n(S^{k_1} \times S^{k_2} \times \cdots \times S^{k_m}) = m(n-1) + l$ *where l is the number of even-dimensional spheres.*

Proof. This follows from theorems 6.1 and 14.1. □

Corollary 14.3. $\mathrm{TC}_n(T^k) = k(n-1)$.

Proof. This is a consequence of either Corollary 14.2 or Theorem 7.6. □

Theorem 14.4 ([BGRT14]). *Let X be a CW complex of finite type, and R a principal ideal domain. Take $u \in H^d(X;R)$ with $d > 0$, d even, and assume that the n-fold iterated self R-tensor product*

$$u^m \otimes \cdots \otimes u^m \in (H^{md}(X;R))^{\otimes n}$$

is an element of infinite additive order. Then $\mathrm{TC}_n(X) \geq mn$.

Corollary 14.5. *For every closed simply connected symplectic manifold M^{2m} we have $\mathrm{TC}_n(M) = nm$.*

Note also the following nice and interesting result.

Theorem 14.6 (Farber-Tabachnikov-Yuzvinsky [FTY03]). *For $n \neq 1, 3, 7$ the number $\mathrm{TC}(\mathbb{R}P^n)$ is the smallest k such that the $\mathbb{R}P^n$ admits an immersion into $\mathbb{R}^k$. Furthermore, for $n = 1, 3, 7$ we have $\mathrm{TC}(\mathbb{R}P^n) = n$.*

15 The sequence $\{\mathrm{TC}_n(X)\}_{n=2}^{\infty}$ as an invariant of X

When the concept of higher topological complexity appeared, the following question arose: Do the invariants $\mathrm{TC}_n(X)$ give us really more information on X than $\mathrm{TC}(X)$? In other words, is it true or not that the sequence $\{\mathrm{TC}_n(X)\}$ is completely determined by $\mathrm{TC}(X)$? The following example shows that the sequenence $\{\mathrm{TC}_n(X)\}$ contains more information of X than merely $\mathrm{TC}(X)$.

Example 15.1. We have

$$\mathrm{TC}(S^2) = \mathrm{TC}(T^2) = 2, \quad \mathrm{TC}_n(S^2) = n, \quad \mathrm{TC}_n(T^2) = 2n - 2.$$

More generally, what can we say about the behavior of the sequence $\{\mathrm{TC}_n\,(X)\}$? As an example, we note the following fact.

Proposition 15.2. *For every CW space X of finite type, the sequence $\{\mathrm{TC}_n(X)\}$ has linear growth with respect to n.*

Proof. This follows from of the inequalities $\mathrm{TC}_n(X) \le \mathrm{cat}(X^n) \le n\,\mathrm{cat}(X)$. □

Given X, we (can) introduce the power series $\sum_{n=0}^{\infty} \mathrm{TC}_{n+2}(X)z^n$ and ask about analytical properties of them.

Example 15.3. For $X = S^{2k+1}$ we have

$$\sum_{n=0}^{\infty} \mathrm{TC}_{n+2}(S^{2k+1})z^n = \sum_{n=0}^{\infty}(n+1)z^n = (1-z)^{-2} + 2(1-z)^{-1}.$$

Generally, we have the following fact:

Proposition 15.4. *For every CW space X of finite type, the radius of convergence of the series $\sum \mathrm{TC}_{n+2}(X)z^n$ is equal to 1.*

Proof. Put $a_n = \mathrm{TC}_n(X)$ and note that $a_n \le \mathrm{cat}\,X^n \le n\,\mathrm{cat}\,X$. Hence $a_n/n \le \mathrm{cat}\,X$. Hence the upper limit $\overline{\lim}(a_n/n)$ exists, and it is positive because $\{a_n\}$ is an increasing sequence of positive numbers. This implies that

$$\overline{\lim}\frac{a_n}{a_{n+1}} = \lim\frac{n+1}{n} = 1. \qquad \square$$

And an open question.

Open Problem 15.5. *Do the power series*

$$\sum \mathrm{cat}(X^n)z^n \quad \textit{and} \quad \sum \mathrm{TC}_{n+2}(X)z^n$$

represent rational functions?

16 Monoidal topological complexity

Consider robot motion planning with the following property: if the initial position of a robot in the configuration space X coincides with the terminal position, then the algorithm keeps the robot still. This leads to the notion of monoidal topological complexity, [IS10, IS12].

Definition 16.1. For a CW space X, the *monoidal topological complexity* $\mathrm{TC}^M(X)$ is the least number m such that there exists a cover of $X \times X$ by $m+1$ open subsets $A_i, i = 0, 1, \dots m$ and, for each A_i, a local section $s_i : U_i \to PX$ for $\zeta_X = \{\pi : X^I \to X \times X\}$ with the following property: $s_i(x, x)$ is the constant path at x for all $x \in X$.

Remark 16.2. Iwase and Sakai [IS10] require that each A_i contains the diagonal $d(X) \subset X \times X$. However, their definition agrees with ours, cf. [Dr14, p.1].

Open Problem 16.3. *Is it true that* $\mathrm{TC}^M(X) = \mathrm{TC}(X)$ *for all* X*?*

In fact, Iwase and Sakai proclaimed the equality $\mathrm{TC}^M(X) = \mathrm{TC}(X)$ in [IS10] and then withdrew the clam in [IS12].

Proposition 16.4. *For any CW space we have* $\mathrm{TC}(X) \le \mathrm{TC}^M(X) \le \mathrm{TC}(X) + 1$.

Proof. See [IS12, Dr14] □

Theorem 16.5. *The equality* $\mathrm{TC}(X) = \mathrm{TC}^M(X)$ *holds true for all* k*-connected simplicial complexes* X *with*

$$(k+1)(\mathrm{TC}(X) + 2) \ge \dim X + 1.$$

Proof. [Dr14, Theorem 2.5]. □

Note also the equalities $\mathrm{TC}(X) = \mathrm{TC}^M(X)$ is X is a sphere S^n or a connected Lie group G, [Dr14].

The following theorem refines Theorem 7.7.

Theorem 16.6 (Dranishnikov [Dr14])**.** Let X, Y be two absolute neighborhood retracts. Then

$$\begin{aligned}\max\{\,&\mathrm{TC}(X), \mathrm{TC}(Y), \mathrm{cat}(X \times Y) \le \mathrm{TC}(X \vee Y) \le \mathrm{TC}^M(X \vee Y)\\ &\le \mathrm{TC}^M(X) + \mathrm{TC}^M(Y) + 1 \le \mathrm{TC}(X) + \mathrm{TC}(Y) + 3.\end{aligned}$$

17 Symmetric topological complexity

This section is an extract from [BGRT14].

In discussing on robotics, it is natural to consider motion planning so that a path α from A to B is equal to the inverse one α^{-1} of the path from B to A. This leads to symmetric version(s) of topological complexity.

We discuss two symmetric versions of TC_n. One of them, TC_n^{Σ}, has the advantage of being a homotopy invariant. The other, TC_n^S, is better for calculations and is a natural generalization of the symmetric topological complexity studied by Farber and Grant in [FG07]. We begin with the $n = 2$ case of the homotopically well-behaved version.

Consider the involutions $\tau : X \times X \to X \times X$ and $\overline{\tau} : X^I \to X^I$ defined by $\tau(x, y) = (y, x)$ and $\overline{\tau}(\alpha)(t) = \alpha(1 - t)$, for $(x, y) \in X \times X$ and $\alpha \in X^I$. We work with symmetric subsets $A \subseteq X \times X$ (i.e. those for which $\tau A = A$), and equivariant maps $s : A \to X^I$ (i.e. those satisfying $\overline{\tau}(s(a)) = s(\tau(a))$ for all $a \in A$).

Definition 17.1. $\mathrm{TC}^{\Sigma}(X)$ is the least integer k such that $X \times X = A_0 \cup A_1 \cup \cdots \cup A_k$ where each A_i is open, symmetric, and admits a continuous equivariant section $s_i : A_i \to X^I$ of the map e_2.

To define Farber–Grant symmetric complexity TC^S, consider the subspace

$$C_2(X) = X^2 \setminus d(X) \subset X \times X$$

of ordered pairs of distinct points in X. The map $\pi : X^I \to X \times X$ restricts to a map

$$\pi' : \pi^{-1}(C_2(X)) \to C_2(X)$$

that is a $\mathbb{Z}/2$-equivariant map with free $\mathbb{Z}/2$-actions on its domain and range. So, the quotient map

$$\varepsilon_2 := \pi'/(\mathbb{Z}/2) : (\pi^{-1}(C_2(X)))/(\mathbb{Z}/2) \to C_2(X)/(\mathbb{Z}/2)$$

is a fibration.

Definition 17.2. $\mathrm{TC}_2^S(X) = 1 + \mathrm{secat}(\varepsilon_2)$.

Proposition 17.3. *For each ENR we have*

$$\mathrm{TC}_2^S(X) - 1 \le \mathrm{TC}^{\Sigma}(X) \le \mathrm{TC}_2^S(X).$$

Example 17.4. For X contractible and not a point we have $\mathrm{TC}_2(X) = \mathrm{TC}^{\Sigma}(X) = 0$ while $\mathrm{TC}_2^S(X) = 1$. In particular, TC_2^S is not a homotopy invariant.

Example 17.5. The numbers $\mathrm{TC}_2^S(S^k)$ and $\mathrm{TC}_2(S^k)$ have been computed in [FG07, Corollary 18] and [Fa03], respectively. Here we use the inequalities $\mathrm{TC}_2 \le \mathrm{TC}^{\Sigma} \le \mathrm{TC}^S$ together with the fact that $\mathrm{TC}_2^S(S^k) = 2 = \mathrm{TC}_2(S^{2k})$ to deduce $\mathrm{TC}^{\Sigma}(S^{2k}) = \mathrm{TC}_2^S(S^{2k}) = 2$ for all k. On the other hand, since $\mathrm{TC}_2(S^{2k+1}) = 1$, the above argument only gives $1 \le \mathrm{TC}^{\Sigma}(S^{2k+1}) \le \mathrm{TC}_2^S(S^{2k+1}) = 2$. Incidentally, note that the construction in [Fa08, Example 4.8] gives an open covering $S^{2k+1} \times S^{2k+1} = A_0 \cup A_1$ by symmetric sets A_i, and continuous sections of e_2 over each A_i, $i = 0, 1$. However, one of these sections is not equivariant, which prevents us from deducing $\mathrm{TC}^{\Sigma}(S^k) = 1$.

Open Problem 17.6. *Evaluate* $\mathrm{TC}^{\Sigma} S^1$.

Remark 17.7. We do not know of an example with $\mathrm{TC} \neq \mathrm{TC}^{\Sigma}$.

We next define higher analogues of TC^{Σ}. Recall that for a given n, the symmetric group Σ_n acts on the right of X^n and X^{J_n} by permuting coordinates and paths, respectively. Further, the fibration e_n in Definition 5.1 is Σ_n-equivariant. We now work with symmetric subsets $A \subseteq X^n$ (i.e. those for which $A\sigma = A$ for all $\sigma \in \Sigma_n$), and equivariant maps $s : A \to X^{J_n}$ (i.e. those satisfying $s(a)\sigma = s(a\sigma)$ for all $a \in A$ and $\sigma \in \Sigma_n$). Definition 17.1 can now be extended to:

Definition 17.8. $\mathrm{TC}_n^{\Sigma}(X)$ is the least integer k such that $X^n = A_0 \cup A_1 \cup \cdots \cup A_k$ where each A_i is open, symmetric and admits a continuous equivariant section $s_i : A_i \to X^{J_n}$ for the map e_n.

Theorem 17.9. *If the spaces X and Y are homotopy equivalent then* $\mathrm{TC}_n^{\Sigma}(X) = \mathrm{TC}_n^{\Sigma}(Y)$.

Now we present the higher analog of TC^S. Let $C_n(X)$ stand for configuration space of n ordered distinct points in X. The symmetric group Σ_n acts on X^{J_n} and X^n in an obvious way, and $e_n : X^{J_n} \to X^n$ is an Σ_n-equivariant map. The Σ_n-actions are free on both domain and range of e_n. Thus, at the level of orbit spaces we get a fibration

$$\varepsilon_n^X = \varepsilon_n : Y_n(X) \to C_n(X)/\Sigma_n$$

where $Y_n := e_n^{-1}(C_n(X)/\Sigma_n$.

Theorem 17.10. *If X is an ENR then*

$$\mathrm{secat}(\varepsilon_n) \leq \mathrm{TC}_n^{\Sigma}(X) \leq \mathrm{secat}(\varepsilon_n) + \cdots + \mathrm{secat}(\varepsilon_2) + n - 1.$$

In view of previous inequality, $\mathrm{TC}_2^S(X) - 1 \leq \mathrm{TC}^{\Sigma}(X) \leq \mathrm{TC}_2^S(X)$. It hints the following definition:

Definition 17.11. For $n \geq 2$ set

$$\mathrm{TC}_n^S(X) = \mathrm{secat}(\varepsilon_n) + \cdots + \mathrm{secat}(\varepsilon_2) + n - 1.$$

18 Topological complexity in presence of group actions

When an interesting topological concept appears, people consider topological groups G and do G-equivariant (G-invariant) generalization of the concept. Topological Complexity is not an exception. I am not able to discuss here different G-versions of Topological Complexity; the interested reader is referred to the papers [BK15, CG12, LM15].

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