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Hochschild homology and cohomology for involutive A_∞-algebras

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Abstract

We present a study of the homological algebra of bimodules over A_∞-algebras endowed with an involution. Furthermore we introduce a derived description of Hochschild homology and cohomology for involutive A_∞-algebras.

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1 Introduction

Hochschild homology and cohomology are homology and cohomology theories developed for associative algebras which appears naturally when one studies its deformation theory. Furthermore, Hochschild homology plays a central role in topological field theory in order to describe the closed states part of a topological field theory.

An involutive version of Hochschild homology and cohomology was developed by Braun in [1] by considering associative and A_∞-algebras endowed with an involution and morphisms which commute with the involution.

This paper takes a step further with regards to [5]. Whilst in the latter paper we develop the homological algebra required to give a derived version of Braun's involutive Hochschild homology and cohomology for

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involutive associative algebras, this research is devoted to develop the machinery required to give a derived description of involutive Hochschild homology and cohomology for A_∞-algebras endowed with an involution.

As in [5], this research has been driven by the author's research on Costello's classification of topological conformal field theories [2], where he proves that an open 2-dimensional theory is equivalent to a Calabi-Yau A_∞-category. In [4], the author extends the picture to unoriented topological conformal field theories, where open theories now correspond to involutive Calabi-Yau A_∞-categories, and the closed state space of the universal open-closed extension turns out to be the involutive Hochschild chain complex of the open state algebra.

2 Basic concepts

2.1 Coalgebras and bicomodules

An *involutive graded coalgebra* over a field $\mathbb{K}$ is a graded $\mathbb{K}$-vector space C endowed with a coproduct $\Delta : C \to C \otimes_{\mathbb{K}} C$ of degree zero together with a counit $\varepsilon : C \to \mathbb{K}$ and an involution $\star : C \to C$ such that:

1. The graded $\mathbb{K}$-vector space C is coassociative and counital, and
2. the involution and Δ are compatible, therefore: $\Delta(c^\star) = (\Delta(c))^\star$, for $c \in C$, where the involution on $C \otimes_{\mathbb{K}} C$ is given by the following expression: $(c_1 \otimes c_2)^\star = (-1)^{|c_1||c_2|} c_2^\star \otimes c_1^\star$, for $c_1, c_2 \in C$.

An *involutive coderivation* on an involutive coalgebra C is a map $L : C \to C$ preserving involutions and making the following diagram commutative:

$$\begin{tikzcd} C \arrow[r, "L"] \arrow[d, "\Delta"'] & C \arrow[d, "\Delta"] \\ C \otimes_{\mathbb{K}} C \arrow[r, "L\otimes \mathrm{Id}_C + \mathrm{Id}_C \otimes L"'] & C \otimes_{\mathbb{K}} C \end{tikzcd}$$

Denote with $\mathrm{iCoder}(-)$ the spaces of coderivations of involutive coalgebras. Observe that $\mathrm{iCoder}(-)$ are Lie subalgebras over $\mathrm{Coder}(-)$ whose Lie bracket is given by the commutator $[n, -]$.

An *involutive differential graded coalgebra* is an involutive coalgebra C equipped with an involutive coderivation $b : C \to C$ of degree -1 such that $b^2 = b \circ b = 0$.

A morphism between two involutive coalgebras C and D is a graded map $C \xrightarrow{f} D$ compatible with the involutions which makes the following diagram commutative:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes_{\mathbb{K}} C \arrow[r, "f\otimes f"'] & D \otimes_{\mathbb{K}} D \end{tikzcd} \tag{1}$$

Example 2.1.1. Let us suppose that A is an associative $\mathbb{K}$-algebra endowed with an involution. An involutive $\mathbb{K}$-bimodule M is a $\mathbb{K}$-bimodule M equipped with an involution satisfying the following condition $(a \cdot m)^\star = m^\star \cdot a^\star$.

For an involutive graded $\mathbb{K}$-bimodule A, we define the cotensor coalgebra of A as

$$TA = \bigoplus_{n\geq 0} A^{\otimes_{\mathbb{K}} n}.$$

We define an involution in $A^{\otimes_{\mathbb{K}} n}$ by stating:

$$(a_1 \otimes \cdots \otimes a_n)^\star := (-1)^{\sum_{i=1}^n |a_i|(\sum_{j=i+1}^n |a_j|)}(a_n^\star \otimes \cdots \otimes a_1^\star).$$

We can endow TA with a coproduct as follows:

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n).$$

Observe that Δ commutes with the involution.

For a given (involutive) graded algebra A, $\mathrm{Hom}_{A-\mathbf{iBimod}}(-,-)$ and $\mathrm{iCoder}(-)$ will denote the spaces of involutive homomorphisms and coderivations of involutive A-bimodules respectively. We will write $\mathrm{Hom}_{A-\mathbf{Bimod}}(-,-)$ for the space of homomorphisms of A-bimodules.

Let us think of A as a (involutive) bimodule over itself. We denote the suspension of A by SA and define it as the graded (involutive) $\mathbb{K}$-bimodule with $SA_i = A_{i-1}$. Given such a bimodule A, we define the following morphism of degree -1 induced by the identity $s : A \to SA$ by $s(a) = a$.

Proposition 2.1. *Let us define* $\mathrm{Bar}(A) := A \otimes_{\mathbb{K}} TSA \otimes_{\mathbb{K}} A$*. For an involutive graded algebra A, the following canonical isomorphism of complexes holds:*

$$\mathrm{iCoder}\,(TSA) \cong \mathrm{Hom}_{A-\mathbf{iBimod}}\,(\mathrm{Bar}(A), A)\,,$$

where the involution we endow $\mathrm{Bar}(A)$ *with is the following:*

$$(a_0 \otimes \cdots \otimes a_{n+1})^\star = a_{n+1}^\star \otimes \cdots \otimes a_0^\star.$$

Proof. The proof follows the arguments in Proposition 4.1.1 [5], where we show the result for the non-involutive setting in order to restrict to the involutive one.

The degree $-n$ part of $\mathrm{Hom}_{A-\mathbf{Bimod}}(\mathrm{Bar}(A), A)$ is the space of degree $-n$ linear maps $TSA \to A$, which is isomorphic to the space of degree $(-n-1)$ linear maps $TSA \to SA$. By the universal property of the tensor coalgebra, there is a bijection between degree $(-n-1)$ linear maps $TSA \to SA$ and degree $(-n-1)$ coderivations on TSA. Hence the degree n part of $\mathrm{Hom}_{A-\mathbf{Bimod}}(\mathrm{Bar}(A), A)$ is isomorphic to the degree n part of $\mathrm{Coder}\,(TSA)$. One checks directly that this isomorphism restricts to an isomorphism of graded vector spaces

$$\mathrm{Hom}_{A-\mathbf{iBimod}}(\mathrm{Bar}(A), A) \cong \mathrm{iCoder}\,(TSA)\,.$$

Finally, one can check that the differentials coincide under the above isomorphism, cf. Section 12.2.4 [7]. □

Remark 2.2. Proposition 2.1 allows us to think of a coderivation on the coalgebra TSA as a map $TA \to A$. Such a map $f : TA \to A$ can be described as a collection of maps $\{f_n : A^{\otimes n} \to A\}$ which will be called the components of f.

If b is a coderivation of degree -1 on TA with $b_n : A^{\otimes_{\mathbb{K}} n} \to A$, then b^2 becomes a linear map of degree -2 with

$$b_n^2 = \sum_{i+j=n+1} \sum_{k=0}^{n-1} b_i \circ \left(\mathrm{Id}^{\otimes k} \circ b_j \circ \mathrm{Id}^{\otimes(n-k-j)}\right).$$

The coderivation b will be a differential for TA if, and only if, all the components b_n^2 vanish.

Lemma 2.1.2 (cf. Lemma 1.3 [6])**.** *If* $b_k : (SA)^{\otimes_{\mathbb{K}} k} \to SA$ *is an involutive linear map of degree* -1*, we define* $m_k : A^{\otimes_{\mathbb{K}} k} \to A$ *as* $m_k = s^{-1} \circ b_k \circ s^{\otimes_{\mathbb{K}} k}$*. Under these conditions:*

$$b_k(sa_1 \otimes \cdots \otimes sa_k) = \sigma m_k(a_1 \otimes \cdots \otimes a_k),$$

where $\sigma := (-1)^{(k-1)|a_1|+(k-2)|a_2|+\cdots+2|a_{k-2}|+|a_{k-1}|+\frac{k(k-1)}{2}}$.

Proof. The proof follows the arguments of Lemma 1.3 [6]. We only need to observe that the involutions are preserved as all the maps involved in the proof are assumed to be involutive. □

Let $\overline{m}_k := \sigma m_k$, then we have $b_k(sa_1 \otimes \cdots \otimes sa_k) = \overline{m}_k(a_1 \otimes \cdots \otimes a_k)$.

Proposition 2.3. *Given an involutive graded $\mathbb{K}$-bimodule A, let $\epsilon_i = |a_1| + \cdots + |a_i| - i$ for $a_i \in A$ and $1 \leq i \leq n$. A boundary map b on TSA is given in terms of the maps $\overline{m}_k$ by the following formula:*

$$\begin{aligned} b_n(sa_1 \otimes \cdots \otimes sa_n) = & \sum_{k=0}^{n} \sum_{i=1}^{n-k+1} (-1)^{\epsilon_{i-1}} (sa_1 \otimes \cdots \\ & \cdots \otimes sa_{i-1} \otimes \overline{m}_k(a_i \otimes \cdots \otimes a_{i+k-1}) \otimes \cdots \otimes sa_n). \end{aligned}$$

Proof. This proof follows the arguments of Proposition 1.4 [6]. The only detail that must be checked is that b_n preserves involutions:

$$\begin{aligned} & b_n((sa_1 \otimes \cdots \otimes sa_n)^\star) \\ & = \sum_{j,k} \pm (sa_n^\star \otimes \cdots \otimes sa_j^\star \otimes \overline{m}_k(a_{j-1}^\star \otimes \cdots \otimes a_{j-k+1}^\star) \otimes \cdots \otimes sa_1^\star) \\ & = \sum_{j,k} \pm (sa_1 \otimes \cdots \otimes \overline{m}_k(a_{j-k+1} \otimes \cdots \otimes a_{j-1}) \otimes sa_j \otimes \cdots \otimes sa_n)^\star \\ & = (b_n(sa_1 \otimes \cdots \otimes sa_n))^\star. \end{aligned}$$

□

Given an involutive coalgebra C with coproduct Δ^C and counit ε, for an involutive graded vector space P, a *left coaction* is a linear map $\Delta^L : P \to C \otimes_{\mathbb{K}} P$ such that

1. $(\mathrm{Id} \otimes \Delta^C) \circ \Delta^L = (\Delta^C \otimes \mathrm{Id}) \circ \Delta^L$;
2. $(\mathrm{Id} \otimes \varepsilon) \circ \Delta^L = \mathrm{Id}$.

Analagously we introduce the concept of *right coaction.*

Given an involutive coalgebra $(C, \Delta^C, \varepsilon)$ with involution $\star$ we define an *involutive C-bicomodule* as an involutive graded vector space P with involution $\dagger$, a left coaction $\Delta^L : P \to C \otimes_{\mathbb{K}} P$ and a right coaction $\Delta^R : P \to P \otimes_{\mathbb{K}} C$ which are compatible with the involutions, that is the diagrams below commute:

(2)
$$\begin{tikzcd} P \arrow[r, "(-)^\star"] \arrow[d, "\Delta^L"'] & P \arrow[d, "\Delta^R"] \\ C \otimes_{\mathbb{K}} P \arrow[r, "(-,-)^\star"'] & P \otimes_{\mathbb{K}} C \end{tikzcd}$$

(3)
$$\begin{tikzcd} P \arrow[r, "\Delta^L"] \arrow[d, "\Delta^R"'] & C \otimes P \arrow[d, "\mathrm{Id}_C \otimes \Delta^R"] \\ P \otimes_{\mathbb{K}} C \arrow[r, "\Delta^L \otimes \mathrm{Id}_C"'] & C \otimes_{\mathbb{K}} P \otimes_{\mathbb{K}} C \end{tikzcd}$$

Here
$$\begin{array}{rccc} (-,-)^\star : & C \otimes_{\mathbb{K}} P & \to & P \otimes_{\mathbb{K}} C \\ & c \otimes p & \mapsto & p^\dagger \otimes c^\star. \end{array}$$

For two involutive C-bicomodules (P_1, Δ_1) and (P_2, Δ_2), a morphism $P_1 \xrightarrow{f} P_2$ is defined as an involutive morphism making diagrams below commute:

(4)
$$\begin{tikzcd} P_1 \arrow[r, "\Delta_1^L"] \arrow[d, "f"'] & C \otimes_{\mathbb{K}} P_1 \arrow[d, "\mathrm{Id}_C \otimes f"] \\ P_2 \arrow[r, "\Delta_2^L"'] & C \otimes_{\mathbb{K}} P_2 \end{tikzcd}$$

(5)
$$\begin{tikzcd} P_1 \arrow[r, "\Delta_1^R"] \arrow[d, "f"'] & P_1 \otimes_{\mathbb{K}} C \arrow[d, "\mathrm{Id}_C \otimes f"] \\ P_2 \arrow[r, "\Delta_2^R"'] & P_2 \otimes_{\mathbb{K}} C \end{tikzcd}$$

2.2 A_∞-algebras and A_∞-quasi-isomorphisms

An *involutive A_∞-algebra* is an involutive graded vector space A endowed with involutive morphisms

(6)
$$b_n : (SA)^{\otimes_{\mathbb{K}} n} \to SA,\ n \geq 1,$$

of degree $n-2$ such that the identity below holds:

(7)
$$\sum_{i+j+l=n} (-1)^{i+jl} b_{i+1+l} \circ (\mathrm{Id}^{\otimes i} \otimes b_j \otimes \mathrm{Id}^{\otimes l}) = 0,\ \forall n \geq 1.$$

An (involutive) A_∞-algebra A is called *strictly unital* if there exists an element $1_A \in A^0$ which is a unit for b_2, satisfying the following conditions $b_n(a_1, \dots, 1, \dots, a_n) = 0$ if $n \neq 2$ and $1_A^\star = 1_A$. If the map $b_0 : \mathbb{K} \to SA$ is non trivial, then we say that A is a *curved* A_∞-algebra, it will be called *flat* otherwise.

Remark 2.4. 1. It is a straight computation to check that condition (7) says, in particular, that $b_1^2 = 0$.

2. Observe that when one applies (7) we need to take care of signs due to Koszul sign rule:

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(x).$$

Example 2.2.1. 1. The concept A_∞-algebra is a generalization for that of a differential graded algebra. Indeed, if the maps $b_n = 0$ for $n \geq 3$ then A is a differential $\mathbb{Z}$-graded algebra and conversely an A_∞-algebra A yields a differential graded algebra if we require $b_n = 0$ for $n \geq 3$.

2. The definition of A_∞-algebra was introduced by Stasheff whose motivation was the study of the graded abelian group of singular chains on the based loop space of a topological space.

For an involutive A_∞-algebra (A, b^A), the *involutive bar complex* is the involutive differential graded coalgebra TSA, endowed with a coderivation defined by $b_i = s^{-1} \circ m_i \circ s^{\otimes_K i}$ (cf. Definition 1.2.2.3 [9]).

Given two involutive A_∞-algebras C and D, a *morphism of involutive A_∞-algebras* $f : C \to D$ is an involutive morphism of degree zero between the associated involutive differential graded coalgebras $TSC \to TSD$.

It follows from Proposition 2.1 that the definition of an involutive A_∞-algebra can be summarized by saying that it is an involutive graded $\mathbb{K}$-vector space A equipped with an involutive coderivation on $\mathrm{Bar}(A)$ of degree -1.

Remark 2.5. It follows from [1], Definition 2.8, we have that a morphism of involutive A_∞-algebras $f : C \to D$ can be given by a series of involutive homogeneous maps of degree zero

$$f_n : (SC)^{\otimes_{\mathbb{K}} n} \to SD,\ n \geq 1,$$

such that

$$\sum_{i+j+l=n} f_{i+l+1} \circ \left(\mathrm{Id}_{SC}^{\otimes i} \otimes b_j \otimes \mathrm{Id}_{SC}^{\otimes l} \right) = \sum_{i_1+\cdots+i_s=n} b_s \circ (f_{i_1} \otimes \cdots \otimes f_{i_s}). \tag{8}$$

The composition $f \circ g$ of two morphisms of involutive A_∞-algebras is given by

$$(f \circ g)_n = \sum_{i_1+\cdots+i_s=n} f_s \circ (g_{i_1} \otimes \cdots \otimes g_{i_s});$$

the identity on SC is defined as $f_1 = \mathrm{Id}_{SC}$ and $f_n = 0$ for $n \geq 2$.

The condition of being involutive means that the following identity holds:

$$f_n(c_1, \ldots, c_n)^\star = \sigma f_n(c_n^\star, \ldots, c_1^\star),$$

where $\sigma := (-1)^{\sum_{i=1}^n |c_i|(\sum_{j=i+1}^n |c_j|)}(-1)^{\frac{n(n+1)}{2}-1}$ (see [1], Definition 2.7).

For an involutive A_∞-algebra A, we define its associated homology algebra $\mathrm{H}_\bullet(A)$ as the homology of the differential b_1 on A, that is: $\mathrm{H}_\bullet(A) = \mathrm{H}_\bullet(A, b_1)$.

Remark 2.6. Endowed with b_2 as multiplication, the homology of an A_∞-algebra A is an associative graded algebra, whereas A is not usually associative.

Let $f : A_1 \to A_2$ be a morphism of involutive A_∞-algebras with components f_n; we note that for $n = 1$, f_1 induces a morphism of algebras $\mathrm{H}_\bullet(A_1) \to \mathrm{H}_\bullet(A_2)$. We say that $f : A_1 \to A_2$ is an *A_∞-quasi-isomorphism* if f_1 is a quasi-isomorphism.

2.3 A_∞-bimodules

Let (A, b^A) be an involutive A_∞-algebra. An *involutive A_∞-bimodule* is a pair (M, b^M) where M is a graded involutive $\mathbb{K}$-vector space and b^M is an involutive differential on the $\mathrm{Bar}(A)$-bicomodule, whose involution will be introduced shortly:

$$\mathrm{B}(M) := \mathrm{Bar}(A) \otimes_{\mathbb{K}} SM \otimes_{\mathbb{K}} \mathrm{Bar}(A).$$

If $\star$ denotes the involution of $\mathrm{Bar}(A)$ and $\dagger$ is the involution for M, we can endow $\mathrm{B}(M)$ with the following involution:

$$(a_1, \ldots, a_n, m, a_1', \ldots, a_n')^\ddagger := ((a_1', \ldots, a_n')^\star, m^\dagger, (a_1, \ldots, a_n)^\star).$$

Let $\left(M, b^M\right)$ and $\left(N, b^N\right)$ be two involutive A_∞-bimodules. We define a *morphism of involutive A_∞-bimodules* $f : M \to N$ as a morphism of $\mathrm{Bar}(A)$-bicomodules $F : \mathrm{B}(M) \to \mathrm{B}(N)$ such that $b^N \circ F = F \circ b^M$.

Proposition 2.7 (cf. [6] Proposition 3.4)**.** *If $f : A_1 \to A_2$ is a morphism of involutive A_∞-algebras, then A_2 becomes an involutive bimodule over A_1.*

Remark 2.8 (Section 5.1 [8])**.** Let **iVect** be the category of involutive $\mathbb{Z}$-graded vector spaces and involutive morphisms. For an involutive A_∞-algebra A, involutive A-bimodules and their respective morphisms form a differential graded category. Indeed, following [8], Definition 5.1.5: let A be an involutive A_∞-algebra and let us define the category $\overline{A - \mathbf{iBimod}}$ whose class of objects are involutive A-bimodules and where $\mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(M,N)$ is:

$$\underline{\mathrm{Hom}}^n_{\mathbf{iVect}}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} SM \otimes_{\mathbb{K}} \mathrm{Bar}(A), \mathrm{Bar}(A)\otimes_{\mathbb{K}} SN \otimes_{\mathbb{K}} \mathrm{Bar}(A)).$$

Let us recall that

$$\underline{\mathrm{Hom}}^n_{\mathbf{iVect}}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} SM \otimes_{\mathbb{K}} \mathrm{Bar}(A), \mathrm{Bar}(A)\otimes_{\mathbb{K}} SN \otimes_{\mathbb{K}} \mathrm{Bar}(A))$$

is by definition the product over $i \in \mathbb{Z}$ of the morphism sets

$$\mathrm{Hom}_{\mathbf{iVect}}((\mathrm{Bar}(A)\otimes_{\mathbb{K}} SM \otimes_{\mathbb{K}} \mathrm{Bar}(A))^i, (\mathrm{Bar}(A)\otimes_{\mathbb{K}} SN \otimes_{\mathbb{K}} \mathrm{Bar}(A))^{i+n}).$$

The morphism

$$\begin{gathered}\underline{\mathrm{Hom}}^n_{\mathbf{iVect}}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} SM \otimes_{\mathbb{K}} \mathrm{Bar}(A), \mathrm{Bar}(A)\otimes_{\mathbb{K}} SN \otimes_{\mathbb{K}} \mathrm{Bar}(A)) \to \\ \underline{\mathrm{Hom}}^{n+1}_{\mathbf{iVect}}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} SM \otimes_{\mathbb{K}} \mathrm{Bar}(A), \mathrm{Bar}(A)\otimes_{\mathbb{K}} SN \otimes_{\mathbb{K}} \mathrm{Bar}(A))\end{gathered}$$

sends a family $\{f_i\}_{i\in\mathbb{Z}}$ to a family $\{b^N \circ f_i - (-1)^n f_{i+1} \circ b^M\}_{i\in\mathbb{Z}}$. Observe that the zero cycles in $\underline{\mathrm{Hom}}^\bullet_{\mathbf{iVect}}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} M \otimes_{\mathbb{K}} \mathrm{Bar}(A), \mathrm{Bar}(A)\otimes_{\mathbb{K}} N \otimes_{\mathbb{K}} \mathrm{Bar}(A))$ are precisely the morphisms of involutive A-bimodules. This morphism defines a differential, indeed: for fixed indices $i, n \in \mathbb{Z}$ we have

$$\begin{aligned} d^2(f_i) &= d\left(b^N f_i - (-1)^n f_{i+1} b^M\right) \\ &= b^N\left(b^N f_i - (-1)^n f_{i+1} b^M\right) - (-1)^{n+1}\left(b^N f_i - (-1)^n f_{i+1} b^M\right) b^M \\ &\overset{(!)}{=} -(-1)^n b^N f_{i+1} b^M - (-1)^{n+1} b^N f_{i+1} b^M = 0, \end{aligned}$$

where (!) points out the fact that $b^N \circ b^N = 0 = b^M \circ b^M$.

For a morphism $\phi \in \underline{\mathrm{Hom}}^n_{\mathbf{iVect}}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} M \otimes_{\mathbb{K}} \mathrm{Bar}(A), \mathrm{Bar}(A)\otimes_{\mathbb{K}} N \otimes_{\mathbb{K}} \mathrm{Bar}(A))$ and an element $x \in \mathrm{Bar}(A)\otimes_{\mathbb{K}} M \otimes_{\mathbb{K}} \mathrm{Bar}(A)$, the complex $\mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(M,N)$ becomes an involutive complex if we endowed it with the involution $\phi^\star(x) = \phi(x^\star)$.

The functor $\mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(M,-)$ sends an involutive A-bimodule F to the involutive $\mathbb{K}$-vector space $\mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(M,F)$ of involutive homomorphisms. Given a homomorphism $f : F \to G$, for

$$F, G \in \mathrm{Obj}\left(\overline{A - \mathbf{iBimod}}\right),$$

$\mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(M,-)$ sends f to the involutive map:

$$\begin{array}{rccc} f_\star : & \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(M,F) & \to & \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(M,G) \\ & \phi & \mapsto & f\circ\phi \end{array}.$$

We prove that $f_\star$ preserves involutions:

$$\begin{aligned}(f_\star\phi^\star)(x) &= (f\circ\phi^\star)(x) = f(\phi(x^\star)) \\ &= f((\phi(x))^\star) = (f(\phi(x)))^\star = (f_\star\phi(x))^\star.\end{aligned}$$

Let us introduce the functor $\mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(-,M)$, which sends an involutive homomorphism $f : F \to G$, for $F, G \in \mathrm{Obj}\left(\overline{A-\mathbf{iBimod}}\right)$, to

$$\begin{array}{rccc} \varphi : & \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(G,M) & \to & \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(F,M) \\ & \phi & \mapsto & \phi\circ f \end{array}$$

Let us check that the involution is preserved:

$$\varphi(\phi^\star)(x) = (\phi^\star\circ f)(x) = \phi(f(x)^\star) = \phi(f(x^\star)) = \varphi(\phi)(x^\star) = (\varphi(\phi))^\star(x).$$

Let A be an involutive A_∞-algebra and let $\left(M, b^M\right)$ and $\left(N, b^N\right)$ be involutive A-bimodules. For $f, g : M \to N$ involutive morphisms of A-bimodules, an A_∞-*homotopy* between f and g is an involutive morphism $h : M \to N$ of A-bimodules satisfying

$$f - g = b^N \circ h + h \circ b^M.$$

We say that two morphisms $u : M \to N$ and $v : N \to M$ of involutive A-bimodules are *homotopy equivalent* if $u \circ v \sim \mathrm{Id}_N$ and $v \circ u \sim \mathrm{Id}_M$.

3 The involutive tensor product

For an involutive A_∞-algebra A and involutive A-bimodules M and N, the involutive tensor product $M\widetilde{\boxtimes}_\infty N$ is the following object in $\mathbf{iVect}_{\mathbb{K}}$:

$$M\widetilde{\boxtimes}_\infty N := \frac{M\otimes_{\mathbb{K}} \mathrm{Bar}(A)\otimes_{\mathbb{K}} N}{(m^\star\otimes a_1\otimes\cdots\otimes a_k\otimes n - m\otimes a_1\otimes\cdots\otimes a_k\otimes n^\star)}.$$

Observe that, for an element of $M\widetilde{\boxtimes}_\infty N$ of the form $m\otimes a_1\otimes\cdots\otimes a_k\otimes n$, we have:

$$(m\otimes a_1\otimes\cdots\otimes a_k\otimes n)^\star = m^\star\otimes a_1\otimes\cdots\otimes a_k\otimes n = m\otimes a_1\otimes\cdots\otimes a_k\otimes n^\star.$$

Proposition 3.1. *For an involutive A_∞-algebra A and involutive A-bimodules M, N and L,* $\mathrm{Hom}_{\mathbf{iVect}}\left(M\widetilde{\boxtimes}_\infty N, L\right)$ *is isomorphic to*

$$\mathrm{Hom}_{\mathbf{iVect}}\left(\frac{M\otimes_{\mathbb{K}}\mathrm{Bar}(A)}{\sim}, \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(N,L)\right),$$

where in $M\otimes_{\mathbb{K}}\mathrm{Bar}(A)$: $(m\otimes a_1\otimes\cdots\otimes a_k)^\star = m^\star\otimes a_1\otimes\cdots\otimes a_k$, $\sim$ denotes the relation

$$m\otimes a_1\otimes\cdots\otimes a_k = m^\star\otimes a_1\otimes\cdots\otimes a_k$$

and $\frac{M\otimes_{\mathbb{K}}\mathrm{Bar}(A)}{\sim}$ has the identity map as involution.

Proof. Let $f : M\widetilde{\boxtimes}_\infty N \to L$ be an involutive map. We define:

$$\tau(f) := \tau_f \in \mathrm{Hom}_{\mathbf{iVect}}\left(\frac{M\otimes_{\mathbb{K}}\mathrm{Bar}(A)}{\sim}, \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(N,L)\right),$$

where $\tau_f(m\otimes a_1\otimes\cdots\otimes a_k) := \tau_f[m\otimes a_1\otimes\cdots\otimes a_k] \in \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(N,L)$. Finally, for $n\in N$ we define:

$$\tau_f[m\otimes a_1\otimes\cdots\otimes a_k](n) := f\left(m\otimes a_1\otimes\cdots\otimes a_k\otimes n\right).$$

We need to check that τ preserves the involutions, indeed:

$$\begin{aligned}\tau_{f^\star}[m\otimes a_1\otimes\cdots\otimes a_k](n) &= f^\star\left(m\otimes a_1\otimes\cdots\otimes a_k\otimes n\right) = \\ &= \left(f\left(m\otimes a_1\otimes\cdots\otimes a_k\otimes n\right)\right)^\star = (\tau_f)^\star[m\otimes a_1\otimes\cdots\otimes a_k](n).\end{aligned}$$

In order to see that τ is an isomorphism, we build an inverse. Let us consider an involutive map

$$\begin{array}{rccc} g_1 : & \frac{M\otimes_{\mathbb{K}}\mathrm{Bar}(A)}{\sim} & \to & \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(N,L) \\ & m\otimes a_1\otimes\cdots\otimes a_k & \mapsto & g_1[m\otimes a_1\otimes\cdots\otimes a_k] \end{array}$$

and define a map

$$\begin{array}{rccc} g_2 : & M\widetilde{\boxtimes}_\infty N & \to & L \\ & m\otimes a_1\otimes\cdots\otimes a_k\otimes n & \mapsto & g_1[m\otimes a_1\otimes\cdots\otimes a_k](n) \end{array}$$

We check that g_2 is involutive:

$$\begin{aligned} g_2((m\otimes a_1\otimes\cdots\otimes a_k\otimes n)^\star) &= g_2(m^\star\otimes a_1\otimes\cdots\otimes a_k\otimes n) = \\ &= g_1[m^\star\otimes a_1\otimes\cdots\otimes a_k](n) = (g_1[m\otimes a_1\otimes\cdots\otimes a_k])^\star(n) \\ &= (g_1[m\otimes a_1\otimes\cdots\otimes a_k](n))^\star = (g_2(m\otimes a_1\otimes\cdots\otimes a_k\otimes n))^\star. \end{aligned}$$

The rest of the proof is standard and follows the steps of Theorem 2.75 [10] or Proposition 2.6.3 [11]. $\square$

For an A-bimodule M, let us define $(-)\widetilde{\boxtimes}_\infty M$ as the covariant functor

$$\begin{array}{ccc} \overline{A-\mathbf{iBimod}} & \xrightarrow{(-)\widetilde{\boxtimes}_\infty M} & \overline{A-\mathbf{iBimod}} \\ B & \rightsquigarrow & B\widetilde{\boxtimes}_\infty M \end{array}.$$

This functor sends a map $B_1 \xrightarrow{f} B_2$ to $B_1\widetilde{\boxtimes}_\infty M \xrightarrow{f\widetilde{\boxtimes}_\infty \mathrm{Id}_M} B_2\widetilde{\boxtimes}_\infty M$.

The functor $(-)\widetilde{\boxtimes}_\infty M$ is involutive: let us consider an involutive map $f : B_1 \to B_2$ and its image under the tensor product functor, $g = f\widetilde{\boxtimes}_\infty \mathrm{Id}_M$. Hence:

$$g((b,a)^\star) = g(b^\star, a) = (f(b^\star), a) = (f(b), a)^\star = (g(b,a))^\star.$$

Given an involutive A_∞-algebra A, we say that an involutive A-bimodule F is *flat* if the tensor product functor

$$(-)\widetilde{\boxtimes}_\infty F : \overline{A-\mathbf{iBimod}} \to \overline{A-\mathbf{iBimod}}$$

is exact, that is: it takes quasi-isomorphisms to quasi-isomorphisms.

Lemma 3.0.1. *If P and Q are homotopy equivalent as involutive A_∞-bimodules then, for every involutive A_∞-bimodule M, the following quasi-isomorphism in the category of involutive A_∞-bimodules holds:*

$$P\widetilde{\boxtimes}_\infty M \simeq Q\widetilde{\boxtimes}_\infty M.$$

Proof. Let $f : P \leftrightarrows Q : g$ be a homotopy equivalence. It is clear that

$$h \sim k \Rightarrow h\widetilde{\boxtimes}_\infty \mathrm{Id}_M \sim k\widetilde{\boxtimes}_\infty \mathrm{Id}_M.$$

Therefore, we have:

$$\begin{array}{ccccc} P\widetilde{\boxtimes}_\infty M & \to & Q\widetilde{\boxtimes}_\infty M & \to & P\widetilde{\boxtimes}_\infty M \\ p\widetilde{\boxtimes} a & \mapsto & f(p)\widetilde{\boxtimes} a & \mapsto & g(f(p))\widetilde{\boxtimes} a \end{array}$$

and

$$\begin{array}{ccccc} Q\widetilde{\boxtimes}_\infty M & \to & P\widetilde{\boxtimes}_\infty M & \to & Q\widetilde{\boxtimes}_\infty M \\ q\widetilde{\boxtimes} a & \mapsto & g(q)\widetilde{\boxtimes} a & \mapsto & f(g(q))\widetilde{\boxtimes} a \end{array}$$

the result follows since $f \circ g \sim \mathrm{Id}_Q$ and $g \circ f \sim \mathrm{Id}_P$. □

Lemma 3.0.2. *Let A be an involutive A_∞-algebra. If P and Q are homotopy equivalent as involutive A-bimodules then, for every involutive A-bimodule M, the following quasi-isomorphism holds:*

$$\mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(P, M) \simeq \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(Q, M).$$

Proof. Consider $f : P \to Q$ a homotopy equivalence and let $g : Q \to P$ be its homotopy inverse. If $[-,-]$ denotes the homotopy classes of morphisms, then both f and g induce the following maps:

$$\begin{array}{rccc} f_\star : & [P,M] & \to & [Q,M] \\ & \alpha & \mapsto & \alpha \circ g \end{array}$$

$$\begin{array}{rccc} g_\star : & [Q,M] & \to & [P,M] \\ & \beta & \mapsto & \beta \circ f \end{array}$$

Now we have:

$$f_\star \circ g_\star \circ \beta = f_\star \circ \beta \circ f = \beta \circ g \circ f \sim \beta;$$

$$g_\star \circ f_\star \circ \alpha = g_\star \circ \alpha \circ g = \alpha \circ f \circ g \sim \alpha. \qquad \square$$

4 Involutive Hochschild (co)homology

4.1 Hochschild homology for involutive A_∞-algebras

We define the *involutive Hochschild chain complex* of an involutive A_∞-algebra A with coefficients in a involutive A-bimodule M as

$$C_\bullet^{\mathrm{inv}}(M,A) = M \widetilde{\boxtimes}_\infty \mathrm{B}(A).$$

The differential is the same given in Section 7.2.4 [8]. The involutive Hochschild homology of A with coefficients in M is

$$\mathrm{HH}_n(M,A) = \mathrm{H}C_n^{\mathrm{inv}}(M,A).$$

Lemma 4.1.1. *For an involutive flat strictly unital A_∞-algebra A and an involutive A-bimodule M, the following quasi-isomorphism holds:*

$$C_\bullet^{inv}(M,A) \simeq M \widetilde{\boxtimes}_\infty A.$$

Proof. The result follows from:

$$M \widetilde{\boxtimes}_\infty A \simeq M \widetilde{\boxtimes}_\infty \mathrm{B}(A) = C_\bullet^{\mathrm{inv}}(M,A).$$

Observe that we are using that there is a quasi-isomorphism, therefore a homotopy equivalence (Proposition 1.3.5.1 [9]), between $\mathrm{B}(A)$ and A (Proposition 2, Section 2.3.1 [3]). $\square$

4.2 Hochschild cohomology for involutive A_∞-algebras

The *involutive Hochschild cochain complex* of an involutive A_∞-algebra A with coefficients on an involutive A-bimodule M is defined as the $\mathbb{K}$-vector space $C^\bullet_{\mathrm{inv}}(A,M) := \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(\mathrm{B}(A),M)$, with the differential defined in section 7.1 of [8].

Proposition 4.1. *For an involutive A_∞-algebra A and an involutive A-bimodule M, we have the following quasi-isomorphism: $C^\bullet_{inv}(A,M) \simeq \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(A,M)$.*

Proof. The result follows from:

$$\begin{aligned} &C^\bullet_{\mathrm{inv}}(A,M) = \mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(\mathrm{B}(A),M) := \\ &\mathrm{Hom}^n_{\mathbf{iVect}}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} S\mathrm{Bar}(A)\otimes_{\mathbb{K}}\mathrm{Bar}(A), \mathrm{Bar}(A)\otimes_{\mathbb{K}} SM\otimes_{\mathbb{K}}\mathrm{Bar}(A)) \overset{(!)}{\simeq} \\ &\mathrm{Hom}^n_{\mathbf{iVect}}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} SA\otimes_{\mathbb{K}}\mathrm{Bar}(A), \mathrm{Bar}(A)\otimes_{\mathbb{K}} SM\otimes_{\mathbb{K}}\mathrm{Bar}(A)) =: \\ &\mathrm{Hom}_{\overline{A-\mathbf{iBimod}}}(A,M). \end{aligned}$$

Here (!) points out the fact that $S\mathrm{Bar}(A)$ is a projective resolution of SA in **iVect** and hence we have the quasi-isomorphism $S\mathrm{Bar}(A) \simeq SA$. Observe that $S\mathrm{Bar}(A)$ is projective in **iVect**, therefore the involved functors in the proof are exact and preserve quasi-isomorphisms. □

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