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A note on distributional equations in discounted risk processes *

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Abstract

In this paper we give an account of the classical discounted risk processes and their limiting distributions. For the models considered, we set the Markov chains embedded in the continuous-time processes; we also set distributional equations for the limit distributions. Additionally, we mention some applications regarding ruin probabilities and optimal premium.

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1 Introduction

Let us introduce the model. We can think of costumers arriving according to a renewal process and at their arrival they bring a reward according to a i.i.d. sequence X_i's. Then the discounted reward at time t with discounted rate δ is

$$(1) \qquad Z^{(\delta)} := \left\{ Z_t^{(\delta)} = \sum_{i=1}^{N_t} X_i e^{-\delta T_i},\ t \geq 0 \right\},$$

where $N := \{N(t), t \geq 0\}$ is a renewal process with interarrival times $\tau_1, \tau_2, \ldots$; which are independent identically distributed positive random

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variables (i.i.d. positive r.v.s.). The process N is defined through

$$N_t := \max\left\{k : \sum_{i=1}^{k} \tau_i \leq t\right\}. \tag{2}$$

Further, $T_i := \sum_{j=1}^{i} \tau_j, i = 1, 2, \ldots$, which represents the arrival times as mentioned before. Variables $X_1, X_2, ...$ are i.i.d. positive r.v.s. and δ is the continuous time interest rate. Throughout this paper we assume that the interarrival times and claim sizes are independent, and we denote by τ and X the generic random variables, such that $\tau \stackrel{d}{=} \tau_1$ and $X \stackrel{d}{=} X_1$. To avoid technical problems we assume that $P(\tau = 0) < 1$.

We use model (1) in the context of insurance; variable τ represent a generic arrival and X a generic claim size, thus $Z_t^{(\delta)}$ is the present value of the claims up to time t. One can see that process (1) is an extension of the well-known *renewal reward processes* (see for instance [2, 32]), and when $\delta = 0$ it resembles a particular instance of the so-called *continuous–time random walk* (one may find a summary on this type of process in [29]). The renewal reward process is also called the *aggregate claim amount* in the insurance jargon (see [26]). Processes that resemble (1) have been studied with other names; for example, the *Markov shot noise processes* in [30, 27], and when the renewal process is Poisson it is called *filtered Poisson processes* in [16]. Process (1) have renewal properties, feature that classifies it in a more general family called *regenerative processes*, as labeled in [2]); or it is a particular instance of a *semi-Markov* process, see e.g. [21]. Specifically, process (1) has been studied previously in [5, 9, 17, 18, 31, 19].

Let Z^* be the limit of $Z_t^{(\delta)}$ when $t \to \infty$, when it is well defined, intuitively one expects the following *distributional equation* to hold,

$$Z^* \stackrel{d}{=} e^{-\delta\tau}(Z^* + X). \tag{3}$$

This equation is derived from a recursive random equation; regarding recursive equations the reader might find interesting the many ideas in [10].

In section 2, we give the details to derive previous equation using the so-called *embedded Markov chains*; this technique has been well used in other papers, see e.g. [12]. Following this idea, in section 3, we do the same for the so-called discounted risk process. Then, we mentioned how one can find the moments recursively (see e.g. [18]). In section 4 it is used the tools developed to give straight forward applications, namely for studying the ruin probability and a perpetual cash flow. In addition, in subsection 4.3 we study a model where rate income and severity are sensitive to the price per contract, so that the insurer should choose an optimal premium.

2 An embedded Markov chain

The term *embedded Markov chain* (EMC) refers to the concept of having a discrete-time process embedded within a continuous-time process. An important feature of model (1) is that it renews/regenerates at the very time of an arrival T_i; this helps to identify Markov chains (MCs) embedded in the process. The EMC helps to study the limit behavior of process (1) by studying the corresponding stationary distributions. The type of MCs that arise here can be compared to the so-called *perpetuities* (see also [20]), and in turn they give rise to *distributional equations* (DEs), also called *stochastic* or *random equations* (some relevant references about DEs are [1, 14, 30]). An important application of the distributional equations is finding properties of stationary distributions, such as moments or even the distribution itself (see e.g. [12, 22]).

Notice first that, since the trajectories are increasing, $\lim_{t\to\infty} Z_t^{(\delta)}$ always exists by monotone convergence theorem, so that it convergences in distribution. Also,

Proposition 2.1. *If X has finite mean, then $Z_t^{(\delta)}$ converges in distribution as $t \to \infty$ to a random variable Z^* with finite mean.*

Proof. By Fatou´s Lemma,

$$E(Z^*) \leq \sum_{i=1}^{\infty} E(X_i)E(e^{-\delta T_i}). \tag{4}$$

Finally, we know that $E(e^{-\delta T_i}) = E^i(e^{-\delta\tau})$ and that $E(e^{-\delta\tau}) < 1$, then

$$E(Z^*) \leq E(X)\sum_{i=1}^{\infty} E^i(e^{-\delta\tau}) < \infty. \tag{5}$$

□

Having $E(X) < \infty$ will be necessary in next section, when asking for the positive safety loading condition.

Proposition 2.2. *Let Y_n be the process $Z^{(\delta)}$ evaluated at the time of the n arrival, that is, $Y_n := Z^{(\delta)}_{T_n}$. Then the following identity holds*

$$Y_{n+1} \overset{d}{=} X_n e^{-\delta\tau_n} + e^{-\delta\tau_n} Y_n, \quad n = 0, 1, \ldots, \tag{6}$$

with $Y_0 := 0$. Here X_n, τ_n and Y_n are independent for each n.

Proof. Using the definition of the process

$$Z^{(\delta)}_{T_{n+1}} = \sum_{i=1}^{N_{T_{n+1}}} X_i e^{-\delta T_i} = \sum_{i=1}^{n+1} X_i e^{-\delta T_i}$$

$$\overset{d}{=} X e^{-\delta\tau} + \sum_{i=2}^{n+1} X_i e^{-\delta T_i} \overset{d}{=} X e^{-\delta\tau} + e^{-\delta\tau} \sum_{i=2}^{n+1} X_i e^{-\delta \sum_{k=2}^{i} \tau_k}$$

$$\overset{d}{=} X e^{-\delta\tau} + e^{-\delta\tau} Z^{(\delta)}_{T_n}.$$

Hence, equation (6) can be set. □

Remark 2.3. It is said that process $Y := \{Y_0, Y_1, \ldots\}$ of previous result is an *embedded Markov chain* of process $Z^{(\delta)}$. There are results regarding ergodic properties of stochastic processes through embedded Markov chains (see for instance [6]). Since the paths of $Z^{(\delta)}$ are piecewise constant between arrivals, we shall study the limit behavior of $Z^{(\delta)}$ by studying the stationary distribution of Y.

Remark 2.4. Relation (6) is an identity of distributions which itself provides a method for approximating samples of Z^* by running the MC. This is possible due to the fact that the stationary distribution of the MC is the limiting distribution of Z_t as $t \to \infty$. An extensive study on stochastic equations of this type can be found in Vervaat [30].

The following result is easy to see from previous results, however the same idea is used later in Theorem 3.1.

Proposition 2.5. *Let*

$$Z^* := \lim_{t\to\infty} Z^{(\delta)}_t. \tag{7}$$

Then we have the following distributional equation

$$Z^* \overset{d}{=} X e^{-\delta\tau} + e^{-\delta\tau} Z^*. \tag{8}$$

Proof. Notice that $(X_i, \tau_i, Y_i) \to (X, \tau, Z^*)$ as $i \to \infty$. Since $f(x,t,y) := xe^{-\delta t} + e^{-\delta t}y$ is a continuous function, we can apply the continuous mapping theorem (see [4]) to obtain (8) from (6) □

Remark 2.6. Often, the following class of distributional equation arises in insurance applications (see for example [12, 14, 30]):

$$Z_\infty \stackrel{d}{=} AZ_\infty + B, \tag{9}$$

where A, B and Z_∞ are random variables, and Z_∞ may or may not be independent of (A, B). The question is to find the distribution of Z_∞ given the distribution of (A, B).

A typical application of the distributional equation (8) is using it for computing the moments of Z^*.

Corollary 2.7. *Suppose that X has all its moments finite and that the Laplace transform of τ exists. Then, the k-moment of Z^* satisfies the following recursive formula*

$$E((Z^*)^k) = \frac{E\left(e^{-k\delta\tau}\right)}{1 - E\left(e^{-k\delta\tau}\right)} \sum_{i=0}^{k-1} \binom{k}{i} E(X^{k-i}) E((Z^*)^i), \tag{10}$$

for $k=1,2,\ldots$.

Proof. This is done by a direct use of the Newton´s binomial theorem to the distributional equation (9). First, expanding the binomial, taking expectations and then solving for the k-moment. □

The procedure described in Corollary 2.7 for finding moments through distributional equations is quite common in the literature, see for example [20, p. 465] or [22, 30]. Notice that using the distributional equations one may also attempt to find the characteristic function or the moment generating function.

3 The present value distribution

Using process (1), a popular model in insurance is the so-called *total surplus* (we also call it *discounted risk process*) given by

$$U_t = \eta + r(t) - Z_t^{(\delta)}, \quad t \geq 0, \tag{11}$$

where $r(t) := \int_0^t \rho e^{-\delta s} ds$ is the present value of the incomes received by the company, which is determined with the premium rate ρ. Variable η is the initial capital of the company. In [31] model (11) is called the renewal risk process. Process $U = \{U_t, t \geq 0\}$ in (11) represents the present value of the total surplus (earnings or losses) of the company up to time t.

It is known that U_t admits the representation

$$U_t := \eta + \int_0^t e^{-\delta s} dY_s,$$

where $Y_s := \rho + \sum_{i=1}^{N(s)} X_i$, see e.g. [15]. At this point, it would be guarantied for the insurance company that $\lim_{t\to\infty} U_t > 0$ almost surely; from previous equation we notice that this condition is achieved if the *positive safety loading* condition holds, that is if

$$\frac{\rho - \lambda E(X)}{\lambda E(X)} > 0,$$

with $\lambda := 1/E(\tau)$ (see [13]). From now on we assume that our model satisfies this condition.

When $t \to \infty$, U_t may converge to a random variable, which is interpreted as the total earnings or losses of the perpetuity, that is to say, the *total outcome of the business*; the interested reader can find a more extensive discussion of this in [24]. A natural question is to find the so-called *present value distribution*, which is defined as the distribution of

$$U^* := \lim_{t\to\infty} U_t. \tag{12}$$

Finding the present value distribution has been done for a general class of models based on the Poisson process; important references are [11, 12, 14, 15]. Specially in [12] one finds a good account.

One can see that if $Z_t^{(\delta)}$ converges in distribution as $t \to \infty$, so does U_t. To this end we have the following (see also [25]).

Theorem 3.1. *Let Z be the process specified in (1). When* $\lim_{t\to\infty} Z_t^{(\delta)}$ *exists in distribution, we have the following distributional equation*

$$U^* \stackrel{d}{=} \alpha - e^{-\delta\tau} \left(X + \alpha - U^*\right), \tag{13}$$

where $\alpha = \eta + \frac{\rho}{\delta}$. *Moreover, the distribution of* U^* *coincides with the stationary distribution of the following MC*

$$W_{n+1} = \alpha - e^{-\delta\tau_{n+1}} \left(X_{n+1} + \alpha - W_n \right), \tag{14}$$

with $W_0 \in \mathbb{R}$.

Proof. First, we will exploit the renewal property to obtain an identity in distribution. Then, we appeal to the continuous mapping theorem to set equation (13), which itself gives rise to the MC (14).

By using the definition of $T_n, n = 1, 2, \ldots$, we have

$$U_{T_n} = \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \sum_{i=0}^{N_{T_n}} X_i e^{-\delta T_i}$$

$$= \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \left(X_1 e^{-\delta\tau_1} + \sum_{i=2}^{n} X_i e^{-\delta T_i} \right)$$

$$\stackrel{d}{=} \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \left(X e^{-\delta\tau} + e^{-\delta\tau} \sum_{i=1}^{n-1} X_i e^{-\delta T_i} \right)$$

$$= \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \left(X e^{-\delta\tau} + e^{-\delta\tau} \left(\sum_{i=1}^{n-1} X_i e^{-\delta T_i} \pm \eta \pm \int_0^{T_{n-1}} \rho e^{-\delta s} ds \right) \right)$$

$$= \eta + \int_0^{T_n} \rho e^{-\delta s} ds - \left(X e^{-\delta\tau} + e^{-\delta\tau} \left(-U_{T_{n-1}} + \eta + \int_0^{T_{n-1}} \rho e^{-\delta s} ds \right) \right).$$

Thus, when taking limits we have distributional equation (13). □

Remark 3.2. Notice that the moments of U^* may be computed using equation (13) as in Corollary 2.7, however this approach does not give a formula as friendly as recurrence (10). It is more convenient to use the fact that

$$U^* = \alpha - Z^*, \text{ with } \alpha = \eta + \frac{\rho}{\delta}, \tag{15}$$

which yields

$$E\left((U^*)^k \right) = (-\alpha)^k \sum_{i=0}^{k} \binom{k}{i} (-1)^i E\left((Z^*)^{k-i} \right), k = 1, 2, \ldots. \tag{16}$$

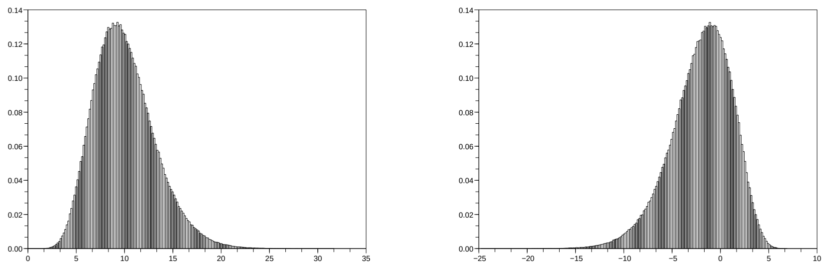

Figure 1: Histograms of Z^* and U^*.

Corollary 2.7 and Remark 3.2 find the moments of Z^* and U^*, respectively. An interesting point is to find the actual limit distribution, i.e. finding the solutions (8) and (13). Next, we use the Markov chains to perform numerical approximations.

Example 3.3. In figure 1 we show the approximations of Z^* and U^* for a model where the claim sizes and the interarrival times are exponential, both with parameters 1; and we have taken $\delta = 0.1$, $\eta = 5$ and $\rho = 0.3$. We have run 10^6 times the corresponding Markov chains (6) and (14), and obtained numerically the histograms with partition 200.

4 Applications

Now we present some applications of the embedded Markov chains and the distributional equations. First, we find a bound for the ruin probability. Then, we discuss about the probability of ending negative in perpetual cash flow. Finally, we give an example to show that the income rate (i.e. ρ) may not be set too high or too low.

4.1 A bound for the ruin probability

Calculating the ruin probability has generated a great deal of interest in risk theory for discounted and non-discounted sums. Since the celebrated works of Lundberg and Crámer, many articles and books have been published to address this problem; few references are [3, 13, 20, 26], and a concise summary can be found in [7].

Consider model (11). The *ruin probability* is defined as follows: Given the initial capital η, variable χ_η is the first time when U_t goes below 0 (when the company goes bankrupt or ruined). It is expressed as $P(\chi_\eta < \infty)$ where

$$\chi_\eta = \inf\{s : U_s < 0\}\,. \tag{17}$$

Ruin probability has been studied extensively with model (1) when the interarrival times are exponential r.v.s and $\delta = 0$ (see e.g. [13]). Here we take $\delta > 0$ but now we assume that τ is exponentially distributed. Under these assumptions, Harrison [15] gives bounds for the ruin probability in terms of the present value distribution.

Proposition 4.1.1. *Using model (11) with τ being an exponential random variable, suppose that $Z^* \overset{d}{=} \lim_{t\to\infty} Z_t^{(\delta)}$ is well defined. Then the following upper bound for the ruin probability holds*

$$P(\chi_\eta < \infty) \le \frac{P\left(Z^* > \eta + \rho/\delta\right)}{P\left(Z^* > \rho/\delta\right)}. \tag{18}$$

Proof. By Corollary 2.4 in [15] we have that

$$P(\chi_\eta < \infty) \le \frac{H(-\eta)}{H(0)}, \tag{19}$$

where H is the distribution function of $-\eta + \lim_{t\to\infty} U_t$. Recall that $\lim_{t\to\infty} r(t) = \rho/\delta$. □

Previous sections give grounds for finding numerically the bound for the ruin probability. This is easily achieved by approximating $P(Z^* > z)$ using the embedded Markov chain of process $Z^{(\delta)}$, which is explained in Remark 2.4 and carried out in Example 3.3.

4.2 The probability of long-run negative dividends

We now turn to the study of the long time behavior of U_t as $t \to \infty$, which is the perpetual cash flow (such as in pension schemes); we may think of this as the "total/final outcome of the business". In this paper we study the probability of ending up loosing at the infinite horizon:

$$P(U^* < 0). \tag{20}$$

We call quantity (20) the *business-ruin probability.* The name is motivated from the fact that U^* may represent the total discounted dividends of a company, and if it is negative, it means that the business did not work.

Despite the aforementioned motivation, in the case of the insurance company, it is quite not realistic to take $t \to \infty$, because the insurer would not continue operating in case of bankruptcy (i.e. when $U_t < 0$).

Compare to the classical concept of ruin probability, the business-ruin probability is a less stringent version. This, due to the fact that process U can go below 0 but may end up positive as $t \to \infty$. Therefore, the event of going business-ruin implies that process U went negative at some point, thus

$$P(U^* < 0) \leq P(\chi_\eta < \infty), \tag{21}$$

where $\chi_\eta = \inf \{s : U_s < 0\}$.

A natural question is to find an income rate ρ that guaranties certain level of total earnings. Moreover, we may find ρ that helps to achieve low probability of ending up loosing or a high probability of ending up earning. Notice that U^* depends on the rate ρ, and we can write $U^*(\rho)$ to emphasize this. The following definition gives criteria to choose an income rate.

Definition 4.2.1. For $\epsilon \in (0, 1)$, whenever it exists we call the quantity ρ_ϵ the ϵ-percentile income rate if it is such that

$$P(U^*(\rho_\epsilon) \in A) = \epsilon, \tag{22}$$

for some Borel set A. And we call ρ_β, with $\beta > 0$, the β-mean income rate if it is such that

$$E(U^*(\rho_\beta)) = \beta. \tag{23}$$

Generally, we would be interested in an income rate that allows us to either minimize the potential loss, maximize profit, or simply such that we reach a minimum level of profit. The previous definition takes into account these ideas, see next remark.

Remark 4.2.2. For the ϵ-percentile income rate, one needs to be more specific. For instance, if we are interested on minimizing loss, a natural choice for A is $(-\infty, \theta]$, where $\theta \geq 0$ is a minimum level of tolerance. Likewise, we can set $A = [\theta, \infty)$ if we want to achieve certain level of profit. In any case, the calculation of ρ_ϵ requires the knowledge of the distribution of U^* (the present value distribution).

The calculation of ρ_β is direct from equation (13). Equating the expectation $E(U^*)$ to β, and solving for ρ_β we have that

Proposition 4.2.3. *The β-mean income rate is given by*

$$\rho_\beta = \delta^2 \frac{(\beta-\eta)\left(1-E(e^{-\delta\tau})\right)+E(X)E(e^{-\delta\tau})}{1-E(e^{-\delta\tau})}. \tag{24}$$

Example 4.2.4. In Example 3.3, for $\rho = 0.3$, we have that $P(U^* < 0) \approx 0.71$. Thus the $\epsilon-$percentile premium rate is $\rho_\epsilon = 0.3$ with $\epsilon \approx 0.71$. This is an unfavorable scenario for the company, because with a high probability the business will end up loosing.

4.3 A control problem for the insurance company

The income rate ρ is a quantity that depends on the price per contract. That is, the rate of income is a factor that can be determined by how cheap or expensive the actual price of the contract is.

Let p be the price per contract. Price p may be so expensive that no one would be able to afford it (and thus no income would be obtained); or, the price could be so cheap that even though many would buy it, the income rate would not be enough to pay the potential losses. The value of p would affect the income rate ρ and the frequency of arrivals, defined by τ. Thus, the company does not want to set a very expensive or very cheap price per contract, rather it needs to find an optimal price.

Consider the distributional equation (13). If we take expectation of both sides of (13), and solve for $E(U^*)$ we obtain

$$E(U^*) = \frac{\eta + \frac{\rho}{\delta} - E(e^{-\delta\tau})(E(X)+\eta+\frac{\rho}{\delta})}{1-E(e^{-\delta\tau})}. \tag{25}$$

Here, ρ and $E(e^{-\delta\tau})$ are functions of p. Then, an optimal price p can be found by maximizing (25).

Now, we give a model that is specified by p.

Suppose that τ is an exponential r.v. with mean $\frac{1}{\lambda(p)}$, where

$$\lambda(p) = \frac{a}{p^b}, \text{ for } a > 0,\ b > 0. \tag{26}$$

Moreover, suppose that $\rho(p)$ is given by

$$\rho(p) = p^c e^{-dp}, \text{ for } c > 0,\ d \geq 0. \tag{27}$$

If p is small (cheap), more people would buy a contract, and thus more potential loses might arrive in the future. If p is large (expensive),

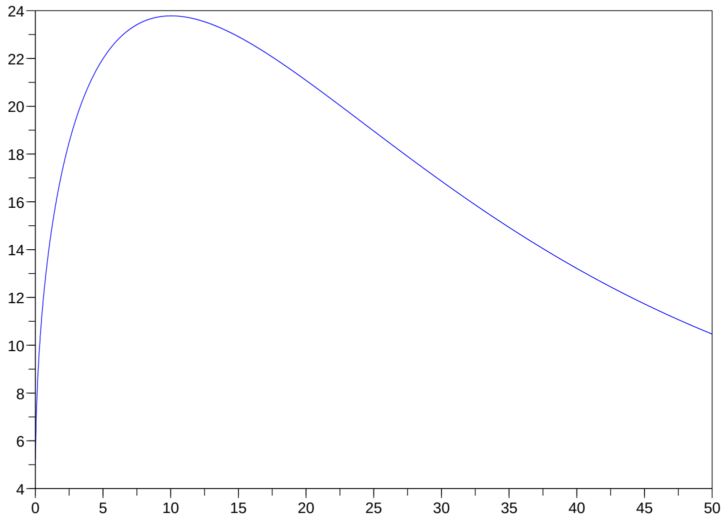

Figure 2: $E(U^*)$ as function of p

less people would buy the insurance, thus the company would have less losses. These phenomena are reflected in (26) and (27).

Now we have that

$$E\left(e^{-\delta\tau}\right) = \frac{\lambda(p)}{\lambda(p)+\delta}.$$

Furthermore, if $X \sim \exp(\mu)$, formula (25) becomes

$$E(U^*) = \left(\eta + \frac{\rho(p)}{\delta} - \frac{\lambda(p)}{\lambda(p)+\delta}\left(\mu + \eta + \frac{\rho(p)}{\delta}\right)\right)\frac{\lambda(p)+\delta}{\delta}. \tag{28}$$

Using Example 3.3 and setting $a = 0.05, b = 0.1, c = 0.5$ and $d = 0.05$ to define functions (26) and (27), in Figure 4.3 we plot $E(U^*)$ as function of p. We can see that $E(U^*)$ attains a maximum: the optimal price per contract. Related to this application, see [28] for control problems in insurance.

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