Morfismos, Vol. 20, No. 1, 2016, pp. 43–51

A bound on the size of irreducible triangulations *

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Abstract

Let S be a closed surface with Euler genus $\gamma(S)$. An irreducible triangulation of S is a simple graph G without contractible edges embedded on S so that each face is a triangle and any two faces share at most one edge. Nakamoto and Ota were the first to give a linear upper bound for the number n of vertices of G in terms of $\gamma(S)$. This bound was recently improved for orientable surfaces. By extending Nakamoto and Ota's method we improve on these bounds by showing that $n \leq 106.5\gamma(S) - 33$ for any closed surface S.

2010 Mathematics Subject Classification: 05C10.
Keywords and phrases: Euler genus, irreducible triangulation.

1 Introduction

The *orientable (non-orientable) closed surface* M_g (N_g) with *genus* g is the sphere with g handles (cross-caps) attached. The *Euler genus* of these surfaces is $\gamma(M_g) = 2g$ for the orientable surface M_g and $\gamma(N_g) = g$ for the non-orientable surface N_g.

A *triangulation* G of a closed surface S is a graph without loops or multiple edges, i.e. a simple graph, embedded on the surface so that

** **In memory of our dear friend Jaime Cruz Sampedro (1955–2015).***

[1] This work is partly supported by Universidad Autónoma de Tlaxcala and by CONACYT doctoral grant 144571. This work is part of the Ph.D. thesis to be presented at the Mathematics Department of CINVESTAV under the supervision of Dr. Isidoro Gitler (CINVESTAV) and Dr. Francisco Javier Zaragoza Martínez (Departamento de Sistemas, Universidad Autónoma Metropolitana Azcapotzalco).

[2] This work is partly supported by Universidad Autónoma Metropolitana Unidad Azcapotzalco research grant SI004-13 and by CONACYT Sistema Nacional de Investigadores grant 33694.

each face is a triangle and any two faces share at most one edge. The *orientable genus* $\bar{\gamma}(G)$ of G is defined as the least g such that G is embeddable in M_g and the *non-orientable genus* $\tilde{\gamma}(G)$ of G is defined as the least g such that G is embeddable in N_g. The *Euler genus* $\gamma(G)$ of G is defined to be $\gamma(G) = \min\{2\bar{\gamma}(G), \tilde{\gamma}(G)\}$. Note that $\gamma(G) = \min\{\gamma(S) : G \text{ is embeddable in } S\}$.

Let G be a triangulation of a closed surface S and let ab be an edge of G; then ab is on two faces, say abc and abd. We say that ab is *contractible* if we can obtain a new triangulation by deleting edges ab, one of ac or bc, one of ad or bd, and identifying a with b, see Figure 1. A triangulation G is said to be *irreducible* if G has no contractible edge.

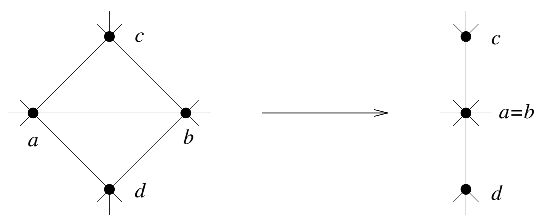

Figure 1: Contracting edge ab

The size of an irreducible triangulation can be measured in terms of the number of its vertices, edges, or triangles (by Euler's formula these are all equivalent and we have chosen to measure the size in terms of the number n of vertices). Ringel [13] obtained an explicit formula for the size of minimum triangulations for all non-orientable surfaces. Later, Jungerman and Ringel [8] obtained an explicit formula for all orientable surfaces.

Steinitz and Rademacher [14] showed that there is only one irreducible triangulation (with 4 vertices) of the sphere. Barnette [3] showed that there are two irreducible triangulations (with 6 or 7 vertices) of the projective plane. Lavrenchenko [9] determined that there are 21 irreducible triangulations (with 9 or 10 vertices) of the torus. Barnette and Edelson [2] showed that any surface has finitely many irreducible triangulations. Gao, Richmond, and Thomassen [6] showed that there are $O(\gamma(S)^4)$ vertices in any irreducible triangulation of the surface S. This bound was improved by Gao, Richter, and Seymour [7] to $O(\gamma(S)^2)$ vertices. Nakamoto and Ota [11] were the first to give a linear bound $n \leq 171\gamma(S) - 72$ and this bound was improved to $n \leq 120\gamma(S)$ for orientable surfaces by Cheng, Dey, and Poon [4]. In this paper we propose a better bound of n for any closed surface S obtained by extending Nakamoto and Ota's method

2 Improving the bound

We use the following bound on the Euler genus of a $1-$ or $2-$sum of graphs.

Lemma 2.1. Miller [10] *Let G_1 and G_2 be two graphs and let $G := G_1 \cup G_2$. If G_1 and G_2 have at most two common vertices, then $\gamma(G) \geq \gamma(G_1) + \gamma(G_2)$.*

From now on, let S be either M_g or N_g with $g \geq 1$ and let G be an irreducible triangulation of S. This implies that G contains no vertex of degree less than four. If v is a vertex of G we define H_v to be the subgraph of G induced by v and its neighbors.

Lemma 2.2. Nakamoto, Ota [11] *Let G be an irreducible triangulation of a closed surface S and v a vertex of G. Then $\gamma(H_v) \geq 1$.*

A set of vertices of G is *independent* if G does not contain any edge between them. The following result appears in [11] for $k = 6$. A similar proof for irreducible quadrangulations appears in [1].

Lemma 2.3. *Let G be an irreducible triangulation of a surface S, and let $k \geq 4$ be an integer. There exists an independent set X of vertices of degree at most k such that*

$$|X| \geq \sum_{i=4}^{k} \frac{|V_i|}{i+1},$$

where V_i is the set of vertices of G with degree i.

Proof. Let X_4 be a maximal independent subset of V_4. For each $i \in \{5, 6, \ldots, k\}$, let X_i be a maximal independent subset of $V_i - \cup_{j=4}^{i-1} A_{i,j}$, where $A_{i,j}$ is the set of vertices of degree i with a neighbor in X_j. We claim that $X = \cup_{i=4}^{k} X_i$ satisfies the required property. If we count each vertex in X_i and each of its neighbors, we obtain:

$$(i+1)|X_i| \geq |V_i| + \sum_{j=i+1}^{k} |A_{j,i}| - \sum_{j=4}^{i-1} |A_{i,j}| \qquad \text{for every } 4 \leq i \leq k.$$

Therefore

$$\begin{aligned}
|X| &= \sum_{i=4}^{k} |X_i| \\
&\geq \sum_{i=4}^{k} \frac{|V_i|}{i+1} + \sum_{i=4}^{k} \sum_{j=i+1}^{k} \frac{|A_{j,i}|}{i+1} - \sum_{i=4}^{k} \sum_{j=4}^{i-1} \frac{|A_{i,j}|}{i+1} \\
&= \sum_{i=4}^{k} \frac{|V_i|}{i+1} + \sum_{j=4}^{k-1} \sum_{i=j+1}^{k} \left[\frac{|A_{i,j}|}{j+1} - \frac{|A_{i,j}|}{i+1} \right] \\
&\geq \sum_{i=4}^{k} \frac{|V_i|}{i+1}.
\end{aligned}$$

□

The proof of the following theorem is very similar to that of Theorem 3 in [11]. We denote by $N_G(v)$ the set of neighbors of the vertex v in the graph G.

Theorem 2.4. *Let G be an irreducible triangulation of a closed surface S with n vertices. Then $n \leq 106.5\gamma(S) - 33$.*

Proof. Let m and f be the number of edges and faces of G, respectively. By Euler's formula

$$n - m + f = 2 - \gamma(S).$$

Since G is a triangulation we have $3f = 2m$ and therefore

$$3n - m = 6 - 3\gamma(S).$$

Since $\sum_{i\geq 4} |V_i| = n$ and $\sum_{i\geq 4} i|V_i| = 2m$ we have that

$$5n + \sum_{i\geq 4} (1-i)|V_i| = 12 - 6\gamma(S).$$

Let $k \geq 6$ be an integer. By adding and subtracting $kn = k\sum_{i\geq 4} |V_i|$ we obtain

$$(5-k)n + \sum_{i\geq 4} (k+1-i)|V_i| = 12 - 6\gamma(S),$$

thus

$$\sum_{i=4}^{k} (k+1-i)|V_i| \geq (k-5)n - 6\gamma(S) + 12. \tag{1}$$

Let X be an independent set as in Lemma 2.3 and define

$$Y := \{y \in V(G) - X : y \in N_G(x) \text{ for some } x \in X\}.$$

Consider the bipartite graph B with bipartition X and Y, where $xy \in E(B)$ for $x \in X$, $y \in Y$ if and only if $xy \in E(G)$.

Let $X' := \{v_1, v_2, \ldots, v_r\}$ be a maximal subset of X satisfying the following condition:

$$\left| \left\{ \bigcup_{1 \le i < j} N_B(v_i) \right\} \cap N_B(v_j) \right| \le 2, \text{ for each } j = 1, 2, \ldots, r.$$

By Lemma 2.1 and Lemma 2.2 we obtain

$$\gamma\left(\bigcup_{i=1}^{r} H_{v_i} \right) \ge \sum_{i=1}^{r} \gamma(H_{v_i}) \ge |X'|.$$

Since $\bigcup_{i=1}^{r} H_{v_i}$ is a subgraph of G, it is embeddable in S, thus

$$\gamma(S) \ge \gamma\left(\bigcup_{i=1}^{r} H_{v_i} \right),$$

and it follows that

$$|X'| \le \gamma(S). \tag{2}$$

Now define $Y' := \{y \in Y : y \in N_B(v) \text{ for some } v \in X'\}$. Let M be the subgraph of B induced by $X \cup Y'$. Since M is a subgraph of G it is embeddable in S, therefore

$$|V(M)| - |E(M)| + |F(M)| \ge 2 - \gamma(S).$$

Since M is bipartite each of its faces has at least 4 edges, therefore $4|F(M)| \le 2|E(M)|$. Hence we have

$$2|V(M)| - |E(M)| \ge 4 - 2\gamma(S). \tag{3}$$

By maximality of X', each vertex $v \in X - X'$ has at least three neighbors in Y'. Also, there are at least $|Y'|$ edges between X' and Y'. Hence $|E(M)| \ge 3(|X| - |X'|) + |Y'|$. By replacing $|V(M)| = |X| + |Y'|$ and

$|E(M)|$ in inequality (3), we obtain

$$\begin{aligned} 4-2\gamma(S) &\le 2|X|+2|Y'|-3(|X|-|X'|)-|Y'| \\ &\le -|X|+|Y'|+3|X'| \\ &\le -|X|+(k+3)|X'| \qquad \text{since } |Y'|\le k|X'| \\ &\le -\sum_{i=4}^{k}\frac{|V_i|}{i+1}+(k+3)|X'| \qquad \text{by Lemma 2.3} \\ &\le -\frac{1}{n_k}\sum_{i=4}^{k}\frac{n_k}{i+1}|V_i|+(k+3)|X'| \end{aligned}$$

In order to use (1) we need

$$\frac{n_k}{i+1}\ge k-i+1, \text{ for every } 4\le i\le k$$

Since $(i+1)(k-i+1)$ has a unique maximum, we take this value as n_k:

$$n_k := \begin{cases} 15 & \text{if } k=6 \\ \left(\frac{k+1}{2}\right)\left(\frac{k+3}{2}\right) & \text{if } k\ge 7 \text{ and } k \text{ is odd,} \\ \left(\frac{k+2}{2}\right)^2 & \text{if } k\ge 8 \text{ and } k \text{ is even.} \end{cases}$$

Thus we obtain

$$4-2\gamma(S)\le -\frac{1}{n_k}[(k-5)n+12-6\gamma(S)]+(k+3)|X'|$$

and therefore

$$\frac{(k-5)n+12+4n_k-(k+3)n_k|X'|}{6+2n_k}\le\gamma(S). \tag{4}$$

Thus, (2) provides a good bound for $\gamma(S)$ when $|X'|$ is *big* and (4) provides a good bound when $|X'|$ is *small*. These two bounds are the same when the left-hand sides of (2) and (4) are equal, that is, when

$$|X'|=\frac{(k-5)n+4n_k+12}{(k+5)n_k+6}.$$

In particular, from (2) we obtain

$$\frac{(k-5)n+4n_k+12}{(k+5)n_k+6}\le\gamma(S),$$

that is

$$n \leq f(k)\gamma(S) - g(k)$$

where $f(k) = \dfrac{(k+5)n_k + 6}{k-5}$ and $g(k) = \dfrac{4n_k + 12}{k-5}$. A straight-forward calculation shows that $f(k)$ attains its minimum at $k = 9$, therefore

$$n \leq 106.5\gamma(S) - 33.$$

□

3 Corollaries

The bound of Nakamoto and Ota has been used to obtain bounds for other problems in Combinatorial Geometry. Using our new bound we can improve those bounds as well.

A *triangulation of S with boundary C* is an embedding of a simple graph on S containing C such that there is a face bounded by C, called the *outer face*, and all other faces are triangles. We say that the vertices are *outer* if they lie in C, and *inner* otherwise. An *outer triangulation* is a triangulation with boundary which has no inner vertices.

Let G be a triangulation, let ac be an edge of G, and let abc and adc be the two faces sharing edge ac in G. The *diagonal flip* of ac is to replace a diagonal ac with bd in the quadrilateral $abcd$, whenever bd is not in G.

Cortés et al. [5] proved that, for any closed surface S, there exists a natural number $N(S)$ such that any two outer-triangulations G_1 and G_2 of S with $|V(G_1)| = |V(G_2)| \geq N(S)$ can be transformed into each other by a sequence of diagonal flips. Furthermore:

Lemma 3.1. Cortés et al. [5] $N(S) \leq 5|V(G)| + 12\gamma(S) - 3$.

With Nakamoto and Ota's bound they obtained $N(S) \leq 867\gamma(S) - 363$. With our new bound this can be improved:

Corollary 3.2. $N(S) \leq 544.5\gamma(S) - 173$.

Negami [12] proved that, for any closed surface S, there exists a natural number $\tilde{N}(S)$ such that any two triangulations G_1 and G_2 of S with $|V(G_1)| = |V(G_2)| \geq \tilde{N}(S)$ can be transformed into each other by a sequence of diagonal flips. Let $V_{irr}(S)$ denote the maximum number of vertices of an irreducible triangulation of a closed surface S.

Lemma 3.3. Negami [12] $\tilde{N}(S) \leq 19V_{irr}(S) + 18\gamma(S) - 36$.

With Nakamoto and Ota's bound he obtained $\tilde{N}(S) \leq 3231\gamma(S) - 1332$. With our new bound this can be improved:

Corollary 3.4. $\tilde{N}(S) \leq 2005.5\gamma(S) - 591$.

Acknowledgement

We are very grateful for the suggestions, contributions, and bug reports offered by Isidoro Gitler, Ernesto Lupercio, Elías Micha, and an anonymous referee.

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