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Baker-Gross theorem revisited

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Abstract

F. Gross conjectured that any meromorphic solution of the Fermat Cubic $F_3\colon\ x^3 + y^3 = 1$ are elliptic functions composed with entire functions. The conjecture was solved affirmatively first by I. N. Baker who found explicit formulas of those elliptic functions and later F. Gross gave another proof proving that in fact one of them uniformize the Fermat cubic. In this paper we give an alternative proof of the Baker and Gross theorems. With our method we obtain other analogous formulas. Some remarks on Fermat curves of higher degree is given.

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Introduction

Consider the Fermat cubic

$$F_3\colon\ x^3 + y^3 = 1. \tag{1}$$

This algebraic curve defines an elliptic curve, i.e., a compact Riemann surface of genus 1 (taking the zeros in $\mathbb{CP}^2$ of its homogenization). A meromorphic solution of this equation is, by definition, a pair of meromorphic functions in the plane such that $f^3 + g^3 = 1$. In his paper [2] F. Gross conjectures that any meromorphic solution of the Fermat cubic

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is obtained by composing elliptic functions with entire functions. The conjecture was solved affirmatively by I. N. Baker in [4]. He proved that any solution is the composition of the following elliptic functions with an entire function:
(2)
$$f(z) = \frac{1}{2\wp(z)}\left(1 - 3^{-1/2}\wp'(z)\right), \quad g(z) = \frac{1}{2\wp(z)}\left(1 + 3^{-1/2}\wp'(z)\right),$$

where $\wp$ is the Weierstrass elliptic function satisfying $(\wp')^2 = 4\wp^3 - 1$. In what follows we denote by Λ' the lattice in $\mathbb{C}$ that defines this $\wp$. In particular these functions are solutions of the Fermat cubic but these formulas differ from the analogous that appear in [2], [3], which seem to contain an error. Later, F. Gross gave another proof in [5], proving in fact that the function f in (2) gives a uniformization of the Fermat cubic (1). In our context we formulate the previous results in the following theorem:

Theorem (Baker-Gross). *Let Λ' and $\wp$ be as above. Then the map $\mathbb{C}/\Lambda' \to F_3$ given in affine coordinates by*

$$z \mapsto \left(\frac{1}{2\wp(z)}\left(1 - 3^{-1/2}\wp'(z)\right), \frac{1}{2\wp(z)}\left(1 + 3^{-1/2}\wp'(z)\right)\right) \tag{3}$$

is a biholomorphism between the two elliptic curves. Then by the lifting property of coverings, any pair of functions F and G, which are meromorphic in the plane and satisfy (1) have the form:

$$F = \frac{1}{2\wp(\alpha)}\left(1 - 3^{-1/2}\wp'(\alpha)\right), \quad G = \frac{1}{2\wp(\alpha)}\left(1 + 3^{-1/2}\wp'(\alpha)\right), \tag{4}$$

where α is an entire function.

In this paper we give a proof of this theorem by using Riemann surface theory and by using an explicit map from a Weierstrass normal form to the Fermat cubic. Our proof could clarify the nature of the previous formulas, which are not obvious. Also, by this method, other formulas analogous to (3) and (4) are obtained (see (13) and (17)).

In Section 1 we recall some basic facts about elliptic curves and compute a Weierstrass normal form of the Fermat cubic, and the corresponding isomorphism as well. In the next section we prove the main theorem. Finally, in the last section we give some remarks on Fermat curves of higher degree.

Recently, N. Steinmetz communicated to the author another proof of the Gross conjecture in [7] (§2.3.5 pp. 56-57) by using Nevanlinna theory. He proved without reference to the Uniformization Theorem the following:

Theorem (Steinmetz). *Suppose that non-constant meromorphic functions f and g parametrize the algebraic curve*

$$F\colon\ x^n + y^m = 1 \quad (n \geq m \geq 2)$$

with $\frac{1}{m} + \frac{1}{n} < 1$*. Then* (m,n) *equals* $(4,2)$ *or* $(3,3)$ *or* $(3,2)$*. In any case* f *and* g *are given by*

$$f = E \circ \psi \quad \textit{and} \quad g = \sqrt[m-1]{E'} \circ \psi,$$

where E *is an elliptic function satisfying*

$$E'^2 = 1 - E^4, \quad E'^3 = (1 - E^3)^2 \quad \textit{and} \quad E'^2 = 1 - E^3,$$

respectively, and ψ *is any non-constant entire function.*

1 The normal form of the Fermat cubic

1.1 Basic facts on elliptic curves

A complex elliptic curve X is by definition a compact Riemann surface of genus 1. The Plücker formula tells us that a non-singular projective curve of degree 3 in $\mathbb{CP}^2$ is a Riemann surface of genus 1 i.e., an elliptic curve. The reciprocal is also true and we will briefly discuss it. For this, we recall the uniformization theorem and the Weierstrass normal form.

The *Uniformization Theorem* says that every simply connected Riemann surface is conformally equivalent to one of the three Riemann surfaces: the Riemann sphere $\overline{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the open unit disk Δ. This theorem combined with the theory of covering spaces give us a classification of Riemann surfaces: every Riemann surface X is conformally equivalent to a quotient $\tilde{X}/G$, where $\tilde{X}$ is the universal holomorphic cover of X (hence isomorphic to one of the three previous Riemann surfaces) and G is a subgroup of holomorphic automorphisms of $\tilde{X}$ which acts on $\tilde{X}$ free and properly discontinuously. In particular, when the Riemann surface is of genus 1, it has the complex plane as its universal holomorphic cover, then X is conformally equivalent to $\mathbb{C}/\Lambda$, for some lattice $\Lambda \subset \mathbb{C}$. For an introduction to Riemann surfaces and a proof of the uniformization theorem see [1].

The homogeneous polynomial with complex coefficients

$$Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3, \tag{5}$$

obtained by homogenization of the polynomial

$$y^2 = 4x^3 - g_2x - g_3, \tag{6}$$

defines an non-singular curve if and only if the discriminant $\Delta = g_2^3 - 27g_3^2$ does not vanish. Hence, (5) defines an elliptic curve if and only if $\Delta \neq 0$. We call a *Weierstrass normal form* of an elliptic curve X an elliptic curve given by an equation of the form (5) which is isomorphic as a Riemann surface to X.

Recall also that given a lattice $\Lambda \subset \mathbb{C}$ we can associate the Weierstrass elliptic function $\wp$ or $\wp_\Lambda$ given by the series:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right). \tag{7}$$

This function satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{8}$$

where g_2 and g_3 are constants depending on Λ given by:

$$g_2 = 60 \sum_{\omega \in \Lambda^*} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda^*} \frac{1}{\omega^6},$$

satisfying $\Delta = g_2^3 - 27g_3^2 \neq 0$. Thus this function gives us a map $\Psi\colon \mathbb{C}/\Lambda \to E$, in affine coordinates given by:

$$\Psi(z) = (\wp(z), \wp'(z)), \tag{9}$$

from $\mathbb{C}/\Lambda$ to the elliptic curve $E\colon y^2 = 4x^3 - g_2x - g_3$. This map is an biholomorphism which sends Λ to the point at infinity $[0:1:0]$.

From the previous results and the Uniformization Theorem we can conclude that every elliptic curve has a Weierstrass normal form. Also, it is true that given a non-singular equation (6), there exists a lattice Λ with the same constants g_2 and g_3. For more information, refer to [6, p. 176].

1.2 Computing the Weierstrass normal form of the Fermat cubic

Although a Weierstrass normal form is in general difficult to compute starting from an abstract Riemann surface of genus 1, the case of the Fermat cubic is relatively easy by choosing suitable changes of variables. Since this process will be applied to other Fermat curves in Section 3, we describe it step-by-step below:

1. Change (x, y) to $(x-y, x+y)$ in order to eliminate the cubic term y^3. Obtaining:
$$E_1\colon\ 2x^3 + 6xy^2 = 1.$$

2. Change (x, y) to $(1/x, y/x)$ to get:
$$E_2\colon\ 2 + 6y^2 = x^3.$$

3. At this point, we could use any change of variables for which the coefficient of y^2 is 1 and the coefficient of x^3 is 4, for instance with $(x, y/\sqrt{24})$ we obtain the case $g_2 = 0$ and $g_3 = 8$:
$$E_3\colon\ y^2 = 4x^3 - 8.$$

Observe that we obtain a map from the curve obtained in the change of variable to the original curve. For example in step 1 we obtain $E_1 \to F_3$, $(x, y) \mapsto (x - y, x + y)$. Then, the maps associated to the previous changes of variables are:

$$\begin{array}{ccc} E_3 \to E_2 & E_2 \to E_1 & E_1 \to F_3 \\ (x,y) \mapsto \left(x, \dfrac{y}{\sqrt{24}}\right), & (x,y) \mapsto \left(\frac{1}{x}, \frac{y}{x}\right), & (x,y) \mapsto (x-y, x+y). \end{array} \tag{10}$$

The inverse maps are (in the reverse order, respectively):

$$\begin{array}{ccc} F_3 \to E_1 & E_1 \to E_2 & E_2 \to E_3 \\ (x,y) \mapsto \left(\dfrac{y+x}{2}, \dfrac{y-x}{2}\right), & (x,y) \mapsto \left(\frac{1}{x}, \frac{y}{x}\right), & (x,y) \to (x, \sqrt{24}y). \end{array} \tag{11}$$

So in each step we have a birrational isomorphism between these non-singular algebraic curves, hence a biholomorphism between their

Riemann surfaces. So we obtain, composing the maps of (11) and (10), respectively, the biholomorphisms $\Phi\colon F_3 \to E_3$ and $\Phi^{-1}\colon E_3 \to F_3$:

$$\begin{aligned} \Phi(x,y) &= \left(\frac{2}{y+x}, \sqrt{24}\frac{y-x}{y+x}\right), \\ \Phi^{-1}(x,y) &= \left(\frac{1}{x} - \frac{y}{\sqrt{24}x}, \frac{1}{x} + \frac{y}{\sqrt{24}x}\right). \end{aligned} \tag{12}$$

2 Proof of the Baker-Gross theorem

From the previous explicit formulas the Baker-Gross theorem follows easily. Consider Λ associated to $g_2 = 0$ and $g_3 = 8$ and consider the biholomorphism $\Psi : \mathbb{C}/\Lambda \to E_3$ defined in (9), then the composition $\Phi^{-1} \circ \Psi\colon \mathbb{C}/\Lambda \to F_3$ is a biholomorphism,

$$\Phi^{-1} \circ \Psi(z) = \left(\frac{1}{\wp(z)} - \frac{1}{\sqrt{24}}\frac{\wp'(z)}{\wp(z)}, \frac{1}{\wp(z)} + \frac{1}{\sqrt{24}}\frac{\wp'(z)}{\wp(z)}\right). \tag{13}$$

where $\wp$ satisfies $(\wp')^2 = 4\wp^3 - 8$.

If we continue from step 3 applying the change of variables $(2x, \sqrt{2^3}y)$ we obtain the curve $E_3'\colon y^2 = 4x^3 - 1$ and the map $\overline{\Phi} = \Phi^{-1}(2x, \sqrt{2^3}y) : E_3' \to F_3$

$$\begin{aligned} \overline{\Phi}(x,y) &= \Phi^{-1}(2x, \sqrt{2^3}y) \\ &= \left(\frac{1}{2x} - \frac{\sqrt{2^3}y}{2\sqrt{24}x}, \frac{1}{2x} + \frac{\sqrt{2^3}y}{2\sqrt{24}x}\right) \\ &= \left(\frac{1}{2x}\left(1 - \frac{y}{\sqrt{3}x}\right), \frac{1}{2x}\left(1 + \frac{y}{\sqrt{3}x}\right)\right), \end{aligned} \tag{14}$$

and taking Λ' associated to $g_2 = 0$ and $g_3 = 1$, and $\Psi' : \mathbb{C}/\Lambda' \to E_3'$ as (9), composing this two isomorphism we obtain the biholomorphism expected in (3) $\overline{\Phi} \circ \Psi'\colon \mathbb{C}/\Lambda' \to F_3$:

$$\overline{\Phi} \circ \Psi'(z) = \left(\frac{1}{2\wp(z)}\left(1 - 3^{-1/2}\wp'(z)\right), \frac{1}{2\wp(z)}\left(1 + 3^{-1/2}\wp'(z)\right)\right),$$

where the Weierstrass elliptic function $\wp$ satisfies here $(\wp')^2 = 4\wp^3 - 1$.

On the other hand, let $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda'$ be the natural projection, this map is an unbranched holomorphic covering, then the map $\overline{\Phi} \circ \Psi' \circ \pi\colon \mathbb{C} \to F_3$ is an unbranched holomorphic covering as well. Hence, given F and G a meromorphic solution of the Fermat cubic, the map $\phi(z) = (F(z), G(z))$ defines a holomorphic map $\phi\colon \mathbb{C} \to F_3$. Since $\mathbb{C}$ is simply connected ϕ has an holomorphic lifting $\alpha\colon \mathbb{C} \to \mathbb{C}$ with respect to this covering, i.e., the following diagram commutes:

$$\begin{array}{ccc} & & \mathbb{C} \\ & \overset{\alpha}{\nearrow} & \big\downarrow {\scriptstyle \overline{\Phi}\circ\Psi'\circ\pi} \\ \mathbb{C} & \xrightarrow{\ \phi\ } & F_3 \end{array} \tag{15}$$

Composing with α we obtain

$$F = \frac{1}{2\wp(\alpha)}\left(1 - 3^{-1/2}\wp'(\alpha)\right), \quad G = \frac{1}{2\wp(\alpha)}\left(1 + 3^{-1/2}\wp'(\alpha)\right), \tag{16}$$

which are the desired formulas. This proves the theorem.

Note that we could use the map $\Phi^{-1} \circ \Psi\colon \mathbb{C}/\Lambda \to F_3$ given in (13) instead of $\overline{\Phi} \circ \Psi'$ in the above argument to obtain that any meromorphic solution of the Fermat cubic is of the form

$$F = \frac{1}{\wp(\alpha)}\left(1 - \frac{1}{\sqrt{24}}\wp'(\alpha)\right), \quad G = \frac{1}{\wp(\alpha)}\left(1 + \frac{1}{\sqrt{24}}\wp'(\alpha)\right), \tag{17}$$

where in this case $\wp$ satisfies $(\wp')^2 = 4\wp^3 - 8$. We could obtain similar solutions depending on which factor we choose in step 3, but we can always obtain one from the other by this process.

3 Some remarks for Fermat curves of higher degree.

We finalize discussing about the application of the changes of variables described in 1.2 to the Fermat curves of higher degrees (see (18)). When the curve is of odd degree the process give us directly an interesting equation, but when the degree is even we need to apply a slight modification in step 1. From these equations we give a meromorphic function on the Fermat curves.

3.1 The odd case

The changes of variables in steps 1 and 2 described in 1.2 can be applied to any Fermat curve,

$$F_n\colon x^n + y^n = 1, \tag{18}$$

but in the case of n odd we get an interesting formula. By a straightforward calculation, following steps 1 and 2, we find the curve E_2:

$$E_2\colon\ 2 + 2\sum_{k=1}^{\frac{n-1}{2}} \binom{n}{2k} y^{2k} = x^n. \tag{19}$$

As we did not modify the above steps we get the same correspondence $\Phi\colon F_n \to E_2$ as in (12) but without step 3, so we get in this case:

$$\begin{aligned} \Phi(x,y) &= \left(\frac{2}{y+x}, \frac{y-x}{y+x}\right), \\ \Phi^{-1}(x,y) &= \left(\frac{1}{x} - \frac{y}{x}, \frac{1}{x} + \frac{y}{x}\right). \end{aligned} \tag{20}$$

Note that E_2 has an holomorphic involution $I(x,y) = (x,-y)$. It is easy to check that it is conjugate by Φ to the canonical involution of F_n, $\overline{I}(x,y) = (y,x)$, i.e., the following diagram commutes

$$\begin{array}{ccc} F_3 & \xrightarrow{\ \overline{I}\ } & F_3 \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ E_2 & \xrightarrow{\ I\ } & E_2 \end{array} \tag{21}$$

Note that the projection in the first coordinate is a meromorphic function of degree $n-1$ on E_2, so composing with Φ we obtain the meromorphic function $2/(y+x)$ on F_n of degree $n-1$, for example in the case $n=3$ we obtain a degree 2 meromorphic function on the elliptic curve F_3.

3.2 The even case

Similar formulas can be obtained in the even case by using the change $(x+\omega y, x+y)$ instead of $(x-y, x+y)$ in the first step, where ω is a root of $x^n = -1$, maintaining the other steps without changes as before. In this case we have

$$E_2\colon 2 + \sum_{k=1}^{n-1} \binom{n}{k} (1+\omega^k) y^k = x^n, \tag{22}$$

and $\Phi : F_n \to E_2$ become

$$\begin{aligned} \Phi(x,y) &= \left(\frac{\omega-1}{\omega y - x}, \frac{x-y}{\omega y - x}\right), \\ \Phi^{-1}(x,y) &= \left(\frac{1}{x} + \omega\frac{y}{x}, \frac{1}{x} + \frac{y}{x}\right). \end{aligned} \tag{23}$$

Similarly as above, the map $(\omega - 1)/(\omega y - x)$ is an meromorphic map of degree $n-1$ on the Fermat curve F_n, for n even.

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