Morfismos, Vol. 22, No. 2, 2018, pp. 1–6

On the equivarant De-Rham cohomology for non-compact Lie groups

Camilo Arias Abad Bernardo Uribe

Abstract

Let G be a connected and non-necessarily compact Lie group acting on the connected manifold M. In this short note we announce the following result: for a G-invariant closed differential form on M, the existence of a closed equivariant extension in the Cartan model for equivariant cohomology is equivalent to the existence of an extension in the homotopy quotient.

2010 Mathematics Subject Classification: 57R91
Keywords and phrases: Equivariant Cartan complex, non-compact Lie group, equivariant cohomology.

1 Introduction

For a manifold M endowed with the action of a Lie group G, the Cartan model for the G-equivariant cohomology of the manifold M could be seen as the De Rham version for the equivariant cohomology. Whenever the Lie group G is compact, Cartan showed an equivariant version of the De Rham Theorem, thus stating that the G-equivariant cohomology of the Cartan complex is canonically isomorphic to the cohomology with real coefficients of the homotopy quotient $M \times_G EG$ [1] cf. [4, Thm. 2.5.1]. When the Lie group is not compact, the G-equivariant cohomology of the Cartan complex fails to be isomorphic to the cohomology of the homotopy quotient; therefore it was not possible to use topological properties of the homotopy quotient in order to obtain information on the Cartan differential forms.

In this short note we investigate the relation between the cohomology of the G-equivariant Cartan complex of M and the cohomology of the homotopy quotient $M \times_G EG$, and we announce that indeed there is a surjective map from the former to the latter. In particular this result implies that for a G-invariant closed differential form on M, the existence of a closed equivariant extension in the Cartan model for equivariant cohomology is equivalent to the existence of an extension in the homotopy quotient.

2 Equivariant Cartan complex for connected Lie groups

Let G be a connected Lie group with lie algebra $\mathfrak{g}$. Let $K \subset G$ be the maximal compact subgroup of G and denote by $\mathfrak{k}$ its Lie algebra. The inclusion of Lie algebras $\mathfrak{k} \hookrightarrow \mathfrak{g}$ induces a dual map $\mathfrak{g}^* \to \mathfrak{k}^*$ which is $\mathfrak{k}$-equivariant. Therefore we have the K-equivariant map

$$S(\mathfrak{g}^*) \to S(\mathfrak{k}^*)$$

between the symmetric algebra on $\mathfrak{g}^*$ to the symmetric algebra on $\mathfrak{k}^*$.

Consider a manifold M endowed with an action of G. The Cartan complex associated to the G-manifold M is

$$\Omega^*_G M := (S(\mathfrak{g}^*) \otimes \Omega^* M)^G, \quad d_G = d + \Omega^a \iota_{X_a}$$

where a runs over a base of $\mathfrak{g}$, Ω^a denotes the element in $\mathfrak{g}^*$ dual to a and X_a is the vector field on M that defines the element $a \in \mathfrak{g}$.

The composition of the natural maps

$$(S(\mathfrak{g}^*) \otimes \Omega^* M)^G \hookrightarrow (S(\mathfrak{g}^*) \otimes \Omega^* M)^K \to (S(\mathfrak{k}^*) \otimes \Omega^* M)^K$$

induces a homomorphism of Cartan complexes

$$\Omega^*_G M \to \Omega^*_K M.$$

Theorem 2.1. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, let $\mathfrak{k}$ be the Lie algebra of the maximal compact subgroup K of G and consider a G-manifold M. Then the map*

$$\Omega^*_G M \to \Omega^*_K M$$

induces a surjective map in cohomology

$$H^*(\Omega^*_G M, d_G) \twoheadrightarrow H^*(\Omega^*_K M, d_K).$$

Since there are canonical isomorphisms $H^*(\Omega^*_K M, d_K) \cong H^*(M \times_K EK, \mathbb{R}) \cong H^*(M \times_G EG, \mathbb{R})$, we conclude that the canonical map

$$H^*(\Omega^*_G M, d_G) \twoheadrightarrow H^*(M \times_G EG, \mathbb{R})$$

is surjective.

Sketch of proof. Consider the complex $C^k(G, S(\mathfrak{g}^*) \otimes \Omega^\bullet M)$ defined in [3, Section 2.1] whose elements are smooth maps

$$f(g_1, \ldots, g_k|X) : G^k \times \mathfrak{g} \to \Omega^\bullet M,$$

which vanish if any of the arguments g_i equals the identity of G. These maps could also be seen as maps

$$f : G^k \to S(\mathfrak{g}^*) \otimes \Omega^\bullet M,$$

and whenever the map has for image a homogeneous polynomial of degree l, then its total degree is $deg(f) = k + l$. The differentials d and ι are defined by the formulas

$$\begin{aligned}(df)(g_1, \ldots, g_k|X) &= (-1)^k df(g_1, \ldots, g_k|X) \qquad \text{and}\\ (\iota f)(g_1, \ldots, g_k|X) &= (-1)^k \iota(X) f(g_1, \ldots, g_k|X),\end{aligned}$$

as in the case of the differentials in Cartan's model for equivariant cohomology [1, 4].

The differential $\bar{d} : C^k \to C^{k+1}$ is defined by the formula

$$\begin{aligned}(\bar{d}f)(g_0, \ldots, g_k|X) &= f(g_1, \ldots, g_k|X)\\ &\quad + \sum_{i=1}^{k} (-1)^i f(g_0, \ldots, g_{i-1}g_i, \ldots, g_k|X)\\ &\quad + (-1)^{k+1} g_k f(g_0, \ldots, g_{k-1}|\mathrm{Ad}(g_k^{-1})X),\end{aligned}$$

and the fourth differential $\bar{\iota} : C^k \to C^{k-1}$ is defined by the formula

$$(\bar{\iota} f)(g_1, \ldots, g_{k-1}|X) = \sum_{i=0}^{k-1} (-1)^i \frac{\partial}{\partial t} f(g_1, \ldots, g_i, e^{tX_i}, g_{i+1} \ldots, g_{k-1}|X),$$

where $X_i = \mathrm{Ad}(g_{i+1} \ldots g_{k-1})X$.

The structural maps $d, \iota, \bar{d}$ and $\bar{\iota}$ are all of degree 1, and the operator

$$d_G = d + \iota + \bar{d} + \bar{\iota}$$

becomes a degree 1 map that squares to zero.

The cohomology of the complex

$$(C^*(G, S(\mathfrak{g}^*) \otimes \Omega^\bullet M), d_G)$$

will be denoted by

$$H^*(G, S(\mathfrak{g}^*) \otimes \Omega^\bullet M)$$

and in [3, Thm. 2.2.3] it was shown that there is a canonical isomorphism of rings

$$H^*(G, S(\mathfrak{g}^*) \otimes \Omega^\bullet M) \cong H^*(M \times_G EG; \mathbb{R})$$

Note that there are natural maps of complexes

$$C^*(G, S(\mathfrak{g}^*) \otimes \Omega^* M) \to C^*(K, S(\mathfrak{k}^*) \otimes \Omega^* M)$$

inducing an isomorphism on its cohomology groups

$$H^*(G, S(\mathfrak{g}^*) \otimes \Omega^* M) \stackrel{\cong}{\to} H^*(K, S(\mathfrak{k}^*) \otimes \Omega^* M).$$

This isomorphism follows from the fact that the inclusion $K \subset G$ is a homotopy equivalence inducing a homotopy equivalence

$$M \times_K EK \simeq M \times_G EG$$

and the fact that

$$H^*(M \times_G EG, \mathbb{R}) \cong H^*(G, S(\mathfrak{g}^*) \otimes \Omega^* M)$$

for any connected Lie group G.

Filtering the double complex $C^*(G, S(\mathfrak{g}^*) \otimes \Omega^* M)$ by the degree of the elements in $S(\mathfrak{g}^*) \otimes \Omega^* M$ we obtain a spectral sequence whose first page is

$$E_1 = H^*_d(G, S(\mathfrak{g}^*) \otimes \Omega^* M),$$

the differentiable cohomology of G with values in the graded representation $S(\mathfrak{g}^*) \otimes \Omega^* M$. Note that in the 0-th row we obtain

$$E_1^{*,0} = (S(\mathfrak{g}^*) \otimes \Omega^* M)^G = \Omega^*_G M.$$

The same degree filtration applied to the complex $C^*(K, S(\mathfrak{k}^*) \otimes \Omega^* M)$ produces a spectral sequence which at the first page is $\overline{E}_1 = H^*_d(K, S(\mathfrak{k}^*) \otimes \Omega^* M)$, and since K is compact this simply becomes

$$\overline{E}_1^{*,0} = (S(\mathfrak{k}^*) \otimes \Omega^* M)^K = \Omega^*_K M$$

with $\overline{E}_1^{p,q} = 0$ for $q \neq 0$.

The first differential of the spectral sequence once restricted to the 0-th row $E_1^{*,0} = \Omega_G^* M$ is precisely the differential of the Cartan complex; therefore we obtain

$$E_2^{*,0} = H^*(\Omega_G^* M).$$

Equivalently we obtain

$$\overline{E}_2^{*,0} = H^*(\Omega_K^* M) \cong H^*(M \times_K EK, \mathbb{R}),$$

but in this case the spectral sequence collapses at the second page and the only non zero elements in $\overline{E}_\infty$ appear on the 0-th row $\overline{E}_\infty^{*,0} \cong H^*(M \times_K EK, \mathbb{R})$.

The canonical map between the complexes

$$C^*(G, S(\mathfrak{g}^*) \otimes \Omega^* M) \to C^*(K, S(\mathfrak{k}^*) \otimes \Omega^* M)$$

induces a map of spectral sequences $E_\bullet \to \overline{E}_\bullet$, and we know that at the pages at infinity it should induce an isomorphism $E_\infty^{*,\star} \stackrel{\cong}{\to} \overline{E}_\infty^{*,\star}$. Therefore the map

$$E_2^{*,0} \to \overline{E}_2^{*,0}$$

must be a surjective map, and hence we have the canonical map

$$\Omega_G^* M = E_1^{*,0} \to \overline{E}_1^{*,0} = \Omega_K^* M$$

inducing the desired surjective map in cohomology

$$H^*(\Omega_G^* M, d_G) \twoheadrightarrow H^*(\Omega_K^* M, d_K).$$ □

The complete and detailed proof of the previous theorem, as well as its applications, will appear in a forthcoming publication.

Finally, from the previous theorem we may conclude:

Corollary 2.2. *Consider a G-invariant closed differential form on M. This differential from may be extended to a closed G-equivariant differential form in the Cartan complex of M if and only if the cohomology class of the differential form may be extended to a cohomology class in the homotopy quotient $M \times_G EG$.*

This result generalizes similar statements that appeared in [2] in the case that the Lie group is reductive.

Camilo Arias Abad
Max-Planck-Institut fuer Mathematik,
Vivatsgasse 7,
53111 Bonn,
Germany
camiloariasabad@gmail.com

Bernardo Uribe
Departamento de Matemáticas y Estadística,
Universidad del Norte,
Km. 5 vía Puerto Colombia,
Barranquilla, Colombia
buribe@gmail.com

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