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Alternative to Euler's formula for $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ with $k \in Z^{+}$ and for even indexed Bernoulli numbers

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Abstract

This paper proposes an alternative mechanism to get an original result for the expression $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}, k \in Z^{+}$, the first related result was obtained by *Leonhard Euler* in 1732; later, we will be able to reproduce even indexed *Bernoulli* numbers from both results.

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1 Introduction

Euler's formula, obtained for the first time in 1736, gives an answer for the sum value $\sum_{n=1}^{+\infty} \frac{1}{n^{2k}}$, nowdays known as the *Riemann Zeta Function* evaluated at even positive integers $\zeta(2k)$. It is interesting to note that Euler's formula has its origin in the *Basilea* problem, which consists in finding the infinite sum of the reciprocals of the squares of positive integers $\sum_{n=1}^{+\infty} \frac{1}{n^2}$. The problem, raised to *Leibniz* by *Oldenburg* who was secretary of the *Royal Society* in 1673, had been addressed years ago by*Pietro Mengoli* and by *Wallis*. This problem was also addressed by *Jacob* and *Johan Bernoulli*, who tried to attack it using triangular numbers, but eventually realizing that such a path could not lead them

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to the correct answer. The problem obsessed *Jacob Bernoulli* to such an extent that he ended up launching it as an open challenge. It should be noted that he proved the non convergence of the *harmonic* series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\infty. \tag{1}$$

There are works ([1]) where the proof of *Euler*'s formula is done by applying probabilistic techniques. In [5] it is shown that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ while in [3] a proof is given directly. Moreover, there are results as in [4] where it is found through formulas which are proved by induction. On the other hand, in [6] and [8], in addition to giving a short proof, they also offer a general expression to calculate $\zeta(2k+1)$.

This work is developed as follows: In the second section we start by giving a synthesized explanation of how *Euler* came up with the expression for $\zeta(2)$. The section closes by presenting the first general expression for $\zeta(2k)$, which was obtained by *Euler* in 1748. In the third section we develop our innovative proposal which involves *Fourier* series and, with it, we obtain a new general formula to calculate $\zeta(2k)$. In the fourth section we give an expression to find B_{2k}.

2 Mechanism used by *Euler* to generate $\zeta(2)$

By the year 1731, the prodigious *Leonhard Euler*, had already been able to calculate the first 20 digits of the problem

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots+\frac{1}{1000^2}=1.6439345666815598031 \tag{2}$$

and of course, it goes without saying that at that time there were no mechanical devices so sophisticated to make such enormous sums and certainly much less calculators. In order to achieve this, he started from the well known *geometric* series, already well established for that time

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots+x^k+\cdots \tag{3}$$

that converges as long as $|x|<1$, which by integrating, can lead to

$$-\ln(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots+\frac{x^k}{k}+\cdots \tag{4}$$

then integrating the expression $\int_0^{\frac{1}{2}} \frac{\ln(1-x)}{x} dx$, and after careful integration by parts, *Euler* manages to come up with the famous expression

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \sum_{k=1}^{\infty} \frac{1}{2^{k-1}k^2} + [\ln 2]^2 \tag{5}$$

to find the expression for $\zeta(2)$ (see [2]). *Euler* starts from the *Taylor* series expansion of the function $\sin x$, it is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)} + \cdots \tag{6}$$

then dividing by $x(x \neq 0)$, we obtain

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + (-1)^{k-1} \frac{x^{2k-2}}{(2k-1)} + \cdots . \tag{7}$$

Now, considering the change $z = x^2$, we get

$$1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \cdots + (-1)^{k-1} \frac{z^{k-1}}{(2k-1)} + \cdots . \tag{8}$$

Since the roots of the sin x function are: $\pm\pi, \pm 2\pi, \pm 3\pi ..., \pm n\pi$, then the roots of the previous expression will be $\pm\pi^2, \pm 4\pi^2, \pm 9\pi^2 ..., \pm n^2\pi^2$, it is,

$$\begin{aligned} &1 - \tfrac{z}{3!} + \tfrac{z^2}{5!} - \tfrac{z^3}{7!} + \cdots + (-1)^{k-1} \tfrac{z^{k-1}}{(2k-1)} + \cdots \\ &= \left(1 - \tfrac{x^2}{\pi^2}\right)\left(1 - \tfrac{x^2}{4\pi^2}\right)\left(1 - \tfrac{x^2}{9\pi^2}\right) \cdots \left(1 - \tfrac{x^2}{n^2\pi^2}\right) \cdots \end{aligned} \tag{9}$$

and therefore, we will have that

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \cdots \left(1 - \frac{x^2}{n^2\pi^2}\right) \cdots . \tag{10}$$

Equating the second coefficient of (9) with the corresponding quadratic term in the previous expression, and after careful calculation, we obtain

$$\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots + \frac{1}{n^2\pi^2} + \cdots . \tag{11}$$

Equivalently,

$$\frac{1}{3!} = \frac{1}{6} = \frac{1}{\pi^2}\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots\right], \tag{12}$$

from where it is directly obtained

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}. \tag{13}$$

Due to some critiques he received (for example from Johan Bernoulli in April 1737 where he points out some deficiencies about the proof such as that the only roots of $\frac{\sin x}{x} = 0$, were $x = n\pi$, with $n = \pm 1, \pm 2, \pm 3, \ldots$), *Euler* found other more convincing solutions about this result to finally publish the generalization in his work *Introductio in analysin infinitorum* in 1748. This was done through the expression

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k!)}, \tag{14}$$

where B_{2k} corresponds to even indexed *Bernoulli* numbers.

3 Formula development

It is well known that *Jean-Baptiste Joseph Fourier* $(1768 - 1830)$ who publishes in 1822 his *Theorie analytique de la chaleur* (Analytical theory of heat), a treatise in which he established the partial differential equation that governs the diffusion of heat giving it a solution through the use of infinite series of trigonometric functions, introduced the concept that later will be called the Fourier expansion of analytic functions; in fact, any analytic function $f(x)$ defined on the interval $[-L, L]$ can be expressed as an infinite series expansion of functions of *sines* and *cosines*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{15}$$

with

$$\begin{aligned} a_0 &= \tfrac{1}{L} \textstyle\int_{-L}^{L} f(x) dx \\ a_n &= \tfrac{1}{L} \textstyle\int_{-L}^{L} f(x) \cos\left(\tfrac{n\pi x}{L}\right) dx \\ b_n &= \tfrac{1}{L} \textstyle\int_{-L}^{L} f(x) \sin\left(\tfrac{n\pi x}{L}\right) dx. \end{aligned} \tag{16}$$

It is clear that *Fourier* never imagined that the implications of his results would have a great impact on the development of engineering: Communications, signal processing, and so on.

The functions that are built in the next section fulfill the conditions that guarantee uniform convergence in the compact. They fulfill the conditions indicated in the following theorems.

Theorem. *Let f be a continuous function on the interval $-L \leq x \leq L$, such that $f(-L) = f(L)$ and whose derivative f' is quasi-continuous in that interval. Then the series*

$$\sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}$$

converges, with a_n and b_n being the Fourier coefficients.

Proof. Can be found in [9]. □

Theorem. *Under the conditions stated in the previous theorem, the convergence of the Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

to $f(x)$ in the interval $-L \leq x \leq L$ is absolute and uniform with respect to x in this interval.

Proof. Can be found in [9]. □

3.1 Obtaining the Formula for $\zeta(2k)$

Let us consider the Fourier series expansion of the function $f(x) = x^2$ defined on the interval $[-2, 2]$; after some direct calculations, we obtain the expression:

$$\begin{aligned} f_2(x) = x^2 &= \tfrac{4}{3} + \tfrac{16}{\pi^2}\textstyle\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{n\pi x}{2}\right)}{n^2} \\ &= C_2 + \tfrac{16}{\pi^2}\textstyle\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{n\pi x}{2}\right)}{n^2} \end{aligned} \tag{17}$$

where

$$C_2 = \tfrac{a_0}{2} = \tfrac{4}{3}$$

(18) $$a_n = \tfrac{(-1)^n 16}{\pi^2 n^2},\ \forall\, n \in Z^+;$$

$$b_n = 0,\ \forall\, n \in Z^+$$

Substituting in the previous expression $x = 2$ or $x = -2$, we get

$$2^2 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi)}{n^2} = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

(19) $$= \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Given that $\cos(n\pi) = \cos(-n\pi) = (-1)^n$, it can be written as

(20) $$2^2 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

when isolating $\sum_{n=1}^{\infty}\frac{1}{n^2}$, we obtain

(21) $$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

This corresponds to *Leonhard Euler*'s famous formula that he obtained in 1734 and whose solution was analyzed in the previous section.

Substituting $x = 0$ in (17) we get another important series, namely,

(22) $$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

From the expression (17), we have

(23) $$f_2(x) - C_2 = x^2 - \frac{4}{3} = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{n\pi x}{2}\right)}{n^2}.$$

Clearly the function is integrable. Let's define

$$
(24)\qquad \begin{aligned}
f_3(x) = &\ \int_0^x (f_2(x) - C_2)dx = \tfrac{x^3}{3} - \tfrac{4x}{3} \\
= &\ \int_0^x \tfrac{16}{\pi^2}\sum_{n=1}^{\infty}\tfrac{(-1)^n\cos\left(\frac{n\pi x}{2}\right)}{n^2} \\
= &\ \tfrac{32}{\pi^3}\sum_{n=1}^{\infty}\tfrac{(-1)^n\left(\sin\left(\frac{n\pi x}{2}\right)-\sin\left(\frac{n\pi 0}{2}\right)\right)}{n^3} \\
= &\ \tfrac{32}{\pi^3}\sum_{n=1}^{\infty}\tfrac{(-1)^n\sin\left(\frac{n\pi x}{2}\right)}{n^3} + C_3.
\end{aligned}
$$

The above expression coincides with the *Fourier* series expansion of the polynomial $f_3(x) = \frac{x^3}{3} - \frac{4x}{3}$ on the interval $[-2, 2]$, in this case we have that C_3 is integration constant and it is equal to zero (the function is odd), thus

$$
(25)\qquad \begin{aligned}
&a_n = 0 \ \ \forall\, n \in Z^+ \\
&b_n = (-1)^n \tfrac{32}{\pi^3 n^3}\ n \in Z^+ \\
&C_3 = \tfrac{a_0}{2} = \tfrac{1}{4}\int_{-2}^{2} f_3(x)dx = \tfrac{1}{2}\int_{-2}^{2}\left(\tfrac{x^3}{3} - \tfrac{4x}{3}\right)dx = 0.
\end{aligned}
$$

As a consequence

$$
(26)\qquad f_3(x) = \frac{x^3}{3} - \frac{4x}{3} = \frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\dfrac{n\pi x}{2}\right)}{n^3}.
$$

If in the previous equation we evaluate at x =1, we get the series

$$
(27)\qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3} = -\frac{\pi^3}{32}
$$

Now, let's consider the function

$$
(28)\qquad \begin{aligned}
f_4(x) = &\ \int_0^x \left(\tfrac{x^3}{3} - \tfrac{4x}{3} - C_3\right)dx = \tfrac{x^4}{12} - \tfrac{4x^2}{6} \\
= &\ -\tfrac{64}{\pi^4}\sum_{n=1}^{\infty}\tfrac{(-1)^n\left(\cos\left(\frac{n\pi x}{2}\right)-\cos\left(\frac{n\pi 0}{2}\right)\right)}{n^4} \\
= &\ -\tfrac{64}{\pi^4}\sum_{n=1}^{\infty}\tfrac{(-1)^n\cos\left(\frac{n\pi x}{2}\right)}{n^4} + \tfrac{64}{\pi^4}\sum_{n=1}^{\infty}\tfrac{(-1)^n}{n^4} \\
= &\ -\tfrac{64}{\pi^4}\sum_{n=1}^{\infty}\tfrac{(-1)^n\cos\left(\frac{n\pi x}{2}\right)}{n^4} + C_4.
\end{aligned}
$$

We have that $C_3 = 0$; but we include the constant to generate the algorithm. We have that the previous expression is the *Fourier* expansion of the polynomial $f_4(x) = \frac{x^4}{12} - \frac{4x^2}{6}$ on the interval $[-2, 2]$, where the *Fourier* coefficients are

$$a_n = -\tfrac{64(-1)^n}{\pi^4 n^4}, \ \ \forall\, n \in Z^+$$

$$b_n = 0, \ \ \forall\, n \in Z^+ \tag{29}$$

$$C_4 = \tfrac{a_0}{2} = \tfrac{1}{4}\int_{-2}^{2} f_4(x)dx = \tfrac{1}{4}\int_{-2}^{2}\left(\tfrac{x^4}{12} - \tfrac{4x^2}{6}\right)dx = -\tfrac{28}{45}.$$

From equations (28) and (29), we get

$$\frac{64}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = -\frac{28}{45}. \tag{30}$$

From the previous expression we conclude that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = -\frac{7}{720}\pi^4. \tag{31}$$

Therefore

$$f_4(x) = \frac{x^4}{12} - \frac{4x^2}{6} = -\frac{64}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{n\pi x}{2}\right)}{n^4} - \frac{28}{45}. \tag{32}$$

Equivalently

$$f_4(x) - C_4 = \frac{x^4}{12} - \frac{4x^2}{6} + \frac{28}{45} = -\frac{64}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{n\pi x}{2}\right)}{n^4}. \tag{33}$$

Substituting in (32) $x = 2$ o $x = -2$, we get

$$\frac{2^4}{12} - \frac{4(2^2)}{6} = -\frac{64}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi)}{n^4} - \frac{28}{45} = -\frac{64}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} - \frac{28}{45}. \tag{34}$$

Equivalently

$$\frac{2^4}{12} - \frac{2^4}{6} = -\frac{64}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} - \frac{28}{45}. \tag{35}$$

Isolating from the above expression $\sum_{n=1}^{\infty}\frac{1}{n^4}$, we have

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}. \tag{36}$$

From equation (33), let us obtain $f_5(x)$:

$$\begin{aligned} f_5(x) &= \textstyle\int_0^x (f_4(x) - C_4)dx = \frac{x^5}{60} - \frac{4x^3}{18} + \frac{28x}{45} \\ &= -\frac{128}{\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^n\left(\sin\left(\frac{n\pi x}{2}\right) - \sin\left(\frac{n\pi 0}{2}\right)\right)}{n^5} \\ &= -\frac{128}{\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^n \sin\left(\frac{n\pi x}{2}\right)}{n^5} + C_5. \end{aligned} \tag{37}$$

Calculating the *Fourier* expansion of the previous expression we get $C_5 = 0$. Evaluating the last equation at $x = 1$, we have the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^5} = -\frac{5\pi^5}{1536}. \tag{38}$$

Let's get the function $f_6(x)$, based on the foregoing

$$\begin{aligned} f_6(x) = &\ \textstyle\int_0^x (f_5(x) - C_5)dx = \frac{x^6}{360} - \frac{x^4}{18} + \frac{14x^2}{45} \\ = &\ \frac{256}{\pi^6}\sum_{n=1}^{\infty}\frac{(-1)^n\left(\cos\left(\frac{n\pi x}{2}\right) - \cos\left(\frac{n\pi 0}{2}\right)\right)}{n^6} \\ = &\ \frac{256}{\pi^6}\sum_{n=1}^{\infty}\frac{(-1)^n \cos\left(\frac{n\pi x}{2}\right)}{n^6} - \frac{256}{\pi^6}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^6} \\ = &\ \frac{256}{\pi^6}\sum_{n=1}^{\infty}\frac{(-1)^n \cos\left(\frac{n\pi x}{2}\right)}{n^6} + C_6. \end{aligned} \tag{39}$$

Since the previous expression corresponds to the Fourier expansion of the function, we have that C_6 is defined by

$$C_6 = \frac{1}{4}\int_{-2}^{2} f_6(x)dx = \int_{-2}^{2}\left(\frac{x^6}{360} - \frac{x^4}{18} + \frac{14x^2}{45}\right)dx = \frac{248}{945}. \tag{40}$$

From the above we obtain that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^6} = -\frac{31\pi^6}{30240}. \tag{41}$$

So we have that $f_6(x)$ is expressed as

$$f_6(x) = \frac{x^6}{360} - \frac{x^4}{18} + \frac{14x^2}{45} = \frac{256}{\pi^6}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{n\pi x}{2}\right)}{n^6} + \frac{248}{945}. \tag{42}$$

Considering that $\cos(n\pi) = (-1)^n$ and substituting in the previous equation $x = 2$ or $x = -2$. Hence, isolating $\sum_{n=1}^{\infty}\frac{1}{n^6}$, we get

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{256}[f_6(2) - C_6] = \frac{\pi^6}{256}\left[f_6(2) - \frac{248}{945}\right] = \frac{\pi^6}{945}, \tag{43}$$

which can be expressed as

$$\zeta(6) = \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}. \tag{44}$$

Using the same notation, we get

$$\begin{aligned} \zeta(2) = \zeta(2(1)) = \textstyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} &= \frac{\pi^2}{16}[f_2(2) - C_2], \\ \zeta(4) = \zeta(2(2)) = \textstyle\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90} &= -\frac{\pi^4}{64}[f_4(2) - C_4], \\ \zeta(6) = \zeta(2(3)) = \textstyle\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945} &= \frac{\pi^6}{256}[f_6(2) - C_6]. \end{aligned} \tag{45}$$

Based on the above, we find the general expression

$$\zeta(2k) = \sum_{n=1}^{\infty}\frac{1}{n^{2k}} = (-1)^{k+1}\frac{\pi^{2k}}{4^{k+1}}[f_{2k}(2) - C_{2k}], \tag{46}$$

where the following expressions are defined recursively:

$$f_2(x) = x^2$$
$$f_n(x) = \int_0^x (f_{n-1}(x) - C_{n-1})dx, \quad \forall\, n \geq 3, \quad \text{with}$$

$$C_{n-1} = \begin{cases} 0 & \text{si } n-1 = 2k+1 \\ \frac{1}{4}\int_{-2}^{2} f_{n-1}(x)dx & \text{si } n-1 = 2k \end{cases} \qquad \forall\, n \geq 3 \tag{47}$$

$$C_{2k} = \frac{1}{4}\int_{-2}^{2} f_{2k}(x)dx = (-1)^k\frac{4^{k+1}}{\pi^{2k}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2k}} \quad \forall\, k \in Z^+$$

Let's apply the above procedure to verify

$$\zeta(8) = \zeta(2(4)) = \sum_{n=1}^{\infty}\frac{1}{n^8} = \frac{\pi^8}{9450}. \tag{48}$$

We must obtain $f_8(2)$ for which we first need to find $f_7(x)$ in terms of $f_6(x)$. From the equation of $f_n(x)$ defined in (47), we obtain

$$
\begin{aligned}
f_7(x) = \int_0^x (f_6(x) - C_6)dx = & \int_0^x \left(\frac{x^6}{360} - \frac{x^4}{18} + \frac{14x^2}{45} - \frac{248}{945} \right) dx \\
= & \frac{x^7}{2520} - \frac{x^5}{90} + \frac{14x^3}{135} - \frac{248x}{945}.
\end{aligned} \tag{49}
$$

Now, let's obtain $f_8(x)$ in terms of $f_7(x)$. We have $C_7 = 0$ because the polynomial is an odd function, thus

$$
\begin{aligned}
f_8(x) = \int_0^x (f_7(x) - C_7)dx = & \int_0^x \left(\frac{x^7}{2520} - \frac{x^5}{90} + \frac{14x^3}{135} - \frac{248x}{945} \right) dx \\
= & \frac{x^8}{20160} - \frac{x^6}{540} + \frac{14x^4}{540} - \frac{124x^2}{945}.
\end{aligned} \tag{50}
$$

Then

$$
\begin{aligned}
& f_8(2) = \frac{2^8}{20160} - \frac{2^6}{540} + \frac{14(2^4)}{540} - \frac{124(2^2)}{945} = -\frac{68}{315}, \\
& C_8 = \frac{1}{4} \int_{-2}^{2} \left(\frac{x^8}{20160} - \frac{x^6}{540} + \frac{14x^4}{540} - \frac{124x^2}{945} \right) dx = -\frac{508}{4725}.
\end{aligned} \tag{51}
$$

From the previous expressions, we get

$$
\begin{aligned}
& (-1)^5 \frac{\pi^8}{4^5} [f_8(2) - C_8] = -\frac{\pi^8}{1024} \left[-\frac{68}{315} + \frac{508}{4725} \right] \\
& = -\frac{\pi^8}{1024} \left[-\frac{512}{4725} \right] = \frac{\pi^8}{9450}.
\end{aligned} \tag{52}
$$

Therefore

$$
\zeta(8) = \frac{\pi^8}{9450} = (-1)^5 \frac{\pi^8}{4^5} [f_8(2) - C_8], \tag{53}
$$

which is what we wanted to prove. We also get

$$
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^8} = -\frac{127\, \pi^8}{1209600}. \tag{54}
$$

In order to verify this, let us apply the procedure indicated in the equations (46) and (47)

$$
\zeta(10) = \zeta(2(5)) = \sum_{n=1}^{\infty} \frac{1}{n^{10}} = \frac{\pi^{10}}{93555}. \tag{55}
$$

We must obtain $f_{10}(2)$ for which we first have to find $f_9(x)$ in terms of $f_8(x)$. From the equation of $f_n(x)$ defined in (47), we obtain

$$
\begin{aligned}
f_9(x) = & \int_0^x (f_8(x) - C_8)dx \\
= & \int_0^x \left(\tfrac{x^8}{20160} - \tfrac{x^6}{540} + \tfrac{14x^4}{540} - \tfrac{124x^2}{945} + \tfrac{508}{4725} \right) dx \\
= & \tfrac{x^9}{181440} - \tfrac{x^7}{3780} + \tfrac{14x^5}{2700} - \tfrac{124x^3}{2835} + \tfrac{508x}{4725}.
\end{aligned} \tag{56}
$$

Now, let us obtain $f_{10}(x)$ in terms of $f_9(x)$. We have $C_9 = 0$ because the polynomial is an odd function, thus

$$
\begin{aligned}
f_{10}(x) = & \int_0^x (f_9(x) - C_9)dx \\
= & \int_0^x \left(\tfrac{x^9}{181440} - \tfrac{x^7}{3780} + \tfrac{14x^5}{2700} - \tfrac{124x^3}{2835} + \tfrac{508x}{4725} \right) dx \\
= & \tfrac{x^{10}}{1814400} - \tfrac{x^8}{30240} + \tfrac{14x^6}{16200} - \tfrac{124x^4}{11340} + \tfrac{508x^2}{9450}.
\end{aligned} \tag{57}
$$

Then

$$
\begin{aligned}
f_{10}(2) = & \tfrac{2^{10}}{1814400} - \tfrac{2^8}{30240} + \tfrac{14(2^6)}{16200} - \tfrac{124(2^4)}{11340} + \tfrac{508(2^2)}{9450} \\
= & \tfrac{248}{2835}, \\
C_{10} = & \tfrac{1}{4} \int_{-2}^{2} \left(\tfrac{x^{10}}{1814400} - \tfrac{x^8}{30240} + \tfrac{14x^6}{16200} - \tfrac{124x^4}{11340} + \tfrac{508x^2}{9450} \right) dx \\
= & \tfrac{584}{13365}.
\end{aligned} \tag{58}
$$

From the above expressions, we get

$$
\begin{aligned}
& (-1)^6 \tfrac{\pi^{10}}{4^6} [f_{10}(2) - C_{10}] = \tfrac{\pi^{10}}{4096} \left[\tfrac{248}{2835} - \tfrac{584}{13365} \right] \\
& = \tfrac{\pi^{10}}{4096} \left[\tfrac{4096}{93555} \right] = \tfrac{\pi^{10}}{93555}.
\end{aligned} \tag{59}
$$

Hence

$$
\zeta(10) = \tfrac{\pi^{10}}{93555} = (-1)^6 \tfrac{\pi^{10}}{4^6} [f_{10}(2) - C_{10}], \tag{60}
$$

which is what we wanted to prove. We also get

$$
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{10}} = -\frac{73\ \pi^{10}}{6842880}. \tag{61}
$$

4 Bernoulli numbers

In Euler's formula

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}, \quad k \in Z^+, \tag{62}$$

even indexed *Bernoulli* numbers B_{2k} appear; they define a sequence of rational numbers. Initially they arise from the search of a formula to know the sum of the k^{th} powers of the first n positive integers and were named in honor of *Jacob Bernoulli* who introduced them for the first time in 1713. In a different way, *Euler* also established a formula to define Bernoulli numbers.

4.1 Formula for Bernoulli numbers B_{2k}

From equation (62) and equation (46), we obtain

$$(-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!} = (-1)^{k+1} \frac{\pi^{2k}}{4^{k+1}} [f_{2k}(2) - C_{2k}]. \tag{63}$$

From the previous expression we deduce

$$B_{2k} = \frac{(2k)!}{2^{2k-1} 4^{k+1}} [f_{2k}(2) - C_{2k}], \quad k \in Z^+. \tag{64}$$

This expression allows us to obtain even indexed *Bernoulli* numbers. In order to illustrate this, we next display calculations for the first four even indexed *Bernoulli* numbers:

$$\begin{array}{l} B_2 = \frac{2!}{2^1 4^2} [f_2(2) - C_2] = \frac{1}{16} \left[4 - \frac{4}{3}\right] = \frac{1}{6} \\ B_4 = \frac{4!}{2^3 4^3} [f_4(2) - C_4] = \frac{3}{64} \left[-\frac{4}{3} + \frac{28}{45}\right] = -\frac{1}{30} \\ B_6 = \frac{6!}{2^5 4^4} [f_6(2) - C_6] = \frac{45}{512} \left[-\frac{8}{15} - \frac{248}{945}\right] = \frac{1}{42} \\ B_8 = \frac{8!}{2^7 4^5} [f_8(2) - C_8] = \frac{315}{1024} \left[-\frac{68}{315} + \frac{508}{4725}\right] = -\frac{1}{30} \end{array} \tag{65}$$

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