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Approximation of general optimization problems*

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Abstract

This paper concerns the approximation of a general optimization problem (OP) for which the cost function and the constraints are defined on a Hausdorff topological space. This degree of generality allows us to consider OPs for which other approximation approaches are not applicable. First we obtain convergence results for a general OP, and then we present two applications of these results. The first application is to *approximation schemes* for infinite-dimensional linear programs. The second is on the approximation of the optimal value and the optimal solutions for the so-called *general capacity* problem in metric spaces.

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1 Introduction

A constrained optimization problem (OP) is, in general, difficult to solve in closed form, and so one is naturally led to consider ways to approximate it. This in turn leads to obvious questions: how "good" are the approximations? Do they "converge" in some suitable sense? These are the questions studied in this paper for a general constrained OP, where general means that the cost function and the constraints are defined on a Hausdorff topological space. This degree of generality

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is important because then our results are applicable to large classes of OPs, even in infinite-dimensional spaces. For instance, as shown in Section 3, we can deal with approximation procedures for infinite linear programming problems in vector spaces with (dual) topologies which are Hausdorff but, say, are not necessarily metrizable.

To be more specific, consider a general constrained (OP)

$$\mathbb{P}_\infty : \quad \text{minimize } \{f_\infty(x) \ : \ x \in \mathcal{F}_\infty\},$$

and a sequence of approximating problems

$$\mathbb{P}_n : \quad \text{minimize } \{f_n(x) \ : \ x \in \mathcal{F}_n\}.$$

(The notation is explained in section 2.) The questions we are interested in are:

(i) the convergence of the sequence of optimal values $\{\min \mathbb{P}_n\}$ —or subsequences thereof— to $\min \mathbb{P}_\infty$, and

(ii) the convergence of sequences of optimal solutions of $\{\mathbb{P}_n\}$ —or subsequences thereof— to optimal solutions of $\mathbb{P}_\infty$.

We give conditions under which the convergence in (i) and (ii) holds —see Theorem 2.3. We also develop two applications of these results. The first one is on *aggregation* (of constraints) *schemes* to approximate infinite-dimensional linear programs (l.p.'s). In the second application we study the approximation of the optimal value and the optimal solutions for the so-called *general capacity* (GC) problem in metric spaces.

This paper is an extended version of [2] which presents the main theoretical results concerning (i) and (ii), including of course detailed proofs. Here, we are mainly interested in the applications mentioned in the previous paragraph. The main motivation for this paper was that the convergence in (i) and (ii) is directly related to some of our work on stochastic control and Markov games [1, 3], but in fact general OPs appear in many branches of mathematics, including probability theory, numerical analysis, optimal control, game theory, mathematical economics and operations research, to name just a few [4, 5, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23].

The problem of finding conditions under which (i) and (ii) hold is of great interest, and it has been studied in many different settings — see e.g. [7, 8, 11, 14, 15, 16, 17, 19, 21, 22, 23] and their references. In particular, the problem can be studied using the notion of Γ-convergence (or epi-convergence) of sequences of functionals [6, 10]. However, the approach used in this paper is more direct and generalizes several known results [6, 10, 11, 19, 21, 22] —see Remark 2.4. Even more, Example 4.7 shows that our assumptions are strictly weaker than those considered in the latter references. Namely, in Example 4.7 we study a particular GC problem in which our assumptions are satisfied, but the assumptions considered in those references fail to hold. The GC problem has been previously analyzed in e.g. [4, 5, 13] from different viewpoints.

The remainder of the paper is organized as follows. In section 2 we present our main results on the convergence and approximation of general OPs. These results are applied in section 3 to the *aggregation schemes* introduced in [15] to approximate infinite l.p.'s. In section 4 our results are applied to the GC problem, and a particular case of the GC problem is analyzed.

2 Convergence of general OPs

We shall use the notation $\mathbb{N} := \{1, 2, \ldots\}$, $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$.

Let $\mathcal{X}$ be a *Hausdorff topological space*. For each $n \in \overline{\mathbb{N}}$, consider a function $f_n : \mathcal{X} \to \overline{\mathbb{R}}$, a set $\mathcal{F}_n \subset \mathcal{X}$, and the optimization problem

$$\begin{aligned} \mathbb{P}_n: \quad & \text{Minimize } f_n(x) \\ & \text{subject to : } \ x \in \mathcal{F}_n. \end{aligned}$$

We call $\mathcal{F}_n$ the set of *feasible solutions* for $\mathbb{P}_n$. If $\mathcal{F}_n$ is nonempty, the (optimum) *value* of $\mathbb{P}_n$ is defined as $\inf \mathbb{P}_n := \inf\{f_n(x) \mid x \in \mathcal{F}_n\}$; otherwise, $\inf \mathbb{P}_n := +\infty$. The problem $\mathbb{P}_n$ is said to be *solvable* if there is a feasible solution x^* that achieves the optimum value. In this case, x^* is called an *optimal solution* for $\mathbb{P}_n$, and the value $\inf \mathbb{P}_n$ is then written as $\min \mathbb{P}_n = f_n(x^*)$. We shall denote by $\mathcal{M}_n$ the *minimum set*, that is, the set of optimal solutions for $\mathbb{P}_n$.

To state our assumptions we will use Kuratowski's [20] concept of *outer* and *inner limits* of $\{\mathcal{F}_n\}$, denoted by $\mathbf{OL}\{\mathcal{F}_n\}$ and $\mathbf{IL}\{\mathcal{F}_n\}$, respectively, and defined as follows.

$$\mathbf{OL}\{\mathcal{F}_n\} := \{x \in \mathcal{X} \mid x = \lim_{i\to\infty} x_{n_i}, \quad \text{where} \quad \{n_i\} \subset \mathbb{N} \text{ is an increasing sequence such that } \ x_{n_i} \in \mathcal{F}_{n_i} \ \text{ for all } i\}.$$

Thus a point $x \in \mathcal{X}$ is in $\mathbf{OL}\{\mathcal{F}_n\}$ if x is an accumulation point of a sequence $\{x_n\}$ with $x_n \in \mathcal{F}_n$ for all n. On the other hand, if x is the limit of the sequence $\{x_n\}$ itself, then x is in the inner limit $\mathbf{IL}\{\mathcal{F}_n\}$, i.e.

$$\mathbf{IL}\{\mathcal{F}_n\} := \{x \in \mathcal{X} \mid x = \lim_{n\to\infty} \ x_n, \text{ where } x_n \in \mathcal{F}_n \ \text{ for all but a finite number of } n'\text{s}\}.$$

In these definitions we may, of course, replace $\{\mathcal{F}_n\}$ with any other sequence of subsets of $\mathcal{X}$. Also note that $\mathbf{IL}\{\cdot\} \subset \mathbf{OL}\{\cdot\}$.

We shall consider two sets of hypotheses.

Assumption 2.1

(a) The minimum sets $\mathcal{M}_n$ satisfy that

$$\mathbf{OL}\{\mathcal{M}_n\} \subset \mathcal{F}_\infty. \tag{1}$$

(b) If x_{n_i} is in $\mathcal{M}_{n_i}$ for all i and $x_{n_i} \to x$ (so that x is in $\mathbf{OL}\{\mathcal{M}_n\}$), then

$$\liminf_{i\to\infty} f_{n_i}(x_{n_i}) \geq f_\infty(x). \tag{2}$$

(c) For each $x \in \mathcal{F}_\infty$ there exist $N \in \mathbb{N}$ and a sequence $\{x_n\}$ with $x_n \in \mathcal{F}_n$ for all $n \geq N$, and such that $x_n \to x$ and $\lim\limits_{n\to\infty} f_n(x_n) = f_\infty(x)$.

Assumption 2.2 Parts (b) and (c) are the same as in Assumption 2.1. Moreover

(a) The minimum sets $\mathcal{M}_n$ satisfy that

$$\mathbf{IL}\{\mathcal{M}_n\} \subset \mathcal{F}_\infty. \tag{3}$$

Note that Assumption 2.1(c) implies, in particular, $\mathcal{F}_\infty \subset \mathbf{IL}\{\mathcal{F}_n\}$, but the equality does not hold necessarily. In fact, in section 4 we give an example in which Assumption 2.1 is satisfied, in particular $\mathcal{F}_\infty \subset \mathbf{IL}\{\mathcal{F}_n\}$, but $\mathcal{F}_\infty \neq \mathbf{IL}\{\mathcal{F}_n\}$ (see Example 4.7).

On the other hand, note that Assumptions 2.2 (a),(c) yield that

$$\mathbf{IL}\{\mathcal{M}_n\} \subset \mathcal{F}_\infty \subset \mathbf{IL}\{\mathcal{F}_n\}.$$

Theorem 2.3 *(a) If Assumption 2.1 holds, then*

$$\mathbf{OL}\{\mathcal{M}_n\} \subset \mathcal{M}_\infty. \tag{4}$$

In other words, if $\{x_n\}$ is a sequence of minimizers of $\{\mathbb{P}_n\}$, and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to $x \in X$, then x is optimal for $\mathbb{P}_\infty$. Furthermore, the optimal values of $\mathbb{P}_{n_i}$ converge to the optimal value of $\mathbb{P}_\infty$, that is,

$$\min \mathbb{P}_{n_i} = f_{n_i}(x_{n_i}) \to f_\infty(x) = \min \mathbb{P}_\infty. \tag{5}$$

(b) Suppose that Assumption 2.2 holds. Then

$$\mathbf{IL}\{\mathcal{M}_n\} \subset \mathcal{M}_\infty.$$

If in addition $\mathbf{IL}\{\mathcal{M}_n\}$ is nonempty, then

$$\min \mathbb{P}_n \;\to\; \min \mathbb{P}_\infty. \tag{6}$$

Proof: We only prove (a) because the proof of (b) is quite similar.

To prove (a), let $x \in \mathcal{X}$ be in the outer limit $\mathbf{OL}\{\mathcal{M}_n\}$. Then there is a sequence $\{n_i\} \subset \mathbb{N}$ and $x_{n_i} \in \mathcal{M}_{n_i}$ for all i such that

$$x_{n_i} \;\to\; x. \tag{7}$$

Moreover, by Assumption 2.1(a), x is in $\mathcal{F}_\infty$. To prove that x is in $\mathcal{M}_\infty$, choose an arbitrary $x' \in \mathcal{F}_\infty$ and let $\{x'_n\}$ and N be as in Assumption 2.1(c) for x', that is, x'_n is in $\mathcal{F}_n$ for all $n \geq N$, $x'_n \to x'$, and $f_n(x'_n) \to f_\infty(x')$. Furthermore, if $\{n_i\} \subset \mathbb{N}$ is as in (7), then the subsequence $\{x'_{n_i}\}$ of $\{x'_n\}$ also satisfies

$$x'_{n_i} \text{ is in } \mathcal{F}_{n_i}, \quad x'_{n_i} \to x', \quad \text{and} \quad f_{n_i}(x'_{n_i}) \to f_\infty(x'). \tag{8}$$

Combining the latter fact with Assumption 2.1(b) and the optimality of each x_{n_i} we get

$$\begin{aligned} f_\infty(x) &\le \liminf_{i\to\infty} f_{n_i}(x_{n_i}) && \text{(by (2))} \\ &\le \liminf_{i\to\infty} f_{n_i}(x'_{n_i}) \\ &= f_\infty(x') && \text{(by (8))}. \end{aligned}$$

Hence, as $x' \in \mathcal{F}_\infty$ was arbitrary, it follows that x is in $\mathcal{M}_\infty$, that is, (4) holds.

To prove (5), suppose again that x is in $\mathbf{OL}\{\mathcal{M}_n\}$ and let $x_{n_i} \in \mathcal{M}_{n_i}$ be as in (7). By Assumption 2.1(c), there exists a sequence $x'_{n_i} \in \mathcal{F}_{n_i}$ that satisfies (8) for x instead of x'; thus

$$\begin{aligned} f_\infty(x) &\le \liminf_{i\to\infty} f_{n_i}(x_{n_i}) && \text{(by (2))} \\ &\le \limsup_{i\to\infty} f_{n_i}(x_{n_i}) \\ &\le \limsup_{i\to\infty} f_{n_i}(x'_{n_i}) \\ &= f_\infty(x) && \text{(by (8))}. \end{aligned}$$

This proves (5). □

In Theorem 2.3 it is not assumed the solvability of each $\mathbb{P}_n$. Thus $\mathbf{OL}\{\mathcal{M}_n\}$ might be empty; in fact, it might be empty even if each $\mathbb{P}_n$ is solvable. In this case, the (convergence of minimizers) inclusion (4) trivially holds. In the convergence of the optimal values (5) and (6), unlike the convergence of minimizers, it is implicitly assumed that $\mathbf{OL}\{\mathcal{M}_n\}$ is nonempty.

Remark 2.4 (i) Parts (a) and (b) of Theorem 2.3 generalize in particular some results in [21, 22] and [11, 19], respectively. Indeed, using our notation, in [11, 19, 21, 22] it is assumed that the cost functions f_n are *continuous* and converge *uniformly* to f_∞. On the other hand, with respect to the feasible sets $\mathcal{F}_n$, in [11] it is assumed that $\mathbf{IL}\{\mathcal{F}_n\} = \mathcal{F}_\infty$, whereas in [19, 21, 22] it is required that $\mathcal{F}_n \to \mathcal{F}_\infty$ in the Hausdorff metric. These hypotheses trivially yield the following conditions:

(C_1) The *inner* and/or the *outer limit* of the feasible sets $\mathcal{F}_n$ coincide with $\mathcal{F}_\infty$, i.e.

(9) $$\mathbf{IL}\{\mathcal{F}_n\} = \mathcal{F}_\infty$$

or

$$\mathbf{OL}\{\mathcal{F}_n\} = \mathbf{IL}\{\mathcal{F}_n\} = \mathcal{F}_\infty. \tag{10}$$

(C_2) For every x in $\mathcal{X}$ and for every sequence $\{x_n\}$ in $\mathcal{X}$ converging to x, it holds that

$$\lim_{n\to\infty} f_n(x_n) = f_\infty(x). \tag{11}$$

However, instead of (10) and (11) we require (the weaker) Assumption 2.1, and instead of (9) and (11) we require (the weaker) Assumption 2.2.

(ii) Theorem 2.3 generalizes the results in [6, 10], where it is used the notion of Γ-convergence. Indeed, [6, 10] study problems of the form

$$\min_{x\in\mathcal{X}} F_n(x). \tag{12}$$

Each of our problems $\mathbb{P}_n$ can be put in the form (12) by letting

$$F_n(x) := \begin{cases} f_n(x) & \text{if} \quad x \in \mathcal{F}_n, \\ \infty & \text{if} \quad x \notin \mathcal{F}_n, \end{cases}$$

and then, when the space $\mathcal{X}$ is *first countable*, the assumptions in [6, 10] can be translated to this context as follows: the sequence $\{F_n\}$ Γ-*converges* to F_∞ —see Theorems 7.8 and 7.18 in [10]. On the other hand, when $\mathcal{X}$ is first countable, the sequence $\{F_n\}$ Γ-converges to F_∞ if and only if

(C_3) For every x in $\mathcal{X}$ and for every sequence $\{x_n\}$ in $\mathcal{X}$ converging to x, it holds that

$$\liminf_{n\to\infty} F_n(x_n) \geq F_\infty(x).$$

(C_4) For every x in $\mathcal{X}$ there exists a sequence $\{x_n\}$ in $\mathcal{X}$ converging to x such that

$$\lim_{n\to\infty} F_n(x_n) = F_\infty(x).$$

See Proposition 8.1 in [10]. It is natural to assume that $f_n(x) < \infty$ for each $x \in \mathcal{F}_n$ and $n \in \overline{\mathbb{N}}$, and that $\mathcal{F}_\infty$ is nonempty. In this case, (C_3) implies part (b) of Assumptions 2.1 and 2.2, (C_4) implies part (c), and (C_3) together with (C_4) imply part (a). Indeed, the last statement can be proved as follows. Let $x \in \mathcal{X}$ be in $\mathbf{OL}\{\mathcal{M}_n\}$. Then there is a sequence $\{n_i\} \subset \mathbb{N}$ and $x_{n_i} \in \mathcal{M}_{n_i}$ for all i such that $x_{n_i} \to x$. Now, as $\mathcal{F}_\infty$ is nonempty we can take x' in $\mathcal{F}_\infty$. For this x', let $\{x'_n\} \in \mathcal{X}$ be as in (C_4), so that

$$\begin{aligned} F_\infty(x) &\leq \liminf_{i\to\infty} F_{n_i}(x_{n_i}) && \text{(by } (C_3)) \\ &\leq \liminf_{i\to\infty} F_{n_i}(x'_{n_i}) && \text{(because } x_{n_i} \text{ is in } \mathcal{M}_{n_i}) \\ &= \lim_{i\to\infty} F_n(x'_n) && \text{(by } (C_4)) \\ &= F_\infty(x') < \infty && \text{(by } (C_4)). \end{aligned}$$

Hence x is in $\mathcal{F}_\infty$.

On the other hand, if in addition we assume that $f_n(x) \leq K$ for all $x \in \mathcal{F}_n$, $n \in \mathbb{N}$ and some $K \in \mathbb{R}$, then (C_3) and (C_4) imply (10). In fact, (C_4) implies the inclusion $\mathcal{F}_\infty \subset \mathbf{IL}\{\mathcal{F}_n\} \subset \mathbf{OL}\{\mathcal{F}_n\}$, and ($C_3$) together with the uniform boundedness condition imply the reverse inclusion.

In the next two sections we present applications of Theorem 2.3. We also show, in Example 4.7, a particular problem in which Assumption 2.1 is satisfied, but the assumptions considered in [6, 10, 11, 19, 21, 22] do not hold.

3 Approximation schemes for l.p.'s

As a first application of Theorem 2.3, in this section we consider the *aggregation* (of constraints) *schemes* introduced in [15] to approximate infinite linear programs (l.p.'s). (See also [17] or chapter 12 in [16] for applications of the aggregation schemes to some stochastic control problems.) Our main objective is to show that the convergence of these schemes can be obtained from Theorem 2.3.

First we introduce the l.p. we shall work with. Let $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Z}, \mathcal{W})$ be two dual pairs of vector spaces. The spaces $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be endowed with the weak topologies $\sigma(\mathcal{X}, \mathcal{Y})$ and $\sigma(\mathcal{Y}, \mathcal{X})$, respectively. Thus, in particular, the topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are Hausdorff. We denote by $\langle \cdot, \cdot \rangle$ the bilinear form on both $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{Z} \times \mathcal{W}$.

Let $A : \mathcal{X} \to \mathcal{Z}$ be a weakly continuous linear map with adjoint

$A^* : \mathcal{W} \to \mathcal{Y}$, i.e.

$$\langle x, A^* w \rangle := \langle Ax, w \rangle \quad \forall\, x \in \mathcal{X},\; w \in \mathcal{W}.$$

We denote by K a positive cone in $\mathcal{X}$. For given vectors $c \in \mathcal{Y}$ and $b \in \mathcal{Z}$, we consider the (primal) l.p.

$$\textbf{LP}: \text{ Minimize } \langle x, c \rangle \tag{13}$$

$$\text{subject to: } Ax = b, \quad x \in K. \tag{14}$$

A vector $x \in \mathcal{X}$ is said to be a *feasible solution* for **LP** if it satisfies (14), and we denote by $\mathcal{F}$ the set of feasible solutions for **LP**. The program **LP** is called *consistent* if it has a feasible solution, i.e. $\mathcal{F}$ is nonempty.

The following assumption ensures that **LP** is solvable.

Assumption 3.1 **LP** *has a feasible solution* x^0 *with* $\langle x^0, c \rangle > 0$ *and, moreover, the set*

$$\Delta^0 := \{x \in K | \langle x, c \rangle \le \langle x^0, c \rangle \}$$

is weakly sequentially compact.

Remark 3.2 Assumption 3.1 implies that the set $\Delta_r := \{x \in K | \langle x, c \rangle \le r\}$ is weakly sequentially compact for every $r > 0$, since $\Delta_r = (r/\langle x^0, c \rangle)\Delta^0$.

Lemma 3.3 *If Assumption 3.1 holds, then* **LP** *is solvable.*

For a proof of Lemma 3.3, see Theorem 2.1 in [15].

If E is a subset of a vector space, then $sp(E)$ denotes the space spanned (or generated) by E.

Aggregation schemes. To realize the aggregation schemes the main assumption is on the vector space $\mathcal{W}$.

Assumption 3.4 *There is an increasing sequence of finite sets* W_n *in* $\mathcal{W}$ *such that* $\mathcal{W}_\infty := \cup_{n=1}^{\infty} \mathcal{W}_n$ *is weakly dense in* $\mathcal{W}$*, where* $\mathcal{W}_n = sp(W_n)$.

For each $n \in \overline{\mathbb{N}}$, let $\mathcal{Z}_n$ be the algebraic dual of $\mathcal{W}_n$, that is, $\mathcal{Z}_n := \{f : \mathcal{W}_n \to \mathbb{R} \mid f \text{ is a linear functional}\}$. Thus $(\mathcal{Z}_n, \mathcal{W}_n)$ is a dual pair of finite-dimensional vector spaces with the natural bilinear form

$$\langle f, w\rangle := f(w) \quad \forall\, w \in \mathcal{W}_n, \quad f \in \mathcal{Z}_n.$$

Now let $A_n : \mathcal{X} \to \mathcal{Z}_n$ be the linear operator given by

$$A_n x(w) := \langle Ax, w\rangle \quad \forall\, w \in \mathcal{W}_n. \tag{15}$$

The adjoint $A_n^* : \mathcal{W}_n \to \mathcal{Y}$ of A_n is the adjoint A^* of A restricted to $\mathcal{W}_n$, that is, $A_n^* := A^*|_{\mathcal{W}_n}$. Finally, we define $b_n \in \mathcal{Z}_n$ by $b_n(\cdot) := \langle b, \cdot\rangle|_{\mathcal{W}_n}$.

With these elements we can define the aggregation schemes as follows. For each $n \in \overline{\mathbb{N}}$,

$$\begin{aligned} \mathbf{LP}_n:\ & \text{Minimize } \langle x, c\rangle \\ & \quad \text{subject to : } A_n x = b_n, \quad x \in K, \end{aligned} \tag{16}$$

which is as our problem $\mathbb{P}_n$ (in section 2) with $f_n(x) := \langle x, c\rangle$ and $\mathcal{F}_n$ the set of vectors $x \in \mathcal{X}$ that satisfy (16). The l.p. $\mathbf{LP}_n$ is called an *aggregation* (of constraints) of **LP**. Moreover, from Proposition 2.2 in [15] we have the following.

Lemma 3.5 *Under the Assumptions 3.1 and 3.4, the l.p.* $\mathbf{LP}_\infty$ *is equivalent to* **LP** *in the sense that (using Lemma 3.3)*

$$\min \mathbf{LP} = \min \mathbf{LP}_\infty. \tag{17}$$

The following lemma provides the connection between the aggregation schemes and Theorem 2.3.

Lemma 3.6 *The Assumptions 3.1 and 3.4 imply that the aggregation schemes* $\mathbf{LP}_n$ *satisfy Assumption 2.1.*

Proof: To check parts (a) and (c) of Assumption 2.1, for each $n \in \mathbb{N}$ let $x_n \in \mathcal{F}_n$ be such that $x_n \to x$ weakly in $\mathcal{X}$. Thus, by definition of the weak topology on $\mathcal{X}$, $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in \mathcal{Y}$, which in particular yields

$$\lim_{n\to\infty} \langle x_n, c\rangle = \langle x, c\rangle.$$

This implies part (b) of Assumption 2.1, and also that the sequence x_n is in the weakly sequentially compact set Δ_r for some $r > 0$ (see Remark 3.2). In particular, x is in K, and from (16) and the definitions of A_n and b_n we get

$$A_\infty x(w) = \lim_{n\to\infty} A_n x_n(w) = \lim_{n\to\infty} b_n(w) = b_\infty(w) \quad \forall\, w \in \mathcal{W}_\infty.$$

Thus x is in $\mathcal{F}_\infty$, which yields that $\mathbf{OL}\{\mathcal{F}_n\} \subset \mathcal{F}_\infty$, and so Assumption 2.1(a) follows.

Finally, to verify part (c) of Assumption 2.1, choose an arbitrary $x \in \mathcal{F}_\infty$. Then, by (15) and the definition of b_n,

$$A_\infty x(w) = \langle Ax, w\rangle = \langle b, w\rangle = b_\infty(w) \quad \forall\, w \in \mathcal{W}_\infty.$$

In particular, if $w \in \mathcal{W}_n$ for some $n \in \overline{\mathbb{N}}$, the latter equation becomes

$$A_n x(w) = b_n(w).$$

Hence $A_n x = b_n$. It follows that $\mathcal{F}_\infty \subset \mathcal{F}_n$ for all $n \in \mathbb{N}$ and, moreover, the sets $\mathcal{F}_n$ form a nonincreasing sequence, i.e.

$$\mathcal{F}_n \supseteq \mathcal{F}_{n+1} \quad \forall\, n \in \mathbb{N}, \tag{18}$$

which implies part (c) of Assumption 2.1. □

To summarize, from Lemma 3.6 and Theorem 2.3, together with (17) and (18) we get the following.

Theorem 3.7 *Suppose that Assumptions 3.1 and 3.4 are satisfied. Then*

(a) The aggregation $\mathbf{LP}_n$ *is solvable for every* $n \in \overline{\mathbb{N}}$.

(b) For every $n \in \mathbb{N}$, *let* $x_n \in \mathcal{F}_n$ *be an optimal solution for* $\mathbf{LP}_n$. *Then, as* $n \to \infty$,

$$\langle x_n, c\rangle \uparrow \min \mathbf{LP}_\infty = \min \mathbf{LP}, \tag{19}$$

and, furthermore, every weak accumulation point of the sequence $\{x_n\}$ *is an optimal solution for* $\mathbf{LP}$.

Proof: (a) It is clear that Assumption 3.1 also holds for each aggregation $\mathbf{LP}_n$. Thus the solvability of $\mathbf{LP}_n$ follows from Lemma 3.3.

(b) From Lemma 3.6, we see that Theorem 2.3(a) holds for the aggregations $\mathbf{LP}_n$. Hence to complete the proof we only need to verify (19). To do this, note that (18) yields $\min \mathbf{LP}_n \leq \min \mathbf{LP}_{n+1}$ for each $n \in \mathbb{N}$, and, moreover, the sequence of values $\min \mathbf{LP}_n$ is bounded above by $\min \mathbf{LP}_\infty$. This fact together with (5) and Lemma 3.5 give (19). □

Theorem 3.7 was obtained in [15] using a different approach.

Remark 3.8 In the aggregation schemes $\mathbf{LP}_n$, the vector spaces $\mathcal{Z}_n$ and $\mathcal{W}_n$ are finite-dimensional for $n \in \mathbb{N}$, and so each $\mathbf{LP}_n$ is a so-called *semi-infinite* l.p. Hence Theorem 3.7 can be seen as a result on the approximation of the infinite-dimensional l.p. **LP** by semi-infinite l.p.'s. On the other hand, a particular semi-infinite l.p. is when the vector space $\mathcal{X}$ of decision variables (or just the cone K) is finite-dimensional, but the vector b lies in an infinite-dimensional space $\mathcal{W}$ [5, 14, 18]. In the latter case, the aggregation schemes would be approximations to **LP** by *finite* l.p.'s.

4 The GC problem

The general capacity (GC) problem is related to the problem of determining the electrostatic capacity of a conducting body. In fact, it originated in the mentioned electrostatic capacity problem —see, for instance, [4, 5].

Let X and Y be metric spaces endowed with their corresponding Borel σ-algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$. We denote by $M(Y)$ the vector space of finite signed measures on Y, and by $M^+(Y)$ the cone of nonnegative measures in $M(Y)$.

Now let $b : X \to \mathbb{R}$, $c : Y \to \mathbb{R}$, and $g : X \times Y \to \mathbb{R}$ be nonnegative Borel-measurable functions. Then the GC problem can be stated as follows.

$$\begin{aligned} \text{GC:} \quad & \text{Minimize } \int_Y c(y)\ \mu(dy) \\ & \text{subject to : } \int_Y g(x,y)\ \mu(dy) \geq b(x) \quad \forall x \in X,\ \mu \in M^+(Y). \end{aligned}$$

In this section we study the convergence problem (see (i) and (ii) in section 1) in which g and c are replaced with sequences of nonnegative measurable functions $g_n : X \times Y \to \mathbb{R}$ and $c_n : Y \to \mathbb{R}$, for $n \in \overline{\mathbb{N}}$, such that $g_n \to g_\infty =: g$ and $c_n \to c_\infty =: c$ uniformly.

Thus we shall deal with GC problems

$$\text{GC}_n: \ \text{Minimize} \int_Y c_n(y)\ \mu(dy)$$

$$\text{subject to}: \int_Y g_n(x,y)\ \mu(dy) \geq b(x) \quad \forall x \in X,\ \mu \in M^+(Y), \tag{20}$$

for $n \in \overline{\mathbb{N}}$. For each $n \in \overline{\mathbb{N}}$, we denote by $\mathcal{F}_n$ the set of feasible solutions for GC$_n$, that is, the set of measures μ that satisfy (20), but in addition $\int_Y c_n\ d\mu < \infty$.

Convergence. We shall study the convergence issue via Theorem 2.3. First, we introduce assumptions that guarantee the solvability of the GC problems. We shall distinguish two cases for the cost functions c_n, the bounded case and the unbounded case, which require slightly different hypotheses. For the bounded case we suppose the following.

Assumption 4.1 (Bounded case) For each $n \in \overline{\mathbb{N}}$:

(a) $\mathcal{F}_n$ is nonempty.

(b) The function $g_n(x, \cdot)$ is bounded above and upper semicontinuous (u.s.c.) for each $x \in X$.

(c) The function c_n is bounded and lower semicontinuous (l.s.c.). Further c_n is *bounded away from zero*, that is, there exist $\delta_n > 0$ such that $c_n(y) \geq \delta_n$ for all $y \in Y$.

In addition,

(d) The space Y is compact.

For the unbounded case, we replace parts (c) and (d) with an inf-compactness hypothesis.

Assumption 4.2 (Unbounded case) Parts (a)-(c) are the same as in Assumption 4.1. Moreover,

(d) For each $n \in \overline{\mathbb{N}}$, the function c_n is *inf-compact*, which means that, for each $r \in \mathbb{R}$, the set $\{y \in Y | c_n(y) \leq r\}$ is compact. Further, c_n is bounded away from zero.

Observe that the inf-compactness condition implies that c_n is l.s.c.

We next introduce the assumptions for our convergence and approximation results. As above, we require two sets of assumptions depending on whether the cost functions c_n are bounded or unbounded. (See Remark 4.8 for alternative sets of assumptions.)

Assumption 4.3 (Bounded case)

(a) (*Slater condition*) There exist $\mu \in \mathcal{F}_\infty$ and $\eta > 0$ such that

$$\int_Y g_\infty(x, y)\ \mu(dy) \geq b(x) + \eta \quad \forall\ x \in X.$$

(b) $g_n \ \to\ g_\infty$ uniformly on $X \times Y$.

(c) $c_n \to c_\infty$ uniformly on Y.

Assumption 4.4 (Unbounded case) Parts (a) and (b) are the same as in Assumption 4.3. Moreover

(c) $c_n \downarrow c_\infty$ uniformly on Y.

Before stating our main result for the GC problem we recall some facts on the weak convergence of measures (for further details see [9] or chapter 12 in [16], for instance).

Definition 4.5 Let Y, $M(Y)$ and $M^+(Y)$ be as at the beginning of this section. A sequence $\{\mu_n\}$ in $M^+(Y)$ is said to be *bounded* if there exists a constant m such that $\mu_n \leq m$ for all n. Let $C_b(Y)$ be the vector space of continuous bounded functions on Y. We say that a sequence $\{\mu_n\}$ in $M(Y)$ *converges weakly* to $\mu \in M(Y)$ if $\mu_n \to \mu$ in the weak topology $\sigma(M(Y), C_b(Y))$, i.e.

$$\int_Y u\ d\mu_n \ \to\ \int_Y u\ d\mu \quad \forall u \in C_b(Y).$$

A subset M_0 of $M^+(Y)$ is said to be *relatively compact* if for any sequence $\{\mu_n\}$ in M_0 there is a subsequence $\{\mu_m\}$ of $\{\mu_n\}$ and a measure μ in $M^+(Y)$ (but not necessarily in M_0) such that $\mu_n \to \mu$ weakly. In the latter case, we say that μ is a *weak accumulation point* of $\{\mu_n\}$.

We now state our main result in this section.

Theorem 4.6 *Suppose that either Assumptions 4.1 and 4.3, or 4.2 and 4.4 hold. Then*

(a) GC_n *is solvable for every* $n \in \overline{\mathbb{N}}$.

(b) The optimal value of GC_n *converges to the optimal value of* GC_∞*, i.e.*

$$\min \mathrm{GC}_n \longrightarrow \min \mathrm{GC}_\infty. \tag{21}$$

Furthermore, if $\mu_n \in M^+(Y)$ *is an optimal solution for* GC_n *for each* $n \in \mathbb{N}$*, then the sequence* $\{\mu_n\}$ *is relatively compact, and every weak accumulation point of* $\{\mu_n\}$ *is an optimal solution for* GC_∞.

(c) If GC_∞ *has a unique optimal solution, say* μ*, then for any* μ_n *in the set of optimal solutions for* GC_n*, with* $n \in \mathbb{N}$*, the sequence* $\{\mu_n\}$ *converges weakly to* μ.

For a proof of Theorem 4.6 the reader is referred to [1].

We shall conclude this section with an example which satisfies our hypotheses, Assumption 2.1, but the hypotheses used in [6, 10, 11, 19, 21, 22] do *not* hold.

Example 4.7 This example shows a particular GC problem in the *unbounded case*, in which our assumptions are satisfied, but the conditions used in [6, 10, 11, 19, 21, 22] do *not* hold. Hence, this example shows that our assumptions are *strictly weaker* than those considered in [6, 10, 11, 19, 21, 22].

We first compare our hypotheses with those in [11, 19, 21, 22]. In our notation, the latter conditions are as follows (see Remark 2.4).

(i) For each sequence $\mu_n \in M^+(Y)$ such that $\mu_n \to \mu$ weakly we have

$$\lim_{n\to\infty} \int_Y c_n \, d\mu_n = \int_Y c_\infty \, d\mu.$$

(ii) $\mathbf{IL}\{\mathcal{F}_n\} = \mathcal{F}_\infty$.

Consider the spaces $X = Y = [0,2]$, and for each $n \in \overline{\mathbb{N}}$ let $g_n \equiv 1$, and

$$c_n(x) := \begin{cases} 1 & \text{if} \quad x = 0, \\ \frac{1}{x} & \text{if} \quad x \in (0,1], \\ 1 & \text{if} \quad x \in (1,2]. \end{cases}$$

Let $b \equiv 0$. With these elements the set $\mathcal{F}_n$ of feasible solutions for each problem GC_n is given by

$$\mathcal{F}_n := \{\mu \in M^+([0,2]) : \int g_n \, d\mu = \mu([0,2]) \geq b, \int c_n \, d\mu < \infty\}.$$

As the cost functions c_n are unbounded, we consider the Assumptions 4.2 and 4.4, which are obviously true in the present case, and which in turn imply Assumption 2.1 —see Lemma 3.11 in [1]. Next we show that (i) and (ii) *do not hold.*

Let μ be the lebesgue measure on $Y = [0,2]$, and for each $n \in \mathbb{N}$ let μ_n be the restriction of μ to $[1/n, 2]$, i.e. $\mu_n(B) := \mu(B \cap [1/n, 2])$ for all $B \in \mathcal{B}(Y)$. Thus μ_n is in $\mathcal{F}_n$ for each $n \geq 2$, and $\mu_n \to \mu$ weakly. Therefore μ is in $\mathbf{IL}\{\mathcal{F}_n\}$, but μ is *not* in $\mathcal{F}_\infty$ because $\int c_\infty \, d\mu = \infty$. Hence (ii) *does not hold.*

Similarly, let $\mu'_n := (1/k_n)\mu_n$ with $k_n := 1 + \ln(n)$. Then μ'_n is in $\mathcal{F}_n$ for all $n \geq 2$, and $\mu'_n \to 0 =: \mu'$ weakly, but

$$\int c_n \, d\mu'_n = 1 \not\to \int c_\infty \, d\mu' = 0.$$

Thus (i) *is not satisfied.*

Now we compare our assumptions with those in [6, 10]. This can be done because, as $M([0,2])$ is metrizable, the space $\mathcal{X} = M([0,2])$ is first countable. See Remark 2.4.

We take X, Y, c_n, g_n and $\mathcal{F}_n$ as above, but now we take $b = 1/2$. As in the former case, Assumption 2.1 holds. Now we slightly modify the set of feasible solutions by

$$\widetilde{\mathcal{F}}_n := \{\mu \in M^+([0,2]) : \int g_n \, d\mu = \mu([0,2]) \geq \frac{1}{2}, \int c_n \, d\mu < 1\}.$$

Notice that $\widetilde{\mathcal{F}}_n \neq \varnothing$ and $\widetilde{\mathcal{F}}_n \subset \mathcal{F}_n$ for all $n \geq 2$. The sequence of modified GC problems, say $\{\widetilde{GC}_n\}$, also satisfies Assumption 2.1. Indeed, parts (a) and (b) of Assumption 2.1 hold because the minimum sets have not changed ($\mathcal{M}_n = \widetilde{\mathcal{M}}_n$), and part (c) is true since it holds for $\mathcal{F}_\infty$, and $\widetilde{\mathcal{F}}_\infty \subset \mathcal{F}_\infty$. Next we show that condition (C_3) in Remark 2.4 does *not hold*. For each $n \in \overline{\mathbb{N}}$, let

$$F_n(\mu) := \begin{cases} \int c_n d\mu & \text{if} \quad \mu \in \widetilde{\mathcal{F}}_n, \\ \infty & \text{if} \quad \mu \notin \widetilde{\mathcal{F}}_n. \end{cases}$$

Moreover, for each $n \in \mathbb{N}$, let μ''_n be the restriction of μ to $[1+1/n, 2]$, and let μ'' be the restriction of μ to $[1, 2]$. Hence we have $\mu''_n([0, 2]) = (n-1)/n \geq 1/2$ and $\int c_n d\mu''_n = (n-1)/n < 1$ for all $n \geq 2$. Therefore, μ''_n is in $\widetilde{\mathcal{F}}_n$ for each $n \geq 2$, and $\mu''_n \to \mu''$ weakly. Thus μ'' is in $\mathbf{IL}\{\widetilde{\mathcal{F}}_n\}$, but μ'' is *not* in $\widetilde{\mathcal{F}}_\infty$ because $\int c_\infty \, d\mu'' = 1$. Hence

$$\liminf_{n\to\infty} F_n(\mu''_n) = \liminf_{n\to\infty} \int c_n d\mu''_n = 1 < \infty = F_\infty(\mu''),$$

and so (C_3) *is not satisfied*. It follows that the F_n do *not* $\Gamma-$converge to F_∞, that is, the assumptions in [6, 10] *do not hold*.

Remark 4.8 Suppose that $c_n \to c_\infty$ uniformly on Y. Then the following holds.

- If c_∞ is bounded away from zero, then so is c_n for all n sufficiently large. Hence, in part (b) of Theorem 4.6 it suffices to require (only) that c_∞ is bounded away from zero, for both cases, bounded and unbounded.
- If the sequence $\{c_n\}$ is uniformly bounded away from zero (that is, there exists $\delta > 0$ such that, for each $n \in \mathbb{N}$, $c_n(y) \geq \delta$ for all $y \in Y$), then also c_∞ is bounded away from zero.

On the other hand, if $c_n \to c_\infty$ uniformly and $g_n \to g_\infty$ uniformly, then the following holds.

- If the Slater condition holds for GC_∞ (see Assumption 4.3 (a)), then GC_n also satisfies the Slater condition for all n large enough. It follows that, for each $n \geq N$, GC_n is consistent, i.e., $\mathcal{F}_n \neq \emptyset$. Then Assumption 4.3 imply part (a) of Assumptions 4.1 and 4.2, for each $n \geq N$. Hence, in part (b) of Theorem 4.6, Assumptions 4.1(a) and 4.2(a) are not required.

- If the Slater condition uniformly holds for the sequence $\{\mathrm{GC}_n\}$ (that is, for each $n \in \mathbb{N}$, there exist $\mu_n \in \mathcal{F}_\infty$ and $\eta > 0$ such that
$$\int_Y g_n(x,y)\ \mu_n(dy) \geq b(x) + \eta \quad \forall\ x \in X),$$
then the Slater condition also holds for GC_∞.

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