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A nonmeasurable set as a union of a family of increasing well–ordered measurable sets *

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Abstract

Given a measurable space $(X, \mathcal{A})$ in which every singleton is measurable and which contains a nonmeasurable subset, we prove the existence of a nonmeasurable set which is the union of a well-ordered increasing family of measurable sets.

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1 Introduction

Using the well order principle (Zermelo's theorem) we prove, for a very general measurable space $(X, \mathcal{A})$, that there exists a well ordered family (under the inclusion) of measurable sets whose union is nonmeasurable. This study is motivated by the determination of the existence of solutions in a Markov decision problem with constraints (see [3] for this topic). The problem we faced was to find an optimal stochastic kernel supported on a measurable function. This led us to try to extend the domain of a measurable function on the union of a well–ordered family of measurable sets. However, the measurability may be missed for the union of the family, as we show below.

We also give an example of a set A contained in a measurable space where each singleton is measurable, but nevertheless A can not be expressed as a well–ordered union of measurable sets.

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Let us start by recalling some basic terminology and the statement of the well order principle.

Let X be a set.

(a) A relation $\preceq$ is called a *partial order* on X if it is reflexive, anti-symmetric and transitive. In this case, X is said to be *partially ordered* by $\preceq$.

(b) Let A be a subset of X. If there exists $x \in A$ such that $x \preceq a$ for all $a \in A$, then x is called the *first* element of A (with respect to the partial order $\preceq$).

(c) A partial order $\preceq$ on X is called a *total order* if for each $x, y \in X$ we have $x \preceq y$ or $y \preceq x$.

(d) A total order $\preceq$ in a set X is called a *well order* if every nonempty subset of X has a first element. In this case, X is said to be *well ordered.*

Theorem 1.1 (Well order principle) *Let X be a set. There is a well order $\preceq$ in X.*

The proof of this principle can be found, for instance, in [1, Well ordering theorem] or [2].

2 The result

Theorem 2.1 *Let $(X, \mathcal{A})$ be a measurable space such that, for each $x \in X$, the set $\{x\}$ is measurable, and X contains a nonmeasurable set. Then there is a collection I of measurable subsets of X, well ordered by contention ($\subset$), such that $\bigcup_{C \in I} C$ is nonmeasurable.*

Proof: Let $A \subset X$ be a nonmeasurable set. By the well order principle, there is a well order $\preceq$ in A. Denote by $\prec$ the relation $a \prec b \Longleftrightarrow (a \preceq b$ and $a \neq b)$.

For each $d \in A$ let us define $A_d := \{x \in A : \ x \preceq d\}$. Set $\mathcal{E} := \{A_d : d \in A\}$ and note that this set is well ordered by $\subset$. If all the A_d are measurable, then we take $I = \mathcal{E}$. Otherwise, there is a $d^* \in A$ such that A_{d^*} is nonmeasurable. Let $A' = \{d \in A : \ A_d$ is nonmeasurable$\}$. Since $A' \subset A$ is nonempty, there exists the first element d' of A'. Now, $A_{d'}$ is nonmeasurable and so is $A_{d'} \setminus \{d'\}$. Moreover, taking $I = \{A_d :\ d \prec d'\}$, we have

$$A_{d'} \setminus \{d'\} = \{d \in A :\ d \prec d'\} = \bigcup_{d \prec d'} A_d = \bigcup_{C \in I} C,$$

and, therefore, we can conclude that the set $\bigcup_{C \in I} C$ is nonmeasurable. Noting again that I is well ordered by $\subset$, the proof is complete. □

3 An example

We shall give an example of a measurable space in which each singleton is measurable, but there exists a nonmeasurable set A that is not the union of measurable sets in a well ordered family (under $\subset$).

For every set B, let $\#B$ denote the cardinality of B and 2^B the power set of B.

Let X be a set such that $\#X > \#\mathbb{R}$ (we can take $X = 2^{\mathbb{R}}$, for instance). Define the σ-algebra $\mathcal{A}$ as the family of subsets A of X such that $A \in \mathcal{A} \iff A$ is countable or $X \setminus A$ is countable. We can take $A \subset X$ such that $\#A > \#\mathbb{R}$ and $\#(X \setminus A) > \#\mathbb{R}$. Let I be a well-ordered index set, and assume that $(A_i)_{i \in I}$ is any strictly increasing net of measurable sets such that $\bigcup_{i \in I} A_i = A$. As each $X \setminus A_i \supset X \setminus A$ is uncountable, each A_i is countable. From Theorem 14, p. 179 in [2], we can see that $\#I = \#A > \#\mathbb{R}$, so the set $J := \{i \in I :\ \#\{j \in I :\ j \preceq i\} > \#\mathbb{N}\}$ is nonempty. Let i^* be the first element of J and observe that $\#\{j \in I :\ j \preceq i^*\} > \#\mathbb{N}$. Now, by the axiom of choice (see [1] or [2]), for each $i \in I$ we can choose $x_i \in A_i \setminus \bigcup_{j \prec i} A_i$, such that the sets $\{j \in I :\ j \preceq i^*\}$ and $\bigcup_{j \preceq i^*} \{x_j\}$ have the same cardinality. However, $\bigcup_{j \preceq i^*} \{x_i\} \subset \bigcup_{j \preceq i^*} A_j = A_{i^*}$, and so $\#A_{i^*} \geq \#\{j \in I :\ j \preceq i^*\} > \#\mathbb{N}$; that is to say, the set A_{i^*} is uncountable, and we arrive at a contradiction because each A_i is countable. Hence, A cannot be the union of measurable sets in a well ordered family.

We would like to conclude by posing a question. Consider the measurable space $(\mathbb{R}, \mathcal{M})$, where $\mathcal{M}$ is the Lebesgue σ-algebra, and let A be an arbitrary nonmeasurable subset of $\mathbb{R}$ (for an example of a non-Lebesgue measurable set see [4]). Is it always possible to express A as the limit of an increasing net $(A_i)_{i \in I}$ of elements in $\mathcal{M}$ for some well ordered set I?

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