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# CONFIGURATION SPACES *

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**Abstract**

This article is intended as a brief introduction to the theory of configuration spaces as well as to some of the basic techniques used in Algebraic Topology. Some recent results about loop spaces of configuration spaces are also presented at the end.

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## 1 Introduction

Given a topological space $X$, let $X^k$ be the $k$-fold cartesian product of $X$ with itself, $X^k = \{(x_1, \ldots, x_k) \mid x_i \in X\}$.

**Definition 1.1** *The configuration space $F(X,k)$ of $k$-ordered tuples in $X$ is given by:*

$$F(X,k) = \{(x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \quad \text{if} \quad i \neq j\}.$$

In other words, $F(X,k)$ is the subspace that results of removing all the "diagonals" from $X^k$.

A natural question that arises now is: Can we describe $F(X,k)$ as a topological space in terms of $X$ ? We will try to give an answer for a wide range of spaces. In this section, we consider some simple examples.

***Invited article.** All the new results that appear here are part of joint work with Frederick R. Cohen.

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**Example:** Let $X = \mathbb{R}^n$ be the $n$-dimensional euclidean space (which is also a vector space) and consider the configuration space $F(\mathbb{R}^n, k)$. If $(x_1, \ldots, x_k) \in F(\mathbb{R}^n, k)$, then $x_i \neq x_j$ or equivalently $x_i - x_j \neq \vec{0}$ for all $i, j$. Thus we can define a continuous map

$$\phi : F(\mathbb{R}^n, k) \longrightarrow \mathbb{R}^n \times F(\mathbb{R}^n - \{\vec{0}\}, k-1)$$

given by $\phi(x_1, \ldots, x_k) = (x_1, x_2 - x_1, \ldots, x_k - x_1)$. One verifies easily that $\phi$ is a homeomorphism. Clearly, the inverse is $\phi^{-1}(y_1, \ldots, y_n) = (y_1, y_2 + y_1, \ldots, y_n + y_1)$.

Now, let $\pi : F(\mathbb{R}^n, k) \to \mathbb{R}^n$ be the projection onto the first coordinate. Then we have:

$$\pi^{-1}(\vec{0}) = \{(\vec{0}, y_1, \ldots, y_{k-1}) \mid y_i \neq \vec{0}, \;\; y_i \neq y_j \;\text{ if }\; i \neq j\}.$$

Thus $\pi^{-1}(\vec{0})$ is homeomorphic to $F(\mathbb{R}^n - \{\vec{0}\}, k-1)$. We have then established that the map $F(\mathbb{R}^n, k) \xrightarrow{\pi} \mathbb{R}^n$ is a fibre bundle (and therefore a fibration) with fibre $F(\mathbb{R}^n - \{\vec{0}\}, k-1)$. Moreover, it is a product bundle since

$$F(\mathbb{R}^n, k) \;\simeq\; \mathbb{R}^n \times F(\mathbb{R}^n - \{\vec{0}\}, k-1).$$

The argument given above depends only on the fact that $(\mathbb{R}^n, +)$ is a topological group. In such a case we have

**Theorem 1.2** *If $G$ is a topological group, and $e \in G$ is the identity. Then there is a homeomorphism:*

$$F(G, k) \simeq G \times F(G - \{e\}, k-1).$$

More generally, in the classical paper [Fad-Neu] Fadell and Neuwirth proved that if $M$ is a manifold and $Q_i$ a subset of $i$ distinct points of $M$, then there is a fibration:

$$\begin{array}{ccc} F(M,k) & \longleftarrow & F(M - Q_i, k-i) \\ \pi\big\downarrow & & \\ F(M,i) & & \end{array}$$

where $i \leq k$ and $\pi$ is the projection on the first $i$ coordinates. In particular, in the case of $M = \mathbb{R}^n$, we have a sequence of fibrations:

(1)

$$\begin{array}{ccccccc} F(\mathbb{R}^n, k) & \longleftarrow & F(\mathbb{R}^n - Q_1, k-1) & \longleftarrow & \cdots \longleftarrow & F(\mathbb{R}^n - Q_{k-1}, 1) \\ \downarrow & & \downarrow & & & \downarrow \approx \\ \mathbb{R}^n & & \mathbb{R}^n - Q_1 & & & \mathbb{R}^n - Q_{k-1} \end{array}$$

where all vertical maps are projections and the horizontal ones are inclusions of the fibres.

## 2 The Homology of $F(R^n, k)$ and $F(S^n, k)$

In algebraic topology we study invariants of topological spaces. A systematic way to do this is to consider certain families of functors $T_q : \mathcal{TOP} \longrightarrow \mathcal{AB} \ \ q \geq 0$ , from the category of topological spaces to the category of abelian groups.

Each functor is a rule that associates to every topological space $X$, an abelian group $T_q(X)$ and to every continuous map $f : X \to Y$ assigns a homomorphism

$$T_q(f) : T_q(X) \longrightarrow T_q(Y)$$

such that:

- $T_q(1_X) = 1_{T_q(X)}$ where $1_X : X \to Y$ and $1_{T_q(X)} : T_q(X) \to T_q(Y)$ are the identity maps.
- $T_q(g \circ f) = T_q(g) \circ T_q(f)$ for all $f : X \to Y$, $g : Y \to Z$.

The kind of functors that are most frequently used in algebraic topology are homotopy functors:

**Definition 2.1** *Let $f, g : X \to Y$ be two maps from $X$ to $Y$. We say that $f$ is homotopic to $g$ if there is a map $H : X \times I \longrightarrow Y$ such that*

$$H(x,0) = f(x) \quad \forall x \in X,$$

$$H(x,1) = g(x) \quad \forall x \in X.$$

The map $H$ is called a homotopy between $f$ and $g$ and we write $f \simeq g$. It is clear that homotopy is an equivalence relation.

**Definition 2.2** *Given a functor $T : \mathcal{TOP} \to \mathcal{AB}$ we say that it is a homotopy functor if $T(f) = T(g)$ whenever $f \simeq g$.*

*Two spaces are of the same homotopy type (or homotopy equivalent) and we write $X \simeq Y$ if there are maps $f : X \to Y$ and $g : Y \to X$ such that*

$$f \circ g \simeq 1_Y$$

$$g \circ f \simeq 1_X$$

As a consequence, if $X \simeq Y$ and $T$ is a homotopy functor, then the groups $T(X)$ and $T(Y)$ are isomorphic.

The homotopy functor we are interested in here is $H_q = H_q(\ \ ;\mathbb{Q})$ the $q$th-homology group with rational coefficients. Thus $H_q(X)$ is a vector space over $\mathbb{Q}$ and we will restrict ourselves to those spaces $X$ for which $H_q(X)$ is a *finite dimensional* vector space.

Associated to the sequence of vector spaces $\{H_q(X)\}_{q=0}^{\infty}$ we have the *Poincaré series* of $X$:

$$P(X) \ = \ \sum_{n=0}^{\infty} a_n(X)\, t^n$$

where $a_n(X) = dim H_n(X)$. $P(X)$ is a formal series on the variable $t$. For example, when $X = \mathbb{R}^n$ or $S^n$ we have

$$\begin{aligned} P(\mathbb{R}^n) &= 1, \\ P(S^n) &= 1 + t^n, \end{aligned}$$

where $S^n$ is the $n$-sphere in $\mathbb{R}^{n+1}$ : $\quad S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid \ \|\vec{x}\| = 1\}$.

The Poincaré series has very nice features:

For example, using the Künneth formulas $H_q(X \times Y) = \bigoplus_{r=0}^{q} H_r(X) \otimes H_{q-r}(Y)$ we deduce that:

$$P(X \times Y) = P(X) \cdot P(Y) \ .$$

Also, if $(X, x_o)$ and $(Y, y_o)$ are well pointed and connected spaces and $X \vee Y$ is the one-point union of $X$ and $Y$

$$X \vee Y = \{(x, y_o) \in X \times Y \mid x \in X\} \cup \{(x_o, y) \in X \times Y \mid y \in Y\}$$

then

$$H_q(X \vee Y) \cong \begin{cases} \mathbb{Q} & \text{if } \ q = 0 \\ H_q(X) \oplus H_q(Y) & \text{if } \ q \geq 1 \ . \end{cases}$$

Thus we have

$$P(X \vee Y) = P(X) + P(Y) - 1.$$

Now, let us return to the fibration diagram (1). It is easy to see that

$$\mathbb{R}^n - Q_j \ \simeq \ \underbrace{S^{n-1} \vee \cdots \vee S^{n-1}}_{j}$$

and therefore $P(\mathbb{R}^n - Q_j) = 1 + jt^{n-1}$.

On the other hand, the fibrations in (1) are not products anymore, but homologically they behave as if they were, this is

$$H_*F(\mathbb{R}^n, k) \ \cong \ H_*(\mathbb{R}^n) \otimes H_*(\mathbb{R}^n - Q_1) \otimes \cdots \otimes H_*(\mathbb{R}^n - Q_{k-1}).$$

In other words:

$$P(F(\mathbb{R}^n, k)) \ = \ \prod_{j=1}^{k-1} (1 + jt^{n-1}). \tag{2}$$

It can also be proven that:

$$P(F(S^n, k)) \;=\; (1+t^n)\prod_{j=1}^{k-2}(1+jt^{n-1}). \tag{3}$$

This is the only known example for the Poincaré series of $F(M,k)$ where $M$ is a closed manifold, that is, compact and without boundary.

## 3 The Loop Space of $F(M,k)$

**Definition 3.1** *Let $(X, x_o)$ be a pointed topological space. The loop space of $X$ based at $x_o$, $\Omega X$ is the space of continuous maps:*

$$\Omega X = \{\alpha : [0,1] \longrightarrow X \mid \alpha(0) = \alpha(1) = x_o\}$$

*equipped with the compact-open topology.*

In $\Omega X$ we can define a multiplication $\Omega X \times \Omega X \xrightarrow{\mu} \Omega X$ , mapping the pair $(\alpha, \beta)$ to the loop $\alpha * \beta$ given by

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \;. \end{cases}$$

The map $\mu$ is continuous, and therefore, it induces for every $q \ge 0$ a homomorphism

$$H_q(\Omega X \times \Omega X) \xrightarrow{H_q(\mu)} H_q(\Omega X).$$

Again by the Künneth formulas we have that

$$H_q(\Omega X \times \Omega X) \;=\; \bigoplus_{i=0}^{q} H_i(\Omega X) \otimes H_{q-i}(\Omega X),$$

and thus, as graded vector spaces

$$H_*(\Omega X \times \Omega X) \cong H_*(\Omega X) \otimes H_*(\Omega X).$$

Therefore, $\mu$ induces an algebra structure on $H_*(\Omega X)$

$$H_*(\mu) \;:\; H_*(\Omega X) \otimes H_*(\Omega X) \longrightarrow H_*(\Omega X).$$

On the other hand, the diagonal map $\Omega X \xrightarrow{\Delta} \Omega X \times \Omega X$ given by $\Delta(\alpha) = (\alpha, \alpha)$ induces a *comultiplication*

$$H_*(\Omega X) \xrightarrow{H_*(\Delta)} H_*(\Omega X) \otimes H_*(\Omega X).$$

Recall the following definition (see for example [Mil-Mo]):

**Definition 3.2** *Let $F$ be a field. A Hopf algebra over $F$ is a graded vector space $H$ together with a multiplication $\mu: H \otimes H \to H$ , a comultiplication $\Delta : H \to H \otimes H$ , a unit $\eta : F \to H$ and an augmentation $\epsilon: H \to F$ , such that*

1. *$(H, \mu, \eta)$ is an algebra over $F$ with augmentation $\epsilon$.*
2. *$(H, \Delta, \epsilon)$ is a coalgebra over $F$ with unit $\eta$.*
3. *The following diagram commutes:*

$$\begin{array}{ccccc} H \otimes H & \xrightarrow{\mu} & H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \uparrow{\scriptstyle \mu\otimes\mu} \\ H \otimes H \otimes H \otimes H & & \xrightarrow{1\otimes T\otimes 1} & & H \otimes H \otimes H \otimes H \end{array}$$

*where $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$ for homogeneous $a$ and $b$.*

Then, by [Mil-Mo], the structures $H_*(\mu)$ and $H_*(\Delta)$ determine a structure of Hopf algebra on $H_*(\Omega X)$.

Another important property of the loop-space functor $\Omega$ is that it preserves fibrations, this is, if $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, then the map $\Omega p$ is a multiplicative fibration

$$\Omega F \xrightarrow{\Omega i} \Omega E \xrightarrow{\Omega p} \Omega B,$$

which means that all the maps are maps of H-spaces.

Fred Cohen and the author are currently studying the space $\Omega F(M,k)$ where $M$ is a manifold without boundary.

In the case of $M = \mathbb{R}^2$, the points in $\Omega F(\mathbb{R}^2, k)$ can be thought of as *braids* in $k$ strands. For general $M$, $\Omega F(M,k)$ is known as the braid space of $M$.

**Definition 3.3** *We say that $M$ is a p-manifold (or a punctured manifold) if $M \simeq M' - Q_1$ where $M'$ is a manifold without boundary and $Q_1$ is a single point.*

Thus for instance, $\mathbb{R}^n \simeq S^n - Q_1$ is a p-manifold.

From the Fadell-Neuwirth fibrations, we have in general:

$$\begin{array}{ccccc} F(M,k) & \longleftarrow & F(M-Q_1,k-1) & \longleftarrow \cdots \longleftarrow & F(M-Q_{k-1},1) \\ \downarrow & & \downarrow & & \downarrow \approx \\ M & & M-Q_1 & & M-Q_{k-1} \end{array}$$

Taking loops we have multiplicative fibrations:

$$\begin{array}{ccccc} \Omega F(M,k) & \longleftarrow & \Omega F(M-Q_1,k-1) & \longleftarrow \cdots \longleftarrow & \Omega F(M-Q_{k-1},1) \\ \downarrow & & \downarrow & & \downarrow \approx \\ \Omega M & & \Omega(M-Q_1) & & \Omega(M-Q_{k-1}) \end{array}$$

**Theorem 3.4** *If $M$ is a p-manifold, then all the fibrations above are homotopy-equivalent to products. In particular,*

$$\Omega F(M,k) \simeq \Omega M \times \Omega(M-Q_1) \times \cdots \times \Omega(M-Q_{k-1}).$$

The homotopy equivalence in Theorem 3.4 is not multiplicative i.e. it is not an equivalence of H-spaces in general. Therefore

$$H_*\Omega F(M,k) \cong H_*(\Omega M) \otimes H_*\Omega(M-Q_1) \otimes \cdots \otimes H_*\Omega(M-Q_{k-1})$$

as graded vector spaces but it is not in general an isomorphism of algebras.

However, in the case of $M = \mathbb{R}^n$ we have obtained the following result:

Let $L(n,k)$ be the *free Lie algebra* on generators $\{B_{i,j}\}_{k\geq i>j\geq 1}$ of degree $n-2$, and consider the relations:

1. $[B_{i,j}, B_{s,t}] = 0$ if $\{i,j\} \cap \{s,t\} = \varnothing$
2. $[B_{i,j}, B_{i,t} + (-1)^n B_{t,j}] = 0$ if $1 \leq j < i \leq k$
3. $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ if $1 \leq j < i \leq k$.

Now, let $B(n,k)$ be the *quotient Lie algebra* of $L(n,k)$ by the relations above. Then we have:

**Theorem 3.5** *The algebra $H_*(\Omega F(\mathbb{R}^n, k))$ is the universal enveloping algebra of $B(n,k)$.*

The relations 1, 2 and 3 are known as the infinitesimal braid relations (or infinitesimal Yang-Baxter relations), and the $B_{i,j}$'s are the primitive elements in $H_*(\Omega F(\mathbb{R}^n, k))$.

The Lie algebra $B(n,k)$ has appeared in a totally different context in the works of Kohno and Drinfeld, when they were studying the work of Knishik - Zamoldchikov about flat connections associated to the Braid groups.

More generally:

For any manifold $M$ we have an inclusion $i : \mathbb{R}^n \hookrightarrow M$ in the neighborhood of a point $m_o \in M$. This map induces inclusions:

$$F(\mathbb{R}^n, k) \longrightarrow F(M,k)$$

$$\Omega F(\mathbb{R}^n, k) \longrightarrow \Omega F(M,k)$$

Moreover, we have:

**Theorem 3.6** *If $M$ is a $p$-manifold, then*

$$\Omega F(M,k) \underset{\varphi}{\simeq} (\Omega M)^k \times \Omega F(\mathbb{R}^n,k) \times \prod_{i=1}^{k-1} \Omega\Sigma(\Omega M \wedge \Omega(\vee_i S^{n-1})),$$

*where $\Sigma X$ is the reduced suspension of $X$ and $X \wedge Y = X \times Y / X \vee Y$.*

*The maps $\Omega M \longrightarrow \Omega F(M,k)$, $\Omega F(\mathbb{R}^n,k) \longrightarrow \Omega F(M,k)$ and*

$$\prod_{i=1}^{k-1} \Omega\Sigma(\Omega M \wedge \Omega(\vee_i S^{n-1})) \longrightarrow \Omega F(M,k)$$

*are multiplicative.*

**Theorem 3.7** *If $M' = M_1' \times M_2' \times M_3'$ and $M = M' - Q_1$, where $M$ is a differentiable manifold, then $\varphi$ induces a multiplicative isomorphism in homology with $F$ coefficients if either:*

*(i)* $w_{n-1}(M) = 0$ *if* $F = \mathbb{F}_2$.

*(ii) The Euler class of $\tau(M) - 1$ is trivial if $F = \mathbb{F}_p$ or $\mathbb{Q}$.*

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