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ERGODIC DECOMPOSITION OF MARKOV OPERATORS ON SIGNED MEASURES *

CÉSAR E. VILLARREAL [1]

Abstract

Let X be a Polish space, and let M_Σ be the Banach space of finite signed measures on the Borel σ-algebra Σ of X. Given a quasi-constrictive Markov operator $P^* : M_\Sigma \to M_\Sigma$, we use the spectral decomposition of P^* to determine the set of P^*-invariant distributions in M_Σ and the set of P^*-ergodic distributions.

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1 Introduction

Komorník and Thomas [2] proved an extension of the spectral decomposition theorem given by Lasota and Mackey [3]. This extension states the existence of the asymptotic periodicity decomposition of constrictive Markov operators on a subset (called a "band") of the Banach space of finite signed measures with the total variation norm; such extension is reproduced without proof in section 3. We use this result to prove an Ergodic Decomposition Theorem (given in section 4), which was previously known for Markov matrices [4].

We begin this paper with some preliminary definitions in the following section. The work finishes with two examples given in section 5.

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[1] Graduate student, CINVESTAV-IPN.

2 Preliminaries

Let (X, Σ) be a measurable space, and let M_Σ be the Banach space of finite signed measures on Σ endowed with the total variation norm $\|\cdot\|$.

Definition 2.1 *A set $M \subset M_\Sigma$ is called a band, if it is a Banach lattice such that $(\mu \in M$ and $\nu \ll \mu) \Rightarrow \nu \in M$.*

Let M be a band, and let D_M be the subset of nonnegative normalized elements of M, called the *distributions* of M. (In other words, D_M is the set of *probability measures* in M.)

Definition 2.2 *A linear operator $P^* : M \to M$ is called a Markov operator if P^* maps D_M into itself, i.e.,*

$$P^*(D_M) \subset D_M.$$

Definition 2.3 *Let M be a band. A Markov operator $P^* : M \to M$ is said to be quasi-constrictive if there exist a weakly compact set $F \subset M$ and a nonnegative number $x < 1$ such that*

$$\limsup_{n\to\infty} d(P^{*n}\mu, F) \leq x \quad \text{for} \quad \mu \in D_M, \tag{1}$$

where $d(\nu, F) := \inf\{\|\nu - \rho\| : \rho \in F\}$.

If P^* is quasi-constrictive and $x = 0$, then P^* is called *constrictive.*

Definition 2.4 *(a) We say that $\mu \in M$ is P^*-periodic if there exists a nonnegative integer n such that $P^{*n}\mu = \mu$.*

(b) We say that a P^*-periodic distribution $\mu \in D_M$ is minimal if for any P^*-periodic measure $\nu \ll \mu$ there exists a scalar t such that $\nu = t\mu$.

3 The Spectral Decomposition Theorem

The following theorem is due to Komorník and Thomas [2]. We will use this theorem in the proof of our main result, which is given in the next section.

Theorem 3.1 (Spectral Decomposition Theorem (*SDT*)) *Let P^* be a quasi-constrictive Markov operator on a band M. Then:*

(a) There exists

- *a finite set* $F_0 = \{\nu_1, ..., \nu_r\}$ *of pairwise orthogonal* P^**-periodic elements of* D_M*,*
- *a set* $\{\lambda_1, ..., \lambda_r\}$ *of continuous linear functionals on* M*, and*
- *a permutation* σ *of the integers* $1, ..., r$ *such that*

$$\lim_{n\to\infty} \|P^{*n}\mu - \sum_{i=1}^{r} \lambda_i(\mu)\nu_{\sigma^n(i)}\| = 0 \quad \text{for each} \quad \mu \in M, \text{ and} \tag{2}$$

$$P^*\nu_i = \nu_{\sigma(i)} \quad \text{for} \quad i = 1, ..., r. \tag{3}$$

(b) The functionals λ_i *are positive, that is,* $\lambda_i(\mu) \geq 0$ *if* $\mu \geq 0$*. Moreover,*

$$\sum_{i=1}^{r} \lambda_i(\nu) = 1 \quad \text{for} \quad \nu \in D_M, \quad \text{and}$$

$$|\lambda_i(\mu)| \leq \|\mu\| \quad \text{for} \quad \mu \in M. \tag{4}$$

(c) The measures ν_i*,* $i = 1, ..., r$ *are minimal.*

(d) The sets $\{\nu_1, ..., \nu_r\}$ *and* $\{\lambda_1, ..., \lambda_r\}$ *satisfying (2) and (3) are unique.*

4 P^*-Invariant Distributions

A topological space is said to be a *Polish* space if it is a complete separable metric space.

Let X be a Borel subset of a Polish space. We denote by Σ the σ-algebra of Borel subsets of X. If $P(x, \cdot)$ is a family of transition probabilities of a Markov process with values in X, we define the *adjoint* operator P^* of P that acts on M_Σ in the following way

$$P^*\rho(G) := \int P(x, G)d\rho(x), \qquad \rho \in M_\Sigma,$$

and P^* is a Markov operator. Conversely, if $P^* : M_\Sigma \to M_\Sigma$ is a Markov operator, then

$$P(x, \cdot) = P^* \mathbb{1}.(x),$$

where $\mathbb{1}_G$ is the indicator function of a Borel set G, defines a Markov transition probability.

We will denote D_{M_Σ} simply by D; that is, D is the set of *probability* measures on X.

Let $P^* : M_\Sigma \to M_\Sigma$ be a quasi-constrictive Markov operator. (See Definition 2.3). Let $D_\infty := \bigcap_{n=1}^{\infty} P^{*n}(D)$ be the set of all the limit points of the sequences $(P^{*n}\mu)_{n=1}^{\infty}$, with $\mu \in D$. By the SDT, $\nu \in D_\infty$ if and only if it is a convex combination of the distributions $\nu_1, ..., \nu_r$. That is, D_∞ is the convex hull of the finite set $F_0 = \{\nu_1, ..., \nu_r\}$ given in the SDT.

Definition 4.1 *(a) A Borel subset $A \subset X$ is called P-invariant (or P-absorbing) if $P(x, A) = 1$ for $x \in A$.*

(b) A distribution $\rho \in D$ is said to be P^*-invariant if $P^*\rho = \rho$.

(c) We say that a P^*-invariant distribution ρ is P^*-ergodic if $\rho(A) = 0$ or $\rho(A) = 1$ for any P-absorbing set A.

We will now identify the set $D_\infty^I \subset D_\infty$ of P^*-invariant distributions and the subset $D_\infty^E \subset D_\infty^I$ of all the P^*-ergodic distributions.

Two integers i and j in $\{1, ..., r\}$ are said to be *equivalent* (denoted by $i \leftrightarrow j$) if $P^{*k}(\nu_i) = \nu_j$ for some positive integer k. Observe that $\leftrightarrow$ is an equivalence relation, and denote by $O_1, O_2, ..., O_d$ the different equivalence classes of $\{1, ..., r\}$. Let $\bar{O}_l := \{\nu_i : i \in O_l\}$. For $j = 1, ..., d$, let $\#O_j$ be the number of elements in O_j, and let

$$\tau_j := \frac{1}{\#O_j} \sum_{i \in O_j} \nu_i \tag{5}$$

be the "average" of the elements in $\bar{O}_j$. Observe that $P^* : \bar{O}_j \to \bar{O}_j$ is bijective and that $P^*\nu_i \in \bar{O}_j \Leftrightarrow \nu_i \in \bar{O}_j$. Therefore,

$$\sum_{i \in O_j} \nu_i = \sum_{\nu \in \bar{O}_j} \nu = \sum_{\nu \in \bar{O}_j} P^*\nu = \sum_{i \in O_j} P^*\nu_i = P^* \sum_{i \in O_j} \nu_i,$$

which gives that τ_j is a P^*-invariant distribution. Also note that $\tau_1, ..., \tau_d$ are mutually singular. The proof of the following theorem proceeds in the same way as in Villarreal [4, Teorema 10], in which X was assumed to be a *countable* set.

Theorem 4.2 (Ergodic Decomposition Theorem) *Let $P^* : M_\Sigma \to M_\Sigma$ be a quasi-constrictive Markov operator and let $D^I_\infty \subset D_\infty$ be the set of all the P^*-invariant distributions. Then D^I_∞ is a convex set and, in fact, it is the convex hull of $\{\tau_1, ..., \tau_d\}$ with τ_j as in (5), i.e.,*

$$(6) \quad D^I_\infty = \{\mu \in D : \mu = \sum_{j=1}^{d} \alpha_j \tau_j \quad \textit{with} \quad \alpha_j \geq 0 \quad \textit{and} \quad \sum_{j=1}^{d} \alpha_j = 1\}.$$

Hence, $D^E_\infty = \{\tau_1, ..., \tau_d\}$ is the collection of all the P^*-ergodic distributions.

Proof: Let C be the convex hull of $\{\tau_1, ..., \tau_d\}$. Since τ_j is a P^*-invariant distribution for $j = 1, 2, ..., d$, any convex combination

$$\mu = \sum_{j=1}^{d} \alpha_j \tau_j$$

is also a P^*-invariant distribution. Hence $C \subset D^I_\infty$. To prove that $D^I_\infty \subset C$, first note that $D^I_\infty \subset D_\infty$ so that if $\mu \in D$ is a P^*-invariant distribution, then, by the SDT, μ is a convex combination of $\nu_1, ..., \nu_r$, that is,

$$(7) \qquad \mu = \sum_{m=1}^{r} \beta_m \nu_m.$$

Also note that (7) is the unique representation of μ as a linear combination of $\nu_1, \nu_2, ..., \nu_r$, because these distributions are mutually singular.

Now, if $i, k \in O_j$, then the coefficients β_i and β_k are equal. Indeed, let $i, k \in O_j$; then there is a nonnegative integer t such that $\nu_k = P^{*t}(\nu_i)$, so that

$$\mu = P^{*t}\mu = \sum_{m=1}^{r} \beta_m P^{*t} \nu_m = \sum_{m=1}^{i-1} \beta_m P^{*t} \nu_m + \beta_i \nu_k + \sum_{m=i+1}^{r} \beta_m P^{*t} \nu_m.$$

However, as the representation given in (7) is unique, we obtain $\beta_i = \beta_k$.

For $j = 1, 2, ..., d$, let s_j be an integer in O_j. Then

$$\mu = \sum_{m=1}^{r} \beta_m \nu_m = \sum_{i \in O_1} \beta_{s_1} \nu_i + \sum_{i \in O_2} \beta_{s_2} \nu_i + \cdots + \sum_{i \in O_d} \beta_{s_d} \nu_i$$

$$= \beta_{s_1}(\# O_1)\tau_1 + \beta_{s_2}(\# O_2)\tau_2 + \cdots + \beta_{s_d}(\# O_d)\tau_d.$$

Finally, if for $j = 1, 2, ..., d$ we take $\alpha_j = \beta_{s_j}(\#O_j)$, then we get

$$\mu = \sum_{j=1}^{d} \alpha_j \tau_j.$$

Moreover, $\alpha_j \geq 0$ and $\sum_{j=1}^{d} \alpha_j = \sum_{j=1}^{d} \beta_{s_j}(\#O_j) = \sum_{j=1}^{d} \sum_{i \in O_j} \beta_i = \sum_{m=1}^{r} \beta_m = 1$. Therefore, $D_{\infty}^{I} \subset C$, which completes the proof of (6).

This in turn yields the last statement in the theorem, $D_{\infty}^{E} = \{\tau_1, ..., \tau_d\}$; see, for instance, Kifer [1, Theorem 1.1 of Appendix A.1]. ■

5 Examples

Example 5.1 Given the Markov matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

With a method given in [4], we can see that the elements $\nu_1, ..., \nu_r$, $\lambda_1, ..., \lambda_r$, and σ of the SDT are: $\nu_1 = e_2$ and $\nu_2 = (0, 0, \frac{1}{2}, \frac{1}{2})^T$; $\lambda_1(\mu) = x_1 + x_2$ and $\lambda_2(\mu) = x_3 + x_4$; $\sigma(1) = 1$ y $\sigma(2) = 2$; where $r = 2$. Hence, from the Ergodic Decomposition Theorem we can obtain that $\tau_1 = \nu_1$ and $\tau_2 = \nu_2$ with $d = 2$.

Example 5.2 Given the Markov matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}.$$

In this example $\nu_1 = e_1$, $\nu_2 = e_2$ y $\nu_3 = e_4$; $\lambda_1(\mu) = x_1$, $\lambda_2(\mu) = x_2$ and $\lambda_3(\mu) = x_3 + x_4$; $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(3) = 3$. Hence, $\tau_1 = (\frac{1}{2}, \frac{1}{2}, 0, 0)^T$ and $\tau_2 = e_4$.

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César E. Villarreal
Departamento de Matemáticas
CINVESTAV-IPN
Apdo. Postal 14-740
México D.F. 07000, México.
cevilla@math.cinvestav.mx

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