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INTERCALATE MATRICES AND ALGEBRAIC VARIETIES *

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Abstract

We give a characterization of intercalate matrices as an algebraic variety over a finite field. We also prove Yuzvinsky's conjecture on the minimum number of colors in an intercalate matrix for matrices with 5 or less rows. Finally, we obtain a set of 37 cases which, if established, would verify Yuzvinsky's conjecture as true for matrices up to order 32×32.

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1 Introduction

A matrix (whose entries are *colors*) is *pseudolatin* if it has no repeated colors along any row or column. If additionally it has two or four colors in each of its 2×2 submatrices then it is called *intercalate* [5]. Equivalently, a matrix is intercalate if all of its 2×2 submatrices are of one of the forms

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{ or } \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are distinct colors.

A 2×2 submatrix of an intercalate matrix is called an *intercalation* if it has two colors and called a *co-intercalation* if it has four colors.

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Two matrices are *isotopic* if one can be obtained from the other using only permutations of rows and columns and relabeling of colors.

An intercalate matrix is of type (r, s, n) if it has r rows, s columns and n distinct colors. In 1981 Yuzvinsky [23] conjectured that any intercalate matrix of type (r, s, n) satisfies that $n \geq r \circ s$, where (in $GF(2)$)

$$r \circ s = \min\{n : (x+y)^n \in (x^r, y^s)\}$$

is the Pfister's function.

2 The Alon–Tarsi lemma

Let F be a field and $f(x)$ a non zero polynomial with coefficients in F, i.e. $f \in F[x]$. The following fact is well known:

Lemma 2.1.1 *If $A \subset F$, $|A| = r$ and $f(a) = 0$ for all $a \in A$, then* $\deg(f) \geq r$.

In 1992, Alon and Tarsi proved a generalization of this result, known as the Alon–Tarsi lemma [1, 4]. After that, Eliahou and Kervaire [10] made equivalent formulations including the following:

Theorem 2.1.2 *Let f be a polynomial in $F[x_1, \dots, x_n]$ and $A_1, \dots, A_n$ be subsets of F such that $|A_1| = r_1, \dots, |A_n| = r_n$ and suppose that $f(A_1 \times \cdots \times A_n) = 0$, then top$(f)$ is in the ideal $(x_1^{r_1}, \dots, x_n^{r_n})$, where top$(f)$ is the homogeneus component of f with maximal degree.*

Proof: By induction on n. For $n = 1$ this is equivalent to the previous lemma. Let $n > 1$ and assume that the result is true for $n - 1$. Let $f \in F[x_1, \dots, x_n]$. We can classify the monomials of top(f) in two classes:

- $u_1, \dots, u_k \notin (x_n^{r_n})$, and
- $v_1, \dots, v_l \in (x_n^{r_n}) \subset (x_1^{r_1}, \dots, x_n^{r_n})$.

The second class is obviously in $(x_1^{r_1}, \dots, x_n^{r_n})$, then it is enough to show that $\{u_1, \dots, u_k\} \subset (x_1^{r_1}, \dots, x_{n-1}^{r_{n-1}})$, and we can assume without loss of generality that $l = 0$.

Consider

$$g(x_n) = \prod_{a \in A_n} (x_n - a) = x_n^{r_n} - h(x_n)$$

where $\deg(h) < r_n$, then we can replace each ocurrence of $x_n^{r_n}$ in f with $h(x_n)$ and this does not change the fact that $f(A_1 \times \cdots \times A_n) = 0$. The new $\text{top}(f) = \{u_1, \ldots, u_k\} \notin (x_n^{r_n})$.

Now write f as a polynomial in x_n

$$f = f_0 + f_1 x_n + \cdots + f_d x_n^d$$

with $d < r_n$.

Let $a \in A_1 \times \cdots \times A_{n-1}$ and write

$$f_a(x_n) = f_0(a) + f_1(a)x_n + \cdots + f_d(a)x_n^d \in F[x_n].$$

It is clear that $f_a(A_n) = 0$, from here, $f_a(x_n)$ is identically zero, as it has at least r_n roots. Then, $f_i(a) = 0$ for $1 \leq i \leq d$ and then $f_i(A_1 \times \cdots \times A_{n-1}) = 0$ and finally $\text{top}(f_i) \in (x_1^{r_1}, \ldots, x_{n-1}^{r_{n-1}})$ for all i. Then

$$\text{top}(f) = \text{top}(\text{top}(f_0) + \text{top}(f_1)x_n + \cdots + \text{top}(f_d)x_n^d) \in (x_1^{r_1}, \ldots, x_{n-1}^{r_{n-1}}).$$

□

3 Yuzvinsky's conjecture in the dyadic case

The Alon–Tarsi lemma proved to be a useful tool to give a short proof of Yuzvinsky's conjecture in the case where the intercalate matrix is a submatrix of the Cayley's table of the dyadic group ($\mathcal{D} = (Z, \oplus)$), as shown in [11].

Theorem 3.1.3 *Let V be a vector space on $GF(2)$. Let $A, B \subset V$ with respective cardinalities r, s. Then $|A \oplus B| \geq r \circ s$.*

Proof: We can assume that V is the field F_q with $q = 2^m$ for some positive integer m. Let $C = A \oplus B$. Now consider the polynomial

$$f(x, y) = \prod_{c \in C} (x + y - c)$$

in $F_q[x, y]$ (remember that $+$ and $-$ are identical with $\oplus$). It is clear that for any $a \in A, b \in B$ we have that $f(a, b) = 0$, i.e. $f(A \times B) = 0$, so f satisfies the conditions of the Alon–Tarsi lemma.

Note also that $\text{top}(f) = (x + y)^{|C|}$. By the Alon–Tarsi lemma this implies that $(x + y)^{|C|} \in (x^r, y^s)$ in $F_q[x, y]$. It can be shown that the previous statement is also true for $F_2[x,y]$. Looking at the definition of $r \circ s$ we can conclude that $|C| \geq r \circ s$. □

It is worth noting that we can always find an intercalate matrix of type $(r, s, r \circ s)$ by taking $A = \{0, 1, \ldots, r-1\}$ and $B = \{0, 1, \ldots, s-1\}$.

4 Algebraic varieties

Let F be a field. A subset of F^n is an *algebraic variety* if it is the set of roots of some polynomial in the ring $F[x_1, \ldots, x_n]$. We will show a characterization of the set of non intercalate matrices of type (r, s, n) as an algebraic variety constructing a polynomial $\mathcal{P}_{rs}$ such that $\mathcal{P}_{rs}(A) = 0$ if and only if the matrix A is non intercalate.

Let x_{ij} be an indeterminate corresponding to the coordinate (i, j) of the matrix A of order $r \times s$. Let F_q be a finite field with cardinality $q \geq rs$ (this implies, of course, that $q = p^m$ for some prime p and integer m). We must note that A has at most rs different colors and then we can always find a matrix isotopic to A such that all of its colors are taken from F_q.

Consider the following polynomials in $F_q[x_{11}, \ldots, x_{rs}]$:

$$\begin{aligned}
\mathcal{R}_i(X) &= \prod_{1 \leq j < k \leq s} (x_{ij} - x_{ik}) \\
\mathcal{C}_j(X) &= \prod_{1 \leq i < k \leq r} (x_{ij} - x_{kj}) \\
\mathcal{Q}^{ij}_{kl}(X) &= H(x_{ij} - x_{kl})H(x_{il} - x_{kj}) + (x_{ij} - x_{kl})(x_{il} - x_{kj})
\end{aligned}$$

where $X = \{x_{11}, \ldots, x_{rs}\}$ and $H(x) = 1 - x^{\phi(q)}$. Euler's theorem implies that $H(x - y) = 1$ if and only if $x = y$, that is, $\delta(x, y) = H(x - y)$ is the characteristic function over F_q.

Now consider the polynomial

$$\mathcal{P}_{rs}(X) = \prod_{i=1}^{r} \mathcal{R}_i \prod_{j=1}^{s} \mathcal{C}_j \prod_{1 \leq j < l \leq s}^{1 \leq i < k \leq r} \mathcal{Q}^{ij}_{kl}$$

Theorem 4.1.4 *Let A be a matrix of order $r \times s$ with colors in F_q, then $\mathcal{P}_{rs}(A) = 0$ if and only if A is non intercalate.*

Proof: First assume that A is non intercalate, then A fails to satisfy one or more of the intercalation conditions:

- for some i the entries a_{ij} and a_{ik} are equal, then $\mathcal{R}_i(A) = 0$,
- for some j the entries a_{ij} and a_{kj} are equal, then $\mathcal{C}_j(A) = 0$,
- for some columns i, k and some rows j, l the 2×2 submatrix induced by them has exactly three colors as follows:

$$\begin{array}{c|cc} & i & k \\ \hline j & a & b \\ l & c & a \end{array}$$

then $\mathcal{Q}^{ij}_{kl}(A) = H(a-a)H(c-b) + (a-a)(c-b) = 0$.

In any case, it follows that $\mathcal{P}_{rs}(A) = 0$.

Now, if A is intercalate then, as it is pseudolatin, all the entries in each row and column are distinct, then $\mathcal{R}_i(A) \neq 0$, $\mathcal{C}_j(A) \neq 0$ for every i, j. Furthermore, every 2×2 submatrix has four or two colors as follows:

$$\begin{array}{c|cc} & i & k \\ \hline j & a & b \\ l & c & d \end{array} \quad \text{or} \quad \begin{array}{c|cc} & i & k \\ \hline j & a & b \\ l & b & a \end{array}$$

In the first case we have:

$$\mathcal{Q}^{ij}_{kl}(A) = H(a-d)H(c-b) + (a-d)(c-b) = (a-d)(c-b) \neq 0$$

and in the second case:

$$\mathcal{Q}^{ij}_{kl}(A) = H(a-a)H(b-b) + (a-a)(b-b) = 1 \neq 0.$$

Then, it follows that $\mathcal{P}_{rs}(A) \neq 0$. □

Note that the degree of $\mathcal{P}_{rs}(A)$ is

$$\frac{1}{2} rs(r+s-2+(q-1)(r-1)(s-1)) \sim \frac{1}{2} r^3 s^3.$$

We will use the following lemma, proposed by Sarmiento [17], to decrease the degree of the previous polynomial.

Lemma 4.1.5 *For every $u, v \in F_q$ we have that $H(u)H(v) + uv = 0$ if and only if $H(u) + H(v) = 1$.*

Proof: Remember that $H(0) = 1$ and $H(x) = 0$ for $x \neq 0$.

($\Leftarrow$) From $H(u) = 1$ and $H(v) = 0$ we get $u = 0$ and then $H(u) + H(v) + uv = 1 \cdot 0 + 0 \cdot v = 0$.

($\Rightarrow$) Assume both $H(u) = H(v) = 1$ then $u = v = 0$ and

$$H(u)H(v) + uv = 1 \cdot 1 + 0 \cdot 0 = 1$$

which is impossible. Then $H(v) = 1$ and $H(u) = 0$. From here $uv = 0$, $v = 0$ and $H(v) = 1$. Finally $H(u) + H(v) = 0 + 1 = 1$. $\square$

Let

$$\mathcal{O}_{kl}^{ij}(X) = H(x_{ij} - x_{kl}) + H(x_{il} - x_{kj}) - 1$$

then it is clear that $\mathcal{O}_{kl}^{ij}(X) = 0$ if and only if $\mathcal{Q}_{kl}^{ij}(X) = 0$.

Lemma 4.1.6 *The* 2×2 *submatrix taken from columns* i, k *and rows* j, l *of the pseudolatin matrix* A *has a number of distinct colors determined by the value of* $\mathcal{O}_{kl}^{ij}(A)$ *as follows:*

$\mathcal{O}_{kl}^{ij}(A)$	*colors*
1	*2*
0	*3*
-1	*4*

Proof: Let $h = \mathcal{O}_{kl}^{ij}(A)$ and a, b, c, d be the colors in the mentioned submatrix.

- If a, b, c, d are all distinct then $h = H(a - c) + H(b - d) - 1 = 0 + 0 - 1 = -1$.
- If $a = c$ and $b = d$ then $h = H(a-a) + h(b-b) - 1 = 1 + 1 - 1 = 1$.
- Finally, if $a = c$ but $b \neq d$ then $h = H(a - a) + H(b - d) - 1 = 1 + 0 - 1 = 0$. $\square$

Now consider the polynomial

$$\mathcal{N}_{rs}(X) = \prod_{i=1}^{r} \mathcal{R}_i \prod_{j=1}^{s} \mathcal{C}_j \prod_{1\leq j<l\leq s}^{1\leq i<k\leq r} \mathcal{O}_{kl}^{ij}$$

Theorem 4.1.7 *Let* A *be a matrix of order* $r \times s$ *with colors in* F_q*, then* $\mathcal{N}_{rs}(A) = 0$ *if and only if* A *is non intercalate. Furthermore,* $\prod_{1\leq j<l\leq s}^{1\leq i<k\leq r} \mathcal{O}_{kl}^{ij} = (-1)^t$ *if* A *is intercalate and has* t *co-intercalations.*

Proof: Immediate from previous lemmas. □

Note that the degree of $\mathcal{N}_{rs}$ is

$$\frac{1}{4}rs(2r+2s-4+(q-1)(r-1)(s-1)) \sim \frac{1}{4}r^3s^3.$$

Now we can reformulate Yuzvinsky's conjecture as follows. Let

$$B^{rs} = \underbrace{B \times \cdots \times B}_{rs \text{ times}}.$$

Conjecture 4.1.8 *Let F_q be a finite field with cardinality $q \geq r \circ s - 1$ and let $B \subseteq F_q$ such that $|B| = r \circ s - 1$, then $\mathcal{N}_{rs}(X) = 0$ for all $X \in B^{rs}$.*

Note that if $q = r \circ s - 1$ then $B = F_q$ and the conjecture states that $\mathcal{N}_{rs}(X)$ is the zero polynomial.

5 Yuzvinsky's conjecture up to 32×32

It has been shown that Yuzvinsky's conjecture is true when either r or s is ≤ 5 and when both $r, s \leq 16$ [8, 22]. Now we will study conditions that must be satisfied in order to show that the conjecture is true for $r, s \leq 32$.

5.1 Signability

The following result is well known [9]:

Theorem 5.1.1 *Let A be a signable intercalate matrix of type (r, s, n), then $n \geq r \circ s$.*

Proof: Under the hypothesis, the matrix A determines a formula

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2.$$

Now, by a theorem of Hopf and Stiefel [13, 20], we have that $n \geq r \circ s$. □

The *incidence matrix* of A is the matrix $\tilde{A}$ that has as rows its coordinates, as columns its intercalations, and as colors $\tilde{A}_{CI} = 1$ if the intercalation I uses coordinate C and $\tilde{A}_{CI} = 0$ in other case.

Let $\vec{1}$ be the vector all whose entries are equal to 1. In [6, 23] we find:

Lemma 5.1.2 *A is signable if and only if the system $x\tilde{A} = \vec{1}$ has a solution over F_2.*

Lemma 5.1.3 *One and only one of the following systems has a solution over F_2: (1) $x\tilde{A} = \vec{1}$ or (2) $\tilde{A}w = 0, \vec{1}w = 1$.*

Now we will show that the intercalate matrices with five or less rows are signable [8]:

Theorem 5.1.4 *Every $r \times s$ intercalate matrix with $r \leq 5$ is signable.*

Proof: Let w be a solution of the system (2) from the previous lemma. Assume that w has a_i intercalations in the i-th column of A. We denote the least of the a_i as the *column type* of w. We define the *column type* of A as the minimal of the column types of the w in it.

We say that an intercalate matrix is *connected* if for every partition of its columns in two non empty sets X, Y there exists a color c that is in both X and Y. We say that an intercalate matrix is *complete* if each of its rows is a permutation of each other.

Let us assume, for a contradiction, that there exists an intercalate non signable matrix A with five rows. We can assume that A has the minimal number possible of columns and, from those non signable matrices with the same number of columns, A has the least possible column type. In particular, A has at least four coordinates in each column, at least four intercalates in the first column and is connected.

Let us assume that A attains its column type in w and that w attains its column type in the first column

1. A cannot contain a 4×3 main submatrix of $\mathcal{D}$. If A has such a submatrix, then A has a 5×8 main submatrix of $\mathcal{D}$. But this submatrix is complete, connected and signable and then A is not connected, a contradiction.

$$\begin{array}{ccc|ccccc}
a & b & c & d & e & f & g & - \\
b & a & d & c & f & e & - & g \\
c & d & a & b & g & - & e & f \\
d & c & b & a & - & g & f & e \\
e & f & g & - & a & b & c & d
\end{array}$$

2. If some coordinate occupied by w in the first row is in four intercalations, then in A there exists a 3×4 main submatrix of $\mathcal{D}$

that uses the first column. Then, using an elementary column operation we can decrease the column type of w and A, which is a contradiction. Then each coordinate of A in the first column is in at most two intercalations of w and, as a consecuence, the column type of A is either 4 or 5.

3. Assume first that A has column type 5. In particular w uses the five coordinates of the first column. Then A necessarily contains a substructure as the following:

$$\begin{array}{cccccc} a & b & c & & & \\ b & a & & d & & \\ c & & a & & e & \\ d & & & b & & e \\ e & & & & c & d \end{array}$$

This matrix can be completed in a unique way as follows:

$$\begin{array}{cccccc} a & b & c & f & g & h \\ b & a & i & d & j & k \\ c & i & a & l & e & m \\ d & f & n & b & m & e \\ e & o & g & k & c & d \end{array}$$

In turn, this matrix can be extended also in a unique way to:

$$\begin{array}{cccccc|cccccccccc} a & b & c & f & g & h & d & e & i & n & o & l & j & k & m & - \\ b & a & i & d & j & k & f & o & c & b & e & n & g & h & - & m \\ c & i & a & l & e & m & n & g & b & d & j & f & o & - & h & k \\ d & f & n & b & m & e & a & h & l & c & k & i & - & o & g & j \\ e & o & g & k & c & d & h & a & j & m & b & - & i & f & n & l \end{array}$$

which is a complete, connected and signable matrix. This contradicts that A is connected.

4. Now assume that A is of column type 4. Then w has exactly four intercalations and ocuppies exactly four coordinates of the first column of A. Then A has the substructure:

$$\begin{array}{ccccc} a & b & c & & \\ b & a & & d & \\ c & & a & & d \\ d & & & b & c \end{array}$$

We can fill that matrix in a unique way as follows:

$$\begin{array}{ccccc} a & b & c & e & f \\ b & a & g & d & h \\ c & g & a & h & d \\ d & e & f & b & c \end{array}$$

(a) If in the fifth row of this submatrix appears one of its colors, then we obtain the only 5×5 intercalate matrix with frecuencies $\{2, 3, 3, 3, 3, 3, 4, 4\}$, that can be extended in a unique way to a 5×8 complete, connected and signable matrix. This is a contradiction.

(b) One of the colors e or g of the second column is in w, then without loss of generality we assume it is g. Then g must appear in a new column. If

$$\begin{array}{cccccc} a & b & c & e & f & h \\ b & a & g & d & h & f \\ c & g & a & h & d & e \\ d & e & f & b & c & g \end{array}$$

then we obtain a 4×6 matrix with 12 intercalations whose symmetric difference is empty, four of them are in the first column. This contradicts the minimality of A. Then g must be also in a new row:

$$\begin{array}{cccccc} a & b & c & e & f & - \\ b & a & g & d & h & - \\ c & g & a & h & d & - \\ d & e & f & b & c & - \\ - & - & - & - & - & g \end{array}$$

This implies in particular that g can only appear three times in A, then every intercalation containing it must be in w. We can complete this submatrix in the unique way:

$$\begin{array}{cccccc} a & b & c & e & f & i \\ b & a & g & d & h & j \\ c & g & a & h & d & k \\ d & e & f & b & c & l \\ m & k & j & n & o & g \end{array}$$

At least one of i, l is in w, and then it must appear at least three times in A. This submatrix can be extended with a column containing at least one of i, l appearing in the second, third or fifth row. The new submatrix can be extended in a unique way to a 5×16 complete, connected and signable matrix, which is a contradiction.

This ends the proof of the theorem. □

Corollary 5.1.5 *Let A be an intercalate matrix of type (r, s, n) with $r \leq 5$, then $n \geq r \circ s$.*

5.2 The 32×32 order

The following result is well known and shows an alternative way to construct the Cayley's table of $\mathcal{D}$:

Lemma 5.2.1 *Let N be an $r \times s$ matrix such that a_{ij} is the least natural that does not appear in the set $\{a_{i0}, \ldots, a_{i,j-1}\} \cup \{a_{0j}, \ldots, a_{i-1,j}\}$, then N is the $r \times s$ main submatrix of $\mathcal{D}$.*

Proof: We will show by induction on i that $a_{ij} = i \oplus j$. For $i = 0$ it is clear that $a_{0j} = j = 0 \oplus j$. Assume that $a_{i'j} = i' \oplus j$ for all $i' < i$ and that $a_{ij'} = i \oplus j'$ for all $j' < j$. It is elementary that $l = i \oplus j \notin \{a_{i0}, \ldots, a_{i,j-1}\} \cup \{a_{0j}, \ldots, a_{i-1,j}\} = \{i \oplus 0, \ldots i \oplus (j-1)\} \cup \{0 \oplus j, \ldots, (i-1) \oplus j\}$.

We only need to show that every $n < l$ is in that set. Let k such that $i, j < 2^k$ and consider the binary expansions of i, j, l, n. Let m be the greatest integer such that $n_m \neq l_m$. As $n < l$, it is easy to see that $n_m = 0$ and that $l_m = 1$. This implies that $i_m \oplus j_m = 1$, that is, one of $\{i_m, j_m\}$ is 1 and the other 0. Assume without loss of generality that $i_m = 0$ and $j_m = 1$ and consider the integer $j' < j$ with binary expansion $j_k, \ldots, j_{m+1}, 0, i_{m-1} \oplus n_{m-1}, \ldots, i_0 \oplus n_0$. Then, $a_{ij'} = i \oplus j' = n$ and we are done. □

We will also need the following:

Lemma 5.2.2 *For every r, s we have $r + s - 1 \geq r \circ s$.*

Proof: Let us assume that $r, s \leq 2^t$. We will do the proof by induction on t. For $t = 0$ it is true. Assume that the inequality is true for every $r, s \leq 2^t$. Now consider $r, s \leq 2^{t+1}$. If $r, s \leq 2^t$ we are done. If

$r \leq 2^t < s \leq 2^{t+1}$ let $z = s - 2^t$. Then $r \circ s = 2^t + r \circ z \leq 2^t + r + z - 1 = r + s - 1$. Finally, if $2^t < r, s \leq 2^{t+1}$ then $r \circ s = 2^{t+1} < r + s$ and then $r \circ s \leq r + s - 1$. □

Now assume that Yuzvinsky's conjecture is false. Let A of type (r, s, n) be a counterexample with minimal $r + s$, that is, with $n < r \circ s$ and let N be the $r \times s$ main submatrix of $\mathcal{D}$ with type $(r, s, r \circ s)$. Let $c = r \circ s - 1$.

Suppose that color c is in the $p \times q$ main submatrix of N, either with $p < r$ or $q < s$. Then, by the minimality of A we have that this submatrix is optimal with type $(p, q, c + 1)$, but then the $p \times q$ main submatrix of A with $m \leq n$ colors has at least $c + 1$ colors, that is, $c + 1 \leq m \leq n < c + 1$, which is a contradiction. Then the color c appears only in the bottom right corner of N.

As c has frequency 1, there are no intercalations in N containing it, and we have that all colors in the last row and column are distinct, so there is at least $(r - 1) + (s - 1) + 1 = r + s - 1$ colors. This, together with the second lemma, says that $r \circ s = r + s - 1$ and then $n \leq r + s - 2$.

Now, it is easily seen that color $c - 1$ must appear either in the $r \times (s - 1)$ or the $(r - 1) \times s$ main submatrix of N in such a way that either $r \circ (s - 1)$ or $(r - 1) \circ s$ is equal to $r + s - 2$. In any case, that submatrix is optimal and any $r \times s$ matrix has at least $r + s - 2$ colors, and finally $n = r + s - 2$. It is also clear that color a_{rs} has frequency at least two (otherwise A would have at least $r + s - 1$ colors.)

As A has $n < r \circ s$ then A is not dyadic and (r, s, n) is not pure. Then there exists k with $n - r < k < s$ such that $\binom{n}{k} \equiv 1$. Then $s - 2 < k < s$ and finally $k = s - 1$. From here we can see that $(r - 1) \oplus (s - 1) = (r - 1) + (s - 1)$.

If we want to extend Yuzvinsky's conjecture up to $r, s \leq 32$ it is enough to show that there are no intercalate matrices of the following 37 types:

- $(17, s, 15 + s)$ for $6 \leq s \leq 16$,
- $(18, 2s + 1, 17 + 2s)$ for $3 \leq s \leq 7$,
- $(19, 6, 23)$, $(19, 9, 26)$, $(19, 10, 27)$, $(19, 13, 30)$, $(19, 14, 31)$,
- $(20, 9, 27)$, $(20, 13, 31)$,
- $(21, s, 19 + s)$ for $9 \leq s \leq 12$,
- $(22, 9, 29)$, $(22, 11, 31)$, $(23, 9, 30)$, $(23, 10, 31)$, $(24, 9, 31)$,

- $(25, s, 23 + s)$ for $6 \leq s \leq 8$,
- $(26, 7, 31)$, $(27, 6, 31)$

as they are the only types with $r, s \leq 32$ such that $r \circ s = r + s - 1$ and are not covered by the previous cases.

We should note that in 25 of these cases we have that $r \circ s - 1$ is a prime power and we may use the reformulation of the conjecture given in section 4.

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