Morfismos, Vol. 3, No. 1, 1999, pp. 1 - 8

LOOP SPACES OF CONFIGURATION SPACES *

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Abstract

This paper gives a brief introduction to the theory of configuration spaces. Some recent results about the homology of their loop spaces are also presented. We also discuss their relationship to Vassiliev invariants for knots.

1991 Mathematics Subject Classification: 55P35, 55R99.
Keywords and phrases: Configuration spaces, Loop spaces, Poincaré series, Hopf algebra, Vassiliev invariants of knots.

In this note, we want to give an idea how the loop spaces of configuration spaces give rise to higher dimensional braidings, and in particular its relation to braids. Recall that if M is a connected m–manifold, then $F(M,k)$ is the open subspace of M^k:

$$F(M,k) = \{(x_1, \cdots, x_k) \mid x_i \in M,\ x_i \neq x_j,\ \text{if } i \neq j\}.$$

Of course this definition does not require M to be a manifold. However, if Q_k denotes a set of k distinct points in M, and M is a manifold, we have locally trivial fibrations:

$$\left.\begin{array}{l} M - Q_k \to F(M,k+1) \to F(M,k) \\ F(M - Q_1, k) \to F(M,k+1) \to M \end{array}\right\} \qquad \text{(I)}$$

*__Invited article.__ All the new results that appear here are part of joint work with Frederick R. Cohen.

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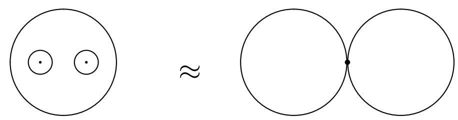
The spaces $M-Q_k$ are all homeomorphic and its homotopy type is that of $(M-Q_1)\vee\bigvee_{k-1} S^{m-1}$. Typically, if $M=D^m$, $D^m-Q_k\approx\bigvee_k S^{m-1}$.



The homology and cohomology of $F(M,k)$ are not known for general M. In fact, if M is closed and compact, the only example which is known is $M=S^n$. The answer is not known even for $S^m\times S^n$. However, for manifolds like $M\times\mathbb{R}$, the answer is known. In particular, it is known for $F(\mathbb{R}^n,k)$.

In general, we construct maps $A_{ij}: S^{m-1}\to F(\mathbb{R}^m,k)$, for $k\geq i> j\geq 1$ as follows: let $q_l=4le_0,\ l=1,2,\cdots,k-1$. Let now $x\in S^{m-1}$ be of norm 1 and define

$$A_{ij}(x)=(z_1,\cdots,z_m),$$

where $z_l=q_l$ if $l\neq i$, and $z_i=x+q_j$. If $x_0\in X$, let ΩX=Map$_*(S^1,X)=\{f:S^1\to X\mid f(u)=x_0\}$. Recall that if $F\to E\to B$ is a fibration, then

$$\Omega F\to\Omega E\to\Omega B$$

is again a fibration. In fact, since ΩF is an H–group, this is a principal fibration, and we have:

$$\begin{array}{ccc} \Omega F\times\Omega E & \xrightarrow{\mu} & \Omega E \\ & \searrow\ \swarrow & \\ & \Omega B & \end{array}$$

Suppose $\Omega F\to\Omega E\xrightarrow{\pi}\Omega B$ has a cross-section s, $\pi s=$id, then $\Omega F\times\Omega B\xrightarrow{1\times s}\Omega F\times\Omega E\xrightarrow{\mu}\Omega E$ induces isomorphisms of homotopy groups, so that if B, E and F are of the homotopy type of CW–complexes, $\Omega F\times\Omega B\to\Omega E$ is a homotopy equivalence. Consider now M to be a *punctured manifold*, $M=M'-Q_1$, where M' is a manifold. We construct a cross–section to $F(M,k+1)\to M$ as follow: choose a neighborhood of Q_1, which is a unit disc D, and take k distinct points $(y_1,\cdots,y_k)$ in D. Then the required cross–section is defined by

$$s(x)=\begin{cases}(x,y_1,\cdots,y_k) & \text{if}\quad x\notin D,\\ (x,|x|y_1,\cdots,|x|y_k) & \text{if}\quad x\in D.\end{cases}$$

Then, $\Omega F(M,k+1) \xrightarrow{\Omega\pi} \Omega M$ has also a cross–section Ωs, so we have a homotopy equivalence $\Omega F(M-Q_1,k)\times\Omega M \to \Omega F(M,k+1)$. Proceeding now with $F(M-Q_1,k)$ we obtain:

Theorem A *For a punctured manifold M we have:*

$$\Omega F(M,k) \approx (\Omega M)^k \times \Omega F(\mathbb{R}^m,k) \times D(M,k)$$

where

$$D(M,k) \approx \prod_{i=1}^{k-1} \Omega\Sigma\left[\Omega M \wedge \Omega\left(\bigvee_{i=1}^{k-1} S^{m-1}\right)\right].$$

This homotopy equivalence, however, is not a homotopy equivalence of H–spaces, i.e. it does not preserve the multiplications.

Corollary *If M is punctured manifold, then*

$$\Sigma\Omega F(M,k) \approx \left(\bigvee \Sigma^{j_\beta}(\Omega M)^{(i_\alpha)}\right) \vee \left(\bigvee S^{k_\gamma}\right)$$

for suitable sets I, J and K, with $j_\beta \in J$, $i_\alpha \in I, k_\gamma \in K$.

$\mathbb{R}^m$ is a punctured manifold, so we also have:

$$\Omega F(\mathbb{R}^m,k) \approx \prod_{j=1}^{k-1} \Omega\left(\bigvee_{t=1}^{j} S^{m-1}\right)$$

where this decomposition is not as H–spaces.

In order to describe the homology $H_*(\Omega F(\mathbb{R}^m,k);\mathbb{Z})$, recall that the multiplication in ΩX and the diagonal $\Omega X \to \Omega X \times \Omega X$ make $H_*(\Omega F(\mathbb{R}^m,k);\mathbb{Z})$ into a Hopf algebra. The primitives form a Lie algebra . If $B_{ij} : S^{m-2} \to \Omega F(\mathbb{R}^m,k)$ are the maps adjoint to the A_{ij}, we let

$$YB(m-2,k)$$

denote the Lie algebra with generators $B_{ij},\ k \geq i > j > k \geq 1$ of dimension $m-2$, modulo the following relations:

$$\begin{array}{ll} [B_{ij},B_{st}] = 0, & \{i,j\}\wedge\{s,t\} = \emptyset; \\ [B_{ij},B_{jt}] = [B_{ij},B_{it}], & i>j>t; \\ [B_{ij},B_{sj}] = [B_{ij},B_{is}], & i>s>j. \end{array}$$

These are the so–called *infinitesimal braid relations* or the *infinitesimal Yang–Baxter relations*. Thus YB stands for Yang–Baxter. This Lie algebra has appeared in several different contexts recently.

Now recall that if L is a Lie algebra, its universal enveloping algebra $U(L)$ is $T(L)/I$, where $T(L)$ is the tensor algebra on L and I is the two sided ideal generated by elements $x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$ where $|\ |$ is dimension and $x, y \in L$.

The following result has also been obtained by Fadell–Huseini:

Theorem B *As a Hopf algebra*

$$H_*(\Omega F(\mathbb{R}^m, k)) = U(YB(m-2, k))$$

where $m \geq 3$ *and* $YB(m-2, k)$ *is the set of primitives.*

In the decomposition of Theorem A, we have maps:

$$\begin{aligned} &\Omega M \xrightarrow{\theta_k} \Omega F(M, k) \\ &\Omega F(\mathbb{R}^m, k) \to \Omega F(M, k) \\ &D(M, k) \to \Omega F(M, k) \end{aligned}$$

that are H–maps. Thus, in order to determine the Pontryagin ring structure of $H_*(\Omega F(M, k); R)$, R say a field, we need to study the commutators among the different factors: If $m \in H_*(\Omega M; R)$, we denote by $m_i = 1 \otimes \cdots \otimes 1 \otimes m \otimes 1 \otimes \cdots \otimes 1$, where m is in the i^{th} position.

We have:

Proposition 1 *a)* $[B_{ij}, m_l] = 0$ *if* $\ell \notin \{i, j\}$.

b) If m *is primitive in* $H_*(\Omega M; R)$, *then* $[B_{ij}, m_i + m_j] = 0$ *provided*

- $R = \mathbb{Z}/2$ *and* $w_{m-1}(\tau M) = 0$, *where* τM *is the tangent bundle to* M *and* w_{m-1} *is its* $(m-1)$ *Stiefel-Whitney class.*
- $R = \mathbb{Z}/p$ *or* $\mathbb{Q}$ *and* $\chi(\tau M) = 0$ *where* p *is an odd prime and* $\chi(\tau M)$ *is the Euler class of* τM.

Proposition 2 $[m_i, m_j] = 0$ *in the following cases:*

a) $M = N \times \mathbb{R}^1$

b) dim $M > 2$ *(homological dim of* M*)*

c) $M = M_1 \times M_2 \times M_3$

However, if $M = S^{n_1} \times S^{n_2}$*, we can show that* $[m_i, m_j] \neq 0$.

Now suppose that M is a manifold such that

$$H_*(\Omega(M - Q_1); R) \to H_*(\Omega M; R)$$

is an epimorphism. We call such a manifold an eR–manifold (eR–epimorphism). The following are examples of eR–manifolds:

a) If M is a punctured manifold, it is an eR–manifold for all R.

b) If $M = M_1 \times M_2$, M is an eR–manifold for R such that $H_*(\Omega(M_1 \times M_2)) \cong H_*(\Omega M_1) \otimes H_*(\Omega M_2)$.

c) If $R = \mathbb{Q}$ and M is compact closed with $H^*(M; \mathbb{Q})$ having more than one cohomology generator [3].

d) If M is the connected sum of simply connected manifolds, where at least one of the summands is an eR–manifold.

e) Certain choices of homogeneous spaces G/H are eR–manifolds.

f) Spheres and complex projective spaces are not $e\mathbb{Q}$–manifolds.

g) $M = N \times \mathbb{R}$.

We now have:

Theorem C *If M is an eR–manifold and M is 1–connected, then there is a short exact sequence of Hopf algebras:*

$$1 \to H_*(\Omega F(M - Q_1, k-1); R) \to H_*(\Omega F(M, k), R) \to H_*(\Omega M; R) \to 1,$$

furthermore $M - Q_i$, $i \geq 1$, *is an eR–manifold and*

$$H_*(\Omega F(M, k); R) \approx \bigotimes_{i=0}^{k-1} H_*(\Omega(M - Q_i); R)$$

is an isomorphism of graded R–modules, while the inclusion $M - Q_1 \subset M$ induces a surjection of Hopf algebras:

$$H_*(\Omega F(M - Q_1, k); R) \to H_*(\Omega F(M, k); R).$$

Thus the structure of Hopf algebra of $H_*(\Omega F(M,k);R)$ can be determined by that of $H_*(\Omega F(M-Q_1,k);R)$. As an interesting example, $SU(n)$ rationally is a product of odd spheres $S^3 \times \cdots \times S^{2n-1}$, yet $H_*(\Omega F(SU(n),k);\mathbb{Q})$ is twisted for $n=3$, but not for $n \neq 3$.

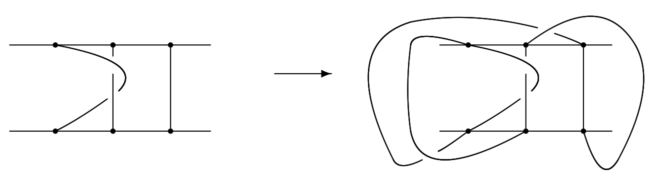
Let us now recall the braid group. If we look at $F(\mathbb{R}^2,k)$, it turns out to be a $K(\pi,1)$–space, where $\pi = B_k$ is Artin's pure braid group on k–strands. It is generated by $x_1,\cdots,x_k$ subject to the following relations:

$$x_i x_j x_i^{-1} = x_j \quad \text{if} \quad |\, i-j \,| > 1$$

$$x_j x_i x_j^{-1} = x_i x_j x_i^{-1} \quad \text{if} \quad |\, i-j \,| = 1$$

$B_k = \pi_0(\Omega F(\mathbb{R}^2,k))$.

There is a theorem of Alexander [1] saying that there is a way to close a pure braid, to produce a knot

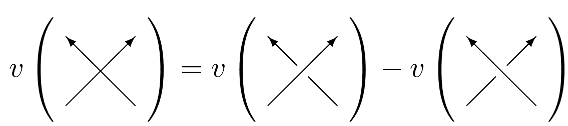


If you do it for all k, you produce all knots. If k is the space of all knots, we thus have

$$A : \amalg B_k \to K$$

onto and one knows when two pure braids produce the same knot: $A(x) = A(y)$, produce the same knot if $x = yx_i$ or when $x = aya^{-1}$ for some $a \in B_k$. We have a complete result on Vassiliev invariants of the pure braids.

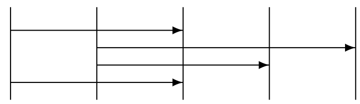
Let $V(B_n)$ be set of invariants over $\mathbb{C}$ of the pure braids with n–strands. We can extend to pure braids B_n^1 with one double point:



and thus extend to B_n^k the set of pure braids on n–strands with k double points. An invariant v is called a Vassiliev invariant of order k if v

vanishes on all pure braids having more than k double points. Let V_k^n be the vector space of Vassiliev invariants on B_k of order k.

Let A_k be the complex vector space spanned by horizontal chord diagrams: a horizontal chord diagram consists of n vertical strands



labeled $1, \cdots, n$ say, and k chords, each chord joining a pair of strands. We take them ordered both with an arrow and their order of appearance in the diagram.

Let A_k^n be the quotient of $\tilde{A}_k^n$ by the relations (I) which are called in the context of knot theory the $4T$ *relations and framing independence*. Then Kohno among others has proved that $V_k^n / V_{k-1}^n \approx \mathrm{Hom}_{\mathbb{C}}(A_k^n, \mathbb{C})$.

What we obtain is:

Theorem D *There is an isomorphism of Hopf algebras*

$$H_*(\Omega F(\mathbb{R}^3, n); \mathbb{C}) \approx A_*^n$$

and the Poincare series of $H_*(\Omega F(\mathbb{R}^3, n); \mathbb{Z})$ is

$$P(H_*(\Omega F(\mathbb{R}^3, n), t) = \left(\prod_{k=1}^{n-1} (1 - kt) \right)^{-1}.$$

Moreover, each A_k^n is spanned by monomials of length k in the B_{ij}.

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