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EACH COPY OF THE REAL LINE IN $\mathbb{C}^2$ IS REMOVABLE

EDUARDO SANTILLÁN ZERÓN [1]

Abstract

In 1994, Professors E.M. Chirka, E.L. Stout and G. Lupacciolu showed that a closed subset of $\mathbb{C}^n$ $(n \geq 2)$ is removable for holomorphic functions, if its cover dimension is less than or equal to $n-2$. Besides, they asked whether closed subsets of $\mathbb{C}^2$ homeomorphic to the real line (the simplest 1-dimensional sets) are removable for holomorphic functions. In this paper we propose a positive answer to that question.

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1 Introduction

In 1994, professors E.M. Chirka, E.L. Stout and G. Lupacciolu showed that a closed subset X of $\mathbb{C}^n$ $(n \geq 2)$ is removable for holomorphic functions, if the cover dimension of X is less than or equal to $n-2$ (see [1], [4] and [5]). That is, each holomorphic function defined on $\mathbb{C}^n - X$ has got a holomorphic extension on $\mathbb{C}^n$. Besides, they asked whether closed subsets of $\mathbb{C}^2$ homeomorphic to the real line (the simplest 1-dimensional sets) are removable for holomorphic functions. We propose answering positively that question by showing the following:

MAIN PROPOSITION. *Let X be a closed subset of $\mathbb{C}^2$ homeomorphic to the real line, then X is removable for holomorphic functions in $\mathbb{C}^2$.*

REMARKS: The real line $\mathbb{R}$ is endowed with the standard topology. **Arcs** are spaces homeomorphic to the closed interval $[0,1] \subset \mathbb{R}$ (see [2, p. 59]). The topology of $\mathbb{C}^2$ is generated by the standard norm $\|\cdot\|$.

[1] Associate Professor, Departamento de Matemáticas del CINVESTAV.

Proof. Let f be a holomorphic function defined on $\mathbb{C}^2 - X$, and let x be an arbitrary point of X. We will show that f extends holomorphically over x by applying Theorem II.7 of [6]: Let $\mathcal{M}$ be a Stein manifold of dimension n ($n \geq 2$) and let W be a relatively compact domain in $\mathcal{M}$ with boundary $\delta W = E \cup \Gamma$ where Γ is a connected $\mathcal{C}^1$ submanifold of $\mathcal{M} - E$ and E is a compact set. If the $\mathcal{O}(\mathcal{M})$-convex hull of E meets $\overline{W}$ only in the set E, then E is removable.

Since X is homeomorphic to the real line, there exists an arc $H \subset X$ such that $x \in H$, but x is not an end-point of H. Note that H has got exactly two end-points, named a and b. We can assume, without loss of generality, that $\|a - b\| \geq 3$ (recall that X is neither compact nor bounded). Set $E = \{a, b\}$ and $H^\diamond = H - E$. It is easy to see that $H^\diamond$ is homeomorphic to $\mathbb{R}$ and is an open subset of X. Whence, there exists an open subset $U \subset \mathbb{C}^2$ such that $H^\diamond = X \cap U$, because X is a subspace of $\mathbb{C}^2$ (see [2, p. 22]).

The arc H is closed and bounded in $\mathbb{C}^2$ because it is compact; so can we demand the open set U to be bounded as well. Therefore, the closure $\overline{U}$ and boundary $\delta U = \overline{U} - U$ are both compact sets; each closure, interior and/or boundary is calculated with respect to $\mathbb{C}^2$ (see [3, pp. 69-72]). We are now going to build (by induction) an open neighborhood $V \subset \mathbb{C}^2$ of $H^\diamond$ which boundary $\delta V \cap X \subset E$ and $\delta V - E$ is a $\mathcal{C}^2$-smooth manifold (it may not be connected).

For every positive integer k, let $B_k(z) \subset \mathbb{C}^2$ be the open ball of radius 1/k and center in $z \in \mathbb{C}^2$, and let $B_k(E) = B_k(a) \cup B_k(b)$ be open in $\mathbb{C}^2$. Besides, consider the compact set $H_k = H - B_k(E)$ (it is closed and bounded). The following facts are satisfied:

- **i)** $E \subset B_m(E) \subset B_k(E)$ for all integers $m \geq k > 0$.
- **ii)** The sequence $B_k(E)$ converges to E when $k \to \infty$ (see page 471 in [2]).
- **iii)** $H_k \subset H_m \subset H^\diamond \subset U$ for all $m \geq k > 0$ by point (i).
- **iv)** The sequence H_k converges to H when $k \to \infty$.

Each H_k is non-empty because the distance between the end-points of H is $\|a - b\| \geq 3$, and the radius of the balls $B_k(a)$ and $B_k(b)$ is $1/k \leq 1$. From point (iii) above: $H_k \cap \delta U = H_k \cap \overline{U} - U = \emptyset$. The distance between H_k and δU is then greater than zero because they are both compact sets (see [2, p. 84]). That is, there exists a real number $\beta > 0$ such that $\|s - t\| > \beta$ for all $s \in H_k$ and $t \in \delta U$. Hence, there exist open sets $V_k \subset \mathbb{C}^2$ such that:

v) $H_k \subset V_k \subset \overline{V_k} \subset U$ for every integer $k > 0$.
vi) The boundary $\delta V_k = \overline{V_k} - V_k$ is a $\mathcal{C}^2$-smooth 3-manifold.

Moreover, we are going to choose the sets V_k inductively. From fact (iii), we have got that $H_k = H_{k+1} - B_k(E)$. That is, the sets H_{k+1} and H_k are equal outside $B_k(E)$, so can we ask $\overline{V_{k+1}}$ and $\overline{V_k}$ to be equal outside $B_k(E)$ as well. Fix $\overline{V_1} \subset U$ and demand that $\overline{V_{k+1}} - B_k(E) = \overline{V_k} - B_k(E)$ holds for every integer $k > 0$. We then deduce that $\overline{V_m} - B_k(E) = \overline{V_k} - B_k(E)$ holds for all integers $m \geq k > 0$, by point (i).

Therefore, from statement (ii), the sequence $\overline{V_k}$ converges to a compact set $\widetilde{V} \subset \mathbb{C}^2$ when $k \to \infty$. Indeed: $\widetilde{V} - B_k(E) = \overline{V_k} - B_k(E)$ for each $k > 0$. Observe that $\widetilde{V}$ is closed (see [2, p. 471]) and bounded in $\mathbb{C}^2$ (recall that each $\overline{V_k} \subset U$ where U is bounded). Now let $V \subset \mathbb{C}^2$ be the interior set of $\widetilde{V}$. The following equalities follows from fact (vi):

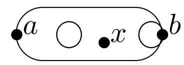
vii) $\delta V - B_k(E) = \delta V_k - B_k(E) = \delta\widetilde{V} - B_k(E)$.
viii) $V - B_k(E) = V_k - B_k(E)$.

We have got that $H^\diamond \subset V$. Indeed, let $w \in H^\diamond$ be an arbitrary point. There exists an integer $\mu > 0$ such that $w \in H_\mu$ by points (iii) and (iv); besides: $H_\mu = H - B_\mu(E) \subset V_\mu - B_\mu(E) \subset V$ (see (v) and (viii)). On the other hand, the set $P = X \cap (\delta V - B_m(E))$ is empty for every integer $m > 0$.

From (v), (vi) and (vii):	$P = X \cap (\delta V_m - B_m(E)) \subset U$.
Since $H^\diamond = U \cap X$:	$P = H^\diamond \cap (\delta V_m - B_m(E))$.
Since $H^\diamond \subset H$:	$P \subset (H - B_m(E)) \cap \delta V_m = H_m \cap \delta V_m$.
From (v) and (vi):	$P \subset (H_m \cap \overline{V_m}) - V_m = \emptyset$.

Moreover: $P = (X \cap \delta V) - B_m(E)$. Whence, the inclusion $X \cap \delta V \subset B_m(E)$ holds for each $m > 0$; and so: $X \cap \delta V \subset E$ by facts (i) and (ii). Finally, we deduce that $\delta V - E$ is a $\mathcal{C}^2$-smooth 3-manifold by using facts (ii), (vi) and (vii). The point $x \in H^\diamond$ has then got a compactly contained open neighborhood $V \subset \mathbb{C}^2$ with the desired boundary. However, the sets V and/or $\delta V - E$ may not be connected. In order to solve this problem we proceed as follows.

Consider W_1 to be the connected component of V which contains to $H^\diamond$ (recall that $H^\diamond$ is connected). Since $\mathbb{C}^2$ is locally connected, we have got that $W_1 \subset \mathbb{C}^2$ is open and $\delta W_1 \subset \delta V$ (see [3, pp. 113 and 118]). Hence, the set W_1 is open, connected and compactly contained in $\mathbb{C}^2$; moreover: $x \in W_1$, its boundary $\delta W_1 \cap X \subset E$ and $\delta W_1 - E$ is a $\mathcal{C}^2$-smooth 3-dimensional manifold. A picture of $\overline{W_1}$ could be the following:

$\overline{W_1} =$ 

Now let W_2 be the only *unbounded* connected component of $\mathbb{C}^2 - \overline{W_1}$ (recall that $\overline{W_1}$ is compact and see [3, p. 356]), and let $W_3 = \mathbb{C}^2 - \overline{W_2}$ be open. The set $W_2 \subset \mathbb{C}^2$ is open by the locally connectedness. Besides:

ix) $\delta W_2 \subset \overline{W_1} - W_1 = \delta W_1$, because W_1 is open and [3, p. 356].
x) $\overline{W_3} = \mathbb{C}^2 - W_2$ is compact (bounded).
xi) $\overline{W_2} \cup \overline{W_3} = \mathbb{C}^2$ and $\overline{W_2} \cap \overline{W_3} = \delta W_2 = \delta W_3$.

Note that $x \in W_1 \subset W_3$. On the other hand, let D be a connected component of W_3. It is easy to see that δD is not a finite set, so can we pick up a point $w \in \delta D - E$. From [3, p. 118] and statements (ix) and (xi), we have got that $\delta D \subset \delta W_3 = \delta W_2 \subset \delta W_1$. Whence, there exists an integer $\mu > 0$ such that $B_\mu(w) \cap \delta W_1$ is *diffeomorphic* to $\mathbb{R}^3$ (recall that $\delta W_1 - E$ is $\mathcal{C}^2$-smooth); and so $B_\mu(w) - \delta W_1$ consists of exactly two connected components.

Furthermore, one connected component of $B_\mu(w) - \delta W_1$ is contained in W_2 (recall that $w \in \delta W_2$) and the other component must be contained in both D and W_1 (because D and W_1 are both contained in $\mathbb{C}^2 - W_2$). Therefore: $D \cap W_1 \neq \emptyset$. That is, the connected set W_1 is contained in W_3 and meets to every connected component of W_3; so is W_3 a connected compactly contained open set of $\mathbb{C}^2$ (see (x)).

We know that $\delta W_3 \subset \delta W_1$. Hence: $\delta W_3 \cap X \subset E$ and $\delta W_3 - E$ is a $\mathcal{C}^2$-smooth 3-manifold. Actually, the equality $\delta W_3 \cap X = E$ holds, but we do not use it in this proof. Recall that $W_2 = \mathbb{C}^2 - \overline{W_3}$ is connected, so are $\overline{W_2}$ and $\overline{W_3}$ (see [3, p. 109]). Since $\mathbb{C}^2$ is unicoherent (see [2, p, 397]) and point (xi), the boundary δW_3 is connected. Reasoning in a similar way and using the fact that $\mathbb{C}^2 - E$ is unicoherent (recall that E consists of only two points), we deduce that $\delta W_3 - E = \overline{W_2} \cap \overline{W_3} \cap (\mathbb{C}^2 - E)$ is connected as well.

Finally, observe that the function f is holomorphic on $\delta W_3 - E$ (because f is holomorphic outside X). Applying Theorem 3 of [4] or Theorem II.7 of [6], we conclude that f can be holomorphically extended over $x \in W_3$. The function f has then got a holomorphic extension on $\mathbb{C}^2$ because the point $x \in X$ was chosen arbitrarily. ■

Moreover, let $\mathcal{M}$ be a connected complex manifold of dimension two. Each closed subset $X \subset \mathcal{M}$ homeomorphic to the real line is removable. Let f be a holomorphic function defined on $\mathcal{M} - X$, and let $x \in X$ be an arbitrary point. It easy to see that there exists a biholomorphism g

from the open unit ball $B_1(0) \subset \mathbb{C}^2$ onto a neighborhood of x, such that $g(0) = x$. Since $B_1(0)$ is homeomorphic to $\mathbb{C}^2$, we can build an open set $W_3 \subset B_1(0)$ such that $0 \in W_3$, its boundary δW_3 meets X in at most two points and satisfies the hypothesis of Theorem 3 of [4]. Thus, the function $f \circ g$ extends holomorphically over 0 (inside $B_1(0)$); and so f extends holomorphically over x as well.

However, in order to show that every closed set homeomorphic to $\mathbb{R}^{n-1}$ is removable in $\mathbb{C}^n$ $(n \geq 3)$, we firstly need to prove that the topological copies of sphere $\mathcal{S}^{n-2}$ are removable for the boundary in $\mathbb{C}^n$. We have strongly used the fact that $\mathcal{S}^0$ contains just two points (it is holomorphically convex).

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R. Eduardo Santillán Z.
Departamento de Matemáticas, Cinvestav.
Apartado Postal 14-740, México 07360, D.F.
eszeron@math.cinvestav.mx

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