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Algebraic K-theory and the η-invariant *

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Abstract

The aim of this paper is to present the main results of J. D. S. Jones and B. W. Westbury on algebraic K-Theory, homology spheres and the η-invariant [6], giving the basic definitions and prerequisites to understand them.

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1 Introduction

In [6] J. D. S. Jones and B. W. Westbury constructed elements in $K_3(\mathbb{C})$, the 3rd algebraic K-theory group of the field of complex numbers, using homology 3-spheres endowed with a representation of their fundamental group. They also computed the image of such elements under the regulator map, using the η-invariant. The aim of this paper is to present the main results of J. D. S. Jones [6], giving the basic definitions and prerequisites to understand them.

The paper is divided in four parts. In section 2 we define the algebraic K-groups of a ring using Quillen's +-construction. We also explain how homology n-spheres equipped with a representation of its fundamental group in the general linear group over a ring R define elements in the K-group $K_n(R)$. In section 3 we give the definition of the η-invariant of a self-adjoint elliptic operator on a closed manifold and its variations. In section 4 we describe the Dirac operator which is a very important example of this kind of operators and the one which we are interested

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in. Finally in section 5 the main results by Jones and Westbury are presented.

2 Algebraic K-Theory

In this section we define the algebraic K-groups and we describe how to construct elements in this groups using homology spheres equipped with a representation of its fundamental group.

2.1 Classifying space of a group

Any discrete group G has a *classifying space* BG which is a pointed space (i.e. it has a base point $*$) unique up to homotopy equivalence such that:

> $\pi_1(BG) = G$ and $\pi_i(BG) = 0$ for $i \neq 1$ i.e. BG is an Eilenberg-Mac Lane space $K(G, 1)$.

From its definition, the universal covering of BG, denoted by EG is contractible. The covering $EG \to BG$ is called the *universal bundle* for G and the space BG satisfies the following universal property:

If $EG \to BG$ is a universal bundle for G and X is of the homotopy type of a CW-complex with base point x_0 (e.g. manifold). Then we have the following one-to-one correspondences

$$[X, BG] \longleftrightarrow \mathrm{Hom}(\pi_1(X, x_0), G) \longleftrightarrow F_G(X)$$

where $[X, BG]$ denotes the homotopy classes of maps from X to BG, $\mathrm{Hom}(\pi_1(X, x_0), G)$ denotes the homomorphisms from $\pi_1(X, x_0)$ to G and $F_G(X)$ the equivalence classes of principal (flat) G-bundles over X.

Note that in the case when $G = GL_N(\mathbb{C})$, $\mathrm{Hom}(\pi_1(X, x_0), G)$ is precisely the set of representations of $\pi_1(X, x_0)$ on $\mathbb{C}^N$.

2.2 Quillen's +-construction

In order to define the algebraic K-theory groups of a ring R, we need the +-construction due to Daniel Quillen in the early 1970's, for which, among other reasons, he was awarded the Fields Medal in 1978.

Theorem 2.2.1 (Quillen). *Let X be a connected CW-complex with base point x_0. Let $A \subset \pi_1(X)$ be a perfect normal subgroup (i.e. $A = [A, A]$ and $A = [\pi_1(X), A]$, where $[\ ,\ ]$ is the commutator). Then there is a space X^+ (depending on A) and a map $i\colon X \to X^+$ such that:*

(a) *The map i induces an isomorphism*

$$i\colon \pi_1(X)/A \to \pi_1(X^+).$$

(b) *For any $\pi_1(X^+)$-module L one has*

$$i_*\colon H_*(X, i_*L) \xrightarrow{\cong} H_*(X^+, L).$$

(c) *The pair (X^+, i) is determined by a) and b) up to homotopy equivalence.*

Let R be a ring with 1. Consider the group $GL_N(R)$ of invertible $N \times N$ matrices over R. The *elementary group* $E_N(R)$ is the subgroup of $GL_N(R)$ generated by the elementary matrices (see [12, 11, 9] for definition).

We have inclusions $GL_N(R) \subset GL_{N+1}(R)$ which restrict to inclusions $E_N(R) \subset E_{N+1}(R)$ and we can define

$$GL(R) = \bigcup_N GL_N(R)$$
$$E(R) = \bigcup_N E_N(R).$$

Let $X = BGL(R)$. Then $\pi_1(X) = GL(R)$ and $A = E(R)$ is perfect. Then applying the +-construction we get $BGL(R)^+$. Define the algebraic K-groups of the ring R by

$$K_n(R) = \pi_n(BGL(R)^+) \quad \text{for } n \geq 1.$$

This definition may seem artificial, the reason is because originally the first three groups $K_0(R)$, $K_1(R)$ and $K_2(R)$ were given by algebraic definitions[2] and for a while seemed to be no good way to define the "higher K-functors" $K_i,\ i \geq 3$, until Quillen's work appeared, for a nice account of this facts see [12].

[2]In the present definition we are not including $K_0(R)$, in this case is called the "reduced" algebraic K-theory of R

2.3 Homology spheres

It is well known that the homology of the n-sphere S^n is given by

$$H_q(S^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & q \neq 0, n. \end{cases}$$

A *homology n-sphere* as its name indicates it, is a path-connected space (say with the homotopy type of a CW-complex) with the same homology groups as S^n ($n \geq 3$).

Let Σ be a homology n-sphere, since

$$0 = H_1(\Sigma, \mathbb{Z}) = \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)]$$

$\pi_1(\Sigma)$ can have no abelian quotients and so is perfect. Given a representation $\alpha\colon \pi_1(X) \to GL_N(R)$, let $f\colon \Sigma \to BGL_N(R)$ be the map which induces α on π_1 (by the universal property of classifying spaces). Composing this map with the inclusion $BGL_N(R) \to BGL(R)$ and applying Quillen's $+$-construction we get

$$S^n \backsimeq \Sigma^+ \to BGL(R)^+,$$

since the $+$-construction is functorial by its universal properties. Here $\backsimeq$ denotes homotopy equivalence. The homotopy class of this map gives us the element in K-theory

$$[\Sigma, \alpha] \in K_n(R) = \pi_n(BGL(R)^+).$$

2.4 The regulator

There is a homomorphism

$$e\colon K_{2n+1}(\mathbb{C}) \to \mathbb{C}/\mathbb{Z}$$

called the regulator map which satisfies the following properties

(i) It is an isomorphism on $K_1(\mathbb{C}) \cong \mathbb{C}^* \to \mathbb{C}/\mathbb{Z}$.

(ii) The homomorphism e gives an isomorphism of the torsion subgroup of $K_{2n+1}(\mathbb{C})$ with $\mathbb{Q}/\mathbb{Z}$.

The aim now is to compute the image of the elements $[\Sigma, \alpha] \in K_3(\mathbb{C})$ under the regulator map. One way to do this is using the η-invariant.

3 The η-invariant

Let X be a closed (compact without boundary) Riemannian manifold and let E be a smooth vector bundle over X with an inner product. We denote by $C^\infty(X, E)$ the space of smooth sections of E and we can endow it with an inner product $\langle\ ,\ \rangle$ using the inner product on E and integration. Let $A\colon C^\infty(X, E) \to C^\infty(X, E)$ be an elliptic differential operator and assume that A is *self-adjoint*, that is

$$\langle s_1, As_2\rangle = \langle As_1, s_2\rangle$$

for every $s_1, s_2 \in C^\infty(X, E)$. Then A has a discrete spectrum with real eigenvalues $\{\lambda\}$ and we define the *η-series* of A by

$$\eta(s; A) = \sum_{\lambda\neq 0} (\operatorname{sign}\lambda)|\lambda|^{-s}$$

where the sum is taken over the non-zero eigenvalues of A. This series converges for $\Re(s)$ sufficiently large. By results of Seeley [13] extends by analytic continuation to a meromorphic function on the whole s-plane and is finite at $s = 0$.

The number $\eta(0; A)$ is called the *η-invariant* of A and is a spectral invariant which measures the asymmetry of the spectrum of A.

We also define a refinement of the η-series which takes into account the zero eigenvalues of A

$$\xi(s; A) = \frac{h + \eta(s; A)}{2}$$

where h is the dimension of the kernel of A or in other words, the multiplicity of the 0-eigenvalue of A.

Now consider a representation $\alpha\colon \pi_1(X) \to GL_N(\mathbb{C})$. Then α defines a flat bundle V_α over X in the following way. Let $\tilde{X}$ be the universal cover of X. Then $V_\alpha = \tilde{X} \times_{\pi_1(X)} \mathbb{C}^N$ i.e. V_α is $\tilde{X} \times \mathbb{C}^N$ modulo the action of $\pi_1(X)$, where $\pi_1(X)$ acts on the first factor with the canonical action of $\pi_1(X)$ on the universal cover and via the representation α on the second factor. The bundle V_α also has a canonical flat connection ∇^α given by the exterior derivative as follows.

A connection is a first order linear differential operator

$$C^\infty(X, V_\alpha) \xrightarrow{\nabla^\alpha} \Omega^1(X, V_\alpha)$$

which satisfies the Leibnitz rule

$$\nabla^\alpha fs = df \otimes s + f \otimes \nabla^\alpha s$$

for every $f \in C^\infty(X, \mathbb{R})$ and every $s \in C^\infty(X, V_\alpha)$.

By the previous construction of the bundle V_α we have that $C^\infty(X, V_\alpha) \cong C^\infty(\tilde{X}, \mathbb{C}^N)^\alpha$ and $\Omega^1(X, V_\alpha) \cong \Omega^1(\tilde{X}, \mathbb{C}^N)^\alpha$, where the spaces $C^\infty(\tilde{X}, \mathbb{C}^N)^\alpha$ and $\Omega^1(\tilde{X}, \mathbb{C}^N)^\alpha$ are, respectively, the sections and 1-forms which are equivariant under the action of $\pi_1(X)$ via the representation α. On the other hand, the exterior derivative

$$C^\infty(\tilde{X}, \mathbb{C}^N) \xrightarrow{d} \Omega^1(\tilde{X}, \mathbb{C}^N)$$

sends invariant sections to invariant 1-forms. Hence the connection ∇^α is given by

$$C^\infty(X, V_\alpha) \cong C^\infty(\tilde{X}, \mathbb{C}^N)^\alpha \xrightarrow{\nabla^\alpha = d} \Omega^1(\tilde{X}, \mathbb{C}^N)^\alpha \cong \Omega^1(X, V_\alpha).$$

Using this connection we can couple the operator A to V_α to get an operator

$$A_\alpha \colon C^\infty(X, E \otimes V_\alpha) \to C^\infty(X, E \otimes V_\alpha)$$

and as before we define the functions[3]

$$\eta(s; \alpha, A) = \eta(s; A_\alpha), \qquad \xi(s; \alpha, A) = \xi(s; A_\alpha)$$

and their reduced forms

$$\tilde{\eta}(s; \alpha, A) = \eta(s; \alpha, A) - N\eta(s; A), \qquad \tilde{\xi}(s; \alpha, A) = \xi(s; \alpha, A) - N\xi(s; A)$$

where N is the dimension of the representation α.

Once more, following [2, Section 2] we can see that the functions $\tilde{\eta}(s; \alpha, A)$ and $\tilde{\xi}(s; \alpha, A)$ are finite at $s = 0$ and if we reduce modulo $\mathbb{Z}$ then

$$\tilde{\eta}(\alpha, A) = \tilde{\eta}(0; \alpha, A) \in \mathbb{C}/\mathbb{Z}, \qquad \tilde{\xi}(\alpha, A) = \tilde{\xi}(0; \alpha, A) \in \mathbb{C}/\mathbb{Z}$$

[3]The operator A_α is not self-adjoint any more, unless the representation α is unitary. Nonetheless, A_α has self-adjoint symbol and that allows us to define the η and ξ functions, see [2, p. 90].

are homotopy invariants of A. The reason for regarding values in $\mathbb{C}/\mathbb{Z}$ and not just in $\mathbb{C}$ is that if we vary A continuously the dimension of $\ker A$ is not a continuous function of A. However the jumps of $\xi(s; A)$ are due to eigenvalues changing sign as they cross zero and therefore the jumps are integer jumps.

Note that if we fix the manifold X and the operator A, the invariant $\tilde{\xi}(\alpha, A)$ only depends on the representation α of the fundamental group of X or equivalently on the flat bundle V_α aver X.

4 The Dirac operator

In this section we describe a particular example of a self-adjoint elliptic differential operator called the *Dirac operator* which is the one we shall use to compute $\mathbb{C}/\mathbb{Z}$-valued invariants of elements of the K-groups of any subring of $\mathbb{C}$. The Dirac operator is very important by itself and plays a central role in the Atiyah-Singer Index Theorem, in the Seiberg-Witten theory and many other things. The main references for the material in this section are [7, 1].

4.1 Clifford algebras

Let V be a finite dimensional real vector space with a non-degenerate, symmetric bilinear form $q\colon V \otimes V \to \mathbb{R}$. Let $\{e_1, \dots, e_n\}$ be an orthogonal basis for V then the *Clifford algebra* $Cl(V, q)$ is the algebra over $\mathbb{R}$, with unit, generated by the e_i, subject to the relations

$$\begin{aligned} e_i^2 &= -q(e_i, e_i) \\ e_i e_j &= -e_j e_i \qquad i \neq j. \end{aligned}$$

For the special case when $V = \mathbb{R}^n$ and q is the standard inner product we denote the algebra $Cl(V, q)$ by Cl_n and its complexification by $Cl_n^{\mathbb{C}} = Cl_n \otimes \mathbb{C}$.

Example 4.1.1.

$$\begin{aligned} Cl_0 &= \mathbb{R} && \textit{with basis} \quad 1 \\ Cl_1 &= \mathbb{C} && \textit{with basis} \quad 1, e_1 \\ Cl_2 &= \mathbb{H} && \textit{with basis} \quad 1, e_1, e_2, e_1 e_2 \end{aligned}$$

The group $\mathrm{Spin}(n)$ is defined as a subgroup of the group of units of Cl_n and it is the non-trivial double covering of $\mathrm{SO}(n)$ and for $n > 2$ it is its universal covering.

Now lets restrict ourselves to odd dimensional vector spaces, in this case, the complexified Clifford algebra $Cl_n^{\mathbb{C}}$ has two inequivalent irreducible complex representations and when they are restricted to $\mathrm{Spin}(n)$ they give isomorphic irreducible complex representations of $\mathrm{Spin}(n)$. We denote such a representation space by S.

4.2 Spin structures

Let X be an odd dimensional oriented closed Riemannian manifold. The Riemannian metric and the orientation give a reduction of the structure group of the tangent bundle TX of X to $\mathrm{SO}(n)$. A *spin structure* on X is a lift of the structure group $\mathrm{SO}(n)$ of TX to $\mathrm{Spin}(n)$.

A spin structure on X provide us with a principal $\mathrm{Spin}(n)$-bundle Q which is a double cover of the principal $\mathrm{SO}(n)$-bundle P associated to the tangent bundle TX. The restriction to the fibre of this double cover $\varpi\colon Q \to P$ is the double covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$.

Now consider the spin representation S of $\mathrm{Spin}(n)$ and let

$$\mathcal{S}(X) = Q \times_{\mathrm{Spin}(n)} S$$

be the vector bundle over X associated to the principal $\mathrm{Spin}(n)$-bundle Q. The bundle $\mathcal{S}(X)$ is called the *spinor bundle* of X and its sections are called *spinor fields*. We denote the space of spinor fields by $C^\infty(X, \mathcal{S}(X))$.

Let $Cl(T^*X)$ be the bundle over X whose fibre at x is $Cl(T_x^*X)$, the Clifford algebra of the cotangent space at x with the inner product given by the Riemannian metric.

There is a pairing

$$C\colon Cl(T^*X) \otimes \mathcal{S}(X) \to \mathcal{S}(X)$$

which is called *Clifford multiplication*. If we consider the inclusion $T^*X \to Cl(T^*X)$ then we get a pairing

$$T^*X \otimes \mathcal{S}(X) \to \mathcal{S}(X).$$

4.3 The Dirac operator

The Riemannian structure of X provides us with the Riemannian connection on the tangent bundle. This connection can be seen as a 1-form

β on the principal SO(n)-bundle P with values in the Lie algebra $\mathfrak{so}(n)$. Since Spin(n) and SO(n) have the same Lie algebra, the double covering $\varpi\colon Q \to P$ given by the spin structure on X gives us a 1-form $\varpi^*(\beta)$ which defines a connection on Q called the *spin connection.* This connection induces a covariant derivative

$$\nabla\colon C^\infty(X, \mathcal{S}(X)) \to C^\infty(X, T^*X \otimes \mathcal{S}(X))$$

on spinor fields.

Composing ∇ with Clifford multiplication

$$C\colon C^\infty(X, T^*X \otimes \mathcal{S}(X)) \to C^\infty(X, \mathcal{S}(X))$$

we obtain the *Dirac operator*

$$D = C \circ \nabla\colon C^\infty(X, \mathcal{S}(X)) \to C^\infty(X, \mathcal{S}(X)).$$

It is a self-adjoint, first order, elliptic partial differential operator.

As in the previous section, a representation $\alpha\colon \pi_1(X) \to GL_N(\mathbb{C})$ defines a bundle V_α with a flat connection ∇^α. In this case we can define the *twisted Dirac operator* D_α by the composition

$$\begin{aligned} C^\infty(X, \mathcal{S}(X) \otimes V_\alpha) &\xrightarrow{\nabla\otimes Id + Id\otimes\nabla^\alpha} C^\infty(X, T^*X \otimes \mathcal{S}(X) \otimes V_\alpha) \\ &\xrightarrow{C\otimes Id} C^\infty(X, \mathcal{S}(X) \otimes V_\alpha) \end{aligned}$$

where $\nabla \otimes \nabla^\alpha$ is the product connection on the bundle $\mathcal{S}(X) \otimes V_\alpha$ and Id is the identity map.

5 The results of Jones and Westbury

The relation between the value of the regulator map on the classes $[\Sigma, \alpha] \in K_3(\mathbb{C})$ and the η-invariant of the Dirac operator of the homology sphere Σ is given by the following theorem:

Theorem 5.1.1 (Jones–Westbury).

$$e([\Sigma, \alpha]) = \tilde{\xi}(\alpha, D)$$

where D is the Dirac operator on Σ.

In [6] Jones and Westbury give a formula to compute $e[\Sigma,\alpha]$ when Σ is a Seifert homology sphere. Let $(a_1,\dots,a_n)$ be an n-tuple of pairwise coprime integers. The Seifert homology 3-sphere $\Sigma(a_1,\dots,a_n)$ is a 3-manifold which admits an action of the circle S^1 which is free except for n exceptional orbits which have isotropy groups $Ca_1,\dots,Ca_n$ where $C_m\subset S^1$ is the cyclic subgroup of order m embedded in S^1 as the mth roots of unity.

In order to give the aforementioned formula we need to know a bit about the fundamental group of $\Sigma(a_1,\dots,a_n)$. Let $T(a_1,\dots,a_n)$ be the generalised triangle group which is defined by the following generators and relations

$$T(a_1,\dots,a_n)=\langle x_1,\dots,x_n|\,x_1^{a_1}=\cdots=x_n^{a_n}=x_1\dots x_n=1\rangle.$$

This group is perfect and it has a universal central extension $\tilde{T}(a_1,\dots,a_n)$ which fits into an exact sequence

$$1\to C_*\to\tilde{T}(a_1,\dots,a_n)\to T(a_1,\dots,a_n)\to 1$$

where C_* is an infinite cyclic group, except for the case of $T(2,3,5)$ where $C_*\cong\mathbb{Z}_2$.

In terms of generators and relations

$$\tilde{T}(a_1,\dots,a_n)=\langle h,x_1,\dots,x_n|\,[x_i,h]=1,x_1^{a_1}=h^{-b_1},\dots,x_n^{a_n}=h^{-b_n},$$

$x_1\dots x_n=h^{-b_0}\rangle$
where h is the generator of the centre of $\tilde{T}(a_1,\dots,a_n)$.

The b_i satisfy the relation

$$a_1\dots a_n\left(-b_0+\frac{b_1}{a_1}+\cdots+\frac{b_n}{a_n}\right)=1$$

and we have that

$$\pi_1(\Sigma(a_1,\dots,a_n))=\tilde{T}(a_1,\dots,a_n).$$

Let $\alpha\colon\pi_1(\Sigma(a_1,\dots,a_n))\to GL_N(\mathbb{C})$ be a representation, since the group $\pi_1(\Sigma(a_1,\dots,a_n))$ is perfect every complex representation α must have image in $SL_N(\mathbb{C})$. We shall consider only those representations in which the central element h acts as a scalar multiple of the identity, for instance, that is the case when α is irreducible and in general for any decomposable representation.

Suppose $\alpha(h) = \lambda_h I$ where λ_h is a scalar, then, since $\alpha(h) \in SL_N(\mathbb{C})$

$$\lambda_h = \zeta_N^{r_h}$$

is a Nth root of unity. Here $\zeta_d = e^{2\pi i/d} \in \mathbb{C}$ is the standard primitive dth root of unity. Now consider the matrices

$$\alpha(x_j), \quad j = 1, \dots, n.$$

In view of the relations $x_j^{a_j} = h^{-b_j}$ the eigenvalues $\lambda_1(j), \dots, \lambda_N(j)$ satisfy the equation

$$\lambda_k(j)^{a_j} = \lambda_h^{-b_j}.$$

There are a_j roots of this equation and we define $s_k(j)$ by

$$\lambda_k(j) = \zeta_{Na_j}^{Ns_k(j) - b_j r_h}.$$

We refer to the numbers

$$s_k(j), \quad 1 \le j \le n,\ 1 \le k \le N$$

as the *type of the representation* α.

Now we have

Theorem 5.1.2 (Jones-Westbury). *Let* $\alpha\colon \pi_1(\Sigma(a_1, \dots, a_n)) \to SL_N(\mathbb{C})$ *be a representation of the fundamental group of the Seifert homology sphere* $\Sigma(a_1, \dots, a_n)$ *in which the central element* h *acts as a scalar multiple of the identity. Let*

$$s_k(j), \quad 1 \le j \le n,\ 1 \le k \le N$$

be the type of the representation α*; then*

$$2N\Re(e[\Sigma(a_1, \dots, a_n), \alpha]) = -\sum_{j=1}^{n} \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{a(s_k(j) - s_l(j))^2}{2a_j^2}$$

where $a = a_1 \dots a_n$.

This formula was obtained using the fact that the invariants $\tilde{\xi}(\alpha, D)$ are cobordism invariants, so it is enough to compute them on a simpler manifold which is cobordant to the Seifert homology sphere (see [6]). The cobordism invariance follows from the index theorem for flat bundles in [2].

Using the previous theorem they also prove the following results

Theorem 5.1.3 (Jones-Westbury). *Every element in $K_3(\mathbb{C})$ of finite order is of the form $[\Sigma(p,q,r),\alpha]$ for some representation*

$$\alpha\colon \pi_1(\Sigma(p,q,r)) \to SL_2(\mathbb{C}).$$

Now let $\mathbb{Z}[\zeta_d]$ be the ring of algebraic numbers in the cyclotomic field $\mathbb{Q}(\zeta_d)$. Then combining the results of Borel [3], Merkurjev and Suslin [10] and Levine [8] we have that

$$K_3(\mathbb{Z}[\zeta_d]) = \mathbb{Z}/w_2(d) \oplus \mathbb{Z}^{r_2}$$

where

$$w_2(d) = \mathrm{lcm}(24, 2d)$$

and r_2 is the number of complex places of $\mathbb{Q}(\zeta_d)$. In particular note that if $(6,d)=1$ the torsion subgroup of $K_3(\mathbb{Z}[\zeta_d])$ is exactly $\mathbb{Z}/24d$.

Theorem 5.1.4 (Jones-Westbury). *If $(6,d) = 1$ there exists a representation $\alpha\colon \pi_1(\Sigma(2,3,d)) \to SL_2(\mathbb{Z}[\zeta_d])$ such that the element $[\Sigma(2,3,d),\alpha] \in K_3(\mathbb{Z}[\zeta_d])$ is a generator of the torsion subgroup.*

Example 5.1.5. *The Seifert homology sphere $P = \Sigma(2,3,5)$ is called the* Poincaré 3-sphere*. Its fundamental group, known as the* binary icosahedral *group, is a subgroup of* SU(2) *and the matrices which occur in this subgroup can all be chosen to have coefficients in the ring $\mathbb{Z}[\zeta_5]$. This gives a representation α of $\pi_1(P)$ in $SL_2(\mathbb{Z}[\zeta_5])$, and using theorem 5.1.2 we get*

$$e[P,\alpha] = \frac{1}{120}.$$

From this we deduce that the generator of the torsion subgroup of $K_3(\mathbb{Z}[\zeta_5])$ is given by $[P,\alpha]$ where α is the natural representation of $\pi_1(P)$.

6 Further research and progress

One could try to compute $\tilde{\xi}(\alpha, D)$ directly from its definition, without using the fact that it is a cobordism invariant and expect an improved formula which works for all the representations of $\pi_1(\Sigma)$ and which also gives the imaginary part. I established a first step in this direction in [4, 5] computing $\tilde{\xi}(\alpha, D)$ directly from its definition for the Poincaré sphere, which is the only homology 3-sphere with finite fundamental group. The method not only works for the Poincaré sphere but for any quotient S^3/Γ of the 3-sphere by a finite subgroup Γ (3-dimensional spherical space forms), so we get a formula to compute the η-invariant of the Dirac operator of S^3/Γ twisted by any representation of Γ, where Γ is any finite subgroup of S^3.

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