Morfismos, Vol. 4, No. 2, 2000, pp. 33–60

Little cubes and homotopy theory *

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Abstract

This is a survey article on the theory of configuration spaces of little cubes, which is naturally related to the homotopy commutativity of double or more highly iterated loop spaces. The article begins with historical background and then surveys important applications to homotopy theory.

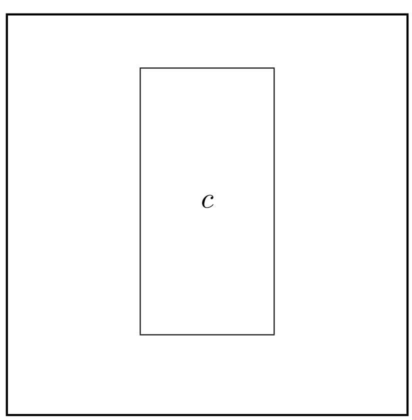
1991 Mathematics Subject Classification: 55-02, 55P35
Keywords and phrases: Configuration space, loop space, quasifibration

1 Introduction

A little n-cube is the image of an affine embedding

$$c : I^n \hookrightarrow I^n$$

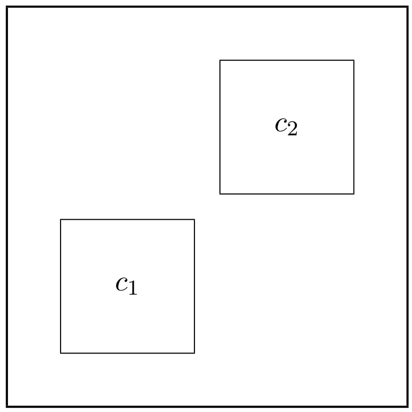
whose image has all edges parallel to the corresponding edges of the standard n-cube I^n.



*Invited article

The configuration space of j little n-cubes $\mathcal{C}_n(j)$ is the set of j-tuples $(c_1, \cdots, c_j)$ of little n-cubes whose images have disjoint interiors from each other. $\mathcal{C}_n(j)$ is topologized with the compact-open topology.

Since a little cube is determined by its image, it is safe to discuss on little cubes by drawing a picture like:



The reader might notice that he or she has seen this kind of picture in a textbook of algebraic topology.

It is an elementary theorem in homotopy theory that the n-th homotopy group of a pointed space X, $\pi_n(X)$, is an Abelian group, or equivalently the n-th loop space on X, $\Omega^n X$, is a homotopy commutative H-space, if $n \geq 2$.

In most textbooks on algebraic topology, this fact is explained as follows: Suppose an element $\alpha \in \pi_n(X)$ is represented by a continuous map

$$f : (I^n, \partial I^n) \longrightarrow (X, *),$$

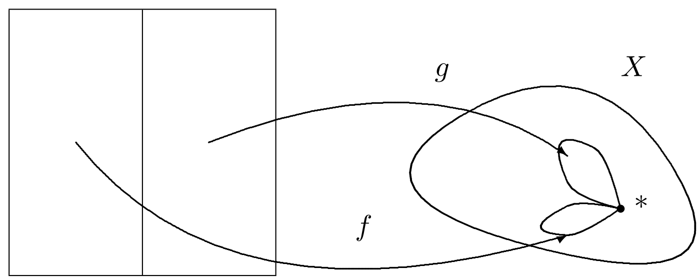
where $*$ denotes the base point of X. Pick up another element $\beta \in \pi_n(X)$ represented by

$$g : (I^n, \partial I^n) \longrightarrow (X, *).$$

The product of α and β in $\pi_n(X)$, $\alpha + \beta$, is represented by the map

$$f + g : (I^n, \partial I^n) \longrightarrow (X, *)$$

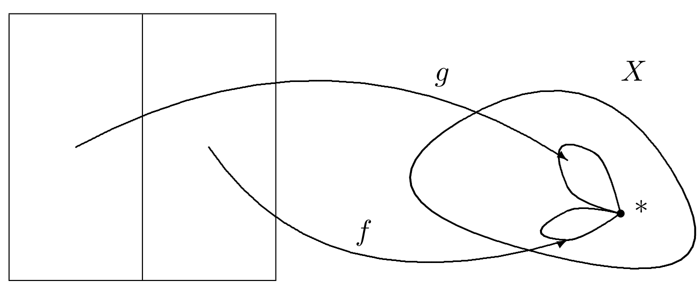
defined by the following picture.



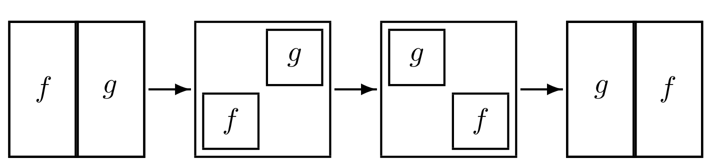
On the other hand, $\beta + \alpha$ is represented by the map

$$g + f : (I^n, \partial I^n) \longrightarrow (X, *)$$

defined by the following picture.



If $n \geq 2$, a homotopy connecting these two maps can be obtained by rotating two (rectangular) cubes in I^n.



Notice that domains of $f+g$ and $g+f$ can be regarded as elements of $\mathcal{C}_n(2)$ and the above picture can be interpreted as a path in $\mathcal{C}_n(2)$, since we need to keep two cubes separated while rotating.

In fact, $\mathcal{C}_n(2)$ is path-connected (more generally $\mathcal{C}_n(j)$ is known to be $(n-2)$-connected). The path-connectivity of $\mathcal{C}_n(2)$ implies the commutativity of $\pi_n(X)$ or the homotopy commutativity of $\Omega^n X$. This can be considered as the first application of the topology of little cubes to homotopy theory.

It is natural to extend this idea to study higher homotopy commutativity of iterated loop spaces. Boardman and Vogt began the first systematic study in this direction [5, 6]. A few years later, Peter May established a concrete relationship between iterated loop spaces and little cubes by proving the recognition principle [40]. The study of little cubes also led him to introduce the notion of operad, which turned out to be an important object naturally appearing in many fields of mathematics and mathematical physics. But we are not going to cover operad in this article. Those who are interested in operad are recommended to take a look at [36], for example.

The following is the organization of this article:

Section 2. Basic Definitions and Fundamental Facts: recalls basic definitions and classical results used in later sections.

Section 3. Stable Splitting of Iterated Loop Spaces: reviews the development of homotopy theory stimulated by Snaith's theorem of stable splitting of $\Omega^n \Sigma^n X$.

Section 4. Constructing Maps and Spaces: shows that various important maps and spaces have been constructed by using little cubes.

Section 5. Problems: is a collection of open problems related to little cubes.

2 Basic Definitions and Fundamental Facts

Let us begin with a precise definition of little cube.

Definition 2.1.1 *For a positive integer n, a little n-cube c is a map $I^n \longrightarrow I^n$ which can be decomposed into the following form:*

$$c = \ell_1 \times \cdots \times \ell_n$$

where each $\ell_i : I \longrightarrow I$ is an orientation preserving Affine embedding. The space of little n-cubes is denoted by $\mathcal{C}_n(1)$ and topologized with compact-open topology. Thus $\mathcal{C}_n(1)$ is a subspace of $\mathrm{Map}(I^n, I^n)$.

The configuration space of j little n-cubes $\mathcal{C}_n(j)$ is defined as follows:

$$\mathcal{C}_n(j) = \{(c_1, \cdots, c_j) \in \mathcal{C}_n(1)^j | c_i(\mathrm{Int} I^n) \cap c_k(\mathrm{Int} I^n) = \emptyset \text{ if } i \neq k\}$$

$\mathcal{C}_n(j)$ admits a natural action of the symmetric group of j-letters, Σ_j, by permuting the indices of little cubes.

$\mathcal{C}_n = \{\mathcal{C}_n(j)\}_j$ is called the little n-cube operad.

We also need the case $n = \infty$ to study infinite loop spaces. Define

$$\mathcal{C}_\infty(j) = \underset{\underset{n}{\longrightarrow}}{\mathrm{colim}}\, \mathcal{C}_n(j)$$

where colimit is taken over the inclusion maps induced by

$$I^n \times \{0\} \hookrightarrow I^{n+1}.$$

Obviously $\mathcal{C}_\infty(j)$ inherits the action of Σ_j.

Thanks to the following fact, we do not have to struggle with the compact-open topology on $\mathcal{C}_n(j)$.

Lemma 2.1.2 *Define a map*

$$\xi_n : \mathcal{C}_n(1) \longrightarrow I^{2n}$$

by $\xi_n(c) = (c(\frac{1}{4}, \cdots, \frac{1}{4}), c(\frac{3}{4}, \cdots, \frac{3}{4}))$, then ξ_n is an embedding of $\mathcal{C}_n(1)$ as an open subset of I^{2n}.

As is stated in Introduction, the following is one of the most fundamental properties of little cube.

Lemma 2.1.3 *$\mathcal{C}_n(j)$ is $(n-2)$-connected. Hence $\mathcal{C}_\infty(j)$ is contractible.*

The following map plays a central role in studying the relationship between the configuration space of little cubes $\mathcal{C}_n(j)$ and an n-fold loop space $\Omega^n X$.

Definition 2.1.4 *For a pointed space X, define*

$$\theta_{n,j} : \mathcal{C}_n(j) \times (\Omega^n X)^j \longrightarrow \Omega^n X$$

as follows: For $(c_1, \cdots, c_j) \in \mathcal{C}_n(j)$ and $\omega_1, \cdots, \omega_j \in \Omega^n X$,

$$\theta_{n,j}(c_1, \cdots, c_j; \omega_1, \cdots, \omega_j) : I^n \longrightarrow X$$

is defined to be ω_i on each cube $c_i(I^n)$ and maps the outside of little cubes to the basepoint. More precisely

$$\begin{aligned} &\theta_{n,j}(c_1, \cdots, c_j; \omega_1, \cdots, \omega_j)(t) = \\ &= \begin{cases} \omega_i(c_i^{-1}(t)) & \text{if } t \in c_i(I^n) \text{ for some } 0 \le i \le j, \\ * & \text{otherwise.} \end{cases} \end{aligned}$$

$\theta_{n,j}$ is obviously Σ_j-equivariant, hence we have an induced map on the quotient, which is denoted by the same notation:

$$\theta_{n,j} : \mathcal{C}_n(j) \times_{\Sigma_j} (\Omega^n X)^j \longrightarrow \Omega^n X$$

These maps $\{\theta_{n,j}\}$ satisfy the following compatibility conditions.

Lemma 2.1.5 *Let $*$ denote the constant loop to the basepoint of $\Omega^n X$, then, for $(c_1, \cdots, c_j) \in \mathcal{C}_n(j)$ and $\omega_1, \cdots, \omega_{i-1}, \omega_{i+1}, \cdots, \omega_j \in \Omega^n X$, we have*

$$\begin{aligned} & \theta_{n,j}(c_1, \cdots, c_j; \omega_1, \cdots, \omega_{i-1}, *, \omega_{i+1}, \cdots, \omega_j) \\ = \ & \theta_{n,j-1}(c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_j; \omega_1, \cdots, \omega_{i-1}, \omega_{i+1}, \cdots, \omega_j) \end{aligned}$$

Definition 2.1.6 *For a pointed space Y, an action of $\mathcal{C}_n$ is a collection of maps*

$$\theta_j : \mathcal{C}_n(j) \times_{\Sigma_j} Y^j \longrightarrow Y$$

for $j = 0, 1, \cdots$ satisfying the same relations as in Lemma 2.1.5.

In other words, an n-fold loop space has an action of $\mathcal{C}_n$. May proved that, conversely, the existence of an action of $\mathcal{C}_n$ on a pointed space Y can be used as a criterion for Y to be equivalent to an n-fold loop space.

Theorem 2.1.7 (Recognition Principle) *Let X be a path-connected space with a nondegenerate basepoint. Then X has a weak homotopy type of an n-fold loop space if and only if it has an action of little n-cube operad.*

The relations in Lemma 2.1.5 can be used to pierce the spaces $\mathcal{C}_n(j) \times_{\Sigma_j} (\Omega^n X)^j$ together to get a single space $C_n(\Omega^n X)$ as follows.

Definition 2.1.8 *Let Y be a pointed space with basepoint $*$. Generate an equivalence relation $\sim$ on $\coprod_j \mathcal{C}_n(j) \times_{\Sigma_j} Y^j$ by relations*

$$\begin{aligned}&(c_1, \cdots, c_j; y_1, \cdots, y_{i-1}, *, y_{i+1}, \cdots, y_j)\\ &\sim (c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_j; y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_j).\end{aligned}$$

Define $C_n(Y)$ by

$$C_n(Y) = \left(\coprod_j \mathcal{C}_n(j) \times_{\Sigma_j} Y^j \right) / \sim .$$

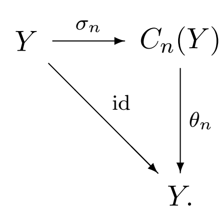
Corollary 2.1.9 $\{\theta_{n,j}\}$ *induces a well-defined continuous map*

$$\theta_n : C_n(\Omega^n X) \longrightarrow \Omega^n X.$$

Note that $C_n(Y)$ can be defined for any pointed space Y and we have a natural map

$$\sigma_n : Y \longrightarrow C_n(Y)$$

induced by the inclusion of $\mathcal{C}_n(1) \times Y$. The reader can easily check that $\theta_n \circ \sigma_n \simeq \mathrm{id}$ if $Y = \Omega^n X$. In other words, if Y is equivalent to an n-fold loop space $\Omega^n X$, Y is a retract of $C_n(Y)$.



Notice that the evaluation map on $\Omega^n X$

$$\mathrm{eval} : \Sigma^n(\Omega^n X) \longrightarrow X$$

and the Freudenthal suspension

$$E^n : X \longrightarrow \Omega^n \Sigma^n X$$

make the following diagram commutative

$$\begin{array}{ccc} \Omega^n X & \xrightarrow{E^n} & \Omega^n\Sigma^n(\Omega^n X) \\ & \searrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \Omega^n \mathrm{eval}} \\ & & \Omega^n X. \end{array}$$

This similarity between the functors C_n and $\Omega^n\Sigma^n$ can be explained by the following important theorem of May.

Theorem 2.1.10 (Approximation Theorem) *Let X denote a path-connected space with a nondegenerate basepoint. Then we have a natural weak homotopy equivalence*

$$C_n(X) \simeq \Omega^n\Sigma^n X.$$

Recognition Principle is important because it is the origin of the theory of operad. But for practical applications in homotopy theory, Approximation Theorem is far more useful, because it gives us a combinatorial model for iterated loop spaces and thus making them easier to handle.

Besides May's model, some other combinatorial models for $\Omega^n\Sigma^n X$ have been discovered. The oldest is the reduced product of James [27] which is a model for $\Omega\Sigma X$. Milgram [41] constructed a model for $\Omega^n\Sigma^n X$ generalizing James' construction. Segal [43] suggested to use the configuration space of distinct points instead of little cubes. The resulting space is equivalent to the little cube model. May's approach was improved by J. Caruso and S. Waner [11] to prove an approximation theorem for nonconnected spaces. There are also simplicial models found by Milnor [42] for $n = 1$, by Barratt and Eccles [2] for $n = \infty$ and by Jeff Smith [44] for general n.

These are essentially equivalent to each other. Among these models, however, the "little cube model" (and its variants, like the one using configuration space of distinct points) has been most popular. This is partly because Victor Snaith used the little cube model to prove the stable splitting theorem, which is the subject of the next section.

3 Stable Splitting of Iterated Loop Spaces

3.1 Snaith's Theorem

A stable splitting (in fact, a splitting after a single suspension) of $\Omega\Sigma X$ was proved by James in 1955 [27] using a combinatorial construction of $\Omega\Sigma X$, called the James construction or reduced product. Important tools in classical homotopy theory, like the (James)-Hopf invariant

$$H : \Omega S^{n+1} \longrightarrow \Omega S^{2n+1}$$

and EHP fibration

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$$

have been constructed by using this splitting.

The little cube model of $\Omega^n\Sigma^n X$ can be regarded as a generalization of the James construction. In fact, Snaith [45] proved a stable splitting theorem for $\Omega^n\Sigma^n X$ by using the little cube model.

Theorem 3.1.1 *For a pointed space X with a nondegenerate basepoint and a positive integer n, we have a weak stable homotopy equivalence*

$$\Omega^n\Sigma^n X \simeq_S \bigvee_{j=1}^{\infty} \mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}. \tag{1}$$

This holds also for $n = \infty$.

Note that $C_n(X)$ is naturally filtered by the number of cubes

$$F_m C_n(X) = \left(\coprod_{j \leq m} \mathcal{C}_n(j) \times_{\Sigma_j} X^j \right) / \sim$$

and the right hand side of the above splitting is the "subquotient" of $C_n(X)$ with respect to this filtration.

Later, Cohen, May and Taylor generalized Snaith's theorem to more general settings [15]. For those who are curious about how to construct these stable splittings, the appendix of [12] will be of interest.

It is also worthwhile to note that these splittings can be obtained by a very simple argument in the category of spectra. This fact, proved by R.L. Cohen [24], depends on the existence of the Σ_j-equivariant half smash product of spectra constructed by Lewis, May and Steinberger [37].

3.2 Applications of Stable Splitting

After the stable splitting theorem was proved by Snaith, the case $n = 2$ and $X = S^{2k-1}$ became a subject of intense study in 70's. Mahowald [38] found a new infinite family in the 2-primary components of the stable homotopy groups of spheres by using the summands of the stable splitting of $\Omega^2 S^9$. An odd primary analogue was obtained by R.L. Cohen [23].

These facts suggest that the cohomology of the summands of the Snaith splitting of $\Omega^2 S^{2k+1}$ is equipped with an important module structure over the Steenrod algebra. To be more precise, we need the following definition.

Definition 3.2.1 *For any prime p and integer $k \geq 0$, define a module $M(k,p)$ over the mod p Steenrod algebra as follows:*

$$M(k,p) = \begin{cases} \mathcal{A}_2/(\chi(Sq^i)|i > k) & \text{if } p = 2, \\ \mathcal{A}_p/(\chi(\beta^\epsilon P^i)|i > k, \epsilon = 0,1) & \text{if } p \text{ is odd,} \end{cases}$$

where χ denotes the canonical anti-automorphism on $\mathcal{A}_p$.

Theorem 3.2.2 *For any prime p and integer $k \geq 0$, there exists a p-local spectrum $B(k,p)$ satisfying the following properties:*

1. *$H^*(B(k,p); \mathbf{Z}/p\mathbf{Z}) \cong M(k,p)$ as modules over the Steenrod algebra.*
2. *Let $j_k : B(k,p) \longrightarrow H\mathbf{Z}/p\mathbf{Z}$ be a generator of $H^0(B(k,p); \mathbf{Z}/p\mathbf{Z})$. Then for any CW complex X, the induced map of generalized homology theories*

$$(j_k)_* : B(k,p)_*(X) \longrightarrow H_*(X; \mathbf{Z}/p\mathbf{Z})$$

   is surjective for $q \leq 2k+1$ if $p = 2$ and for $q \leq 2p(k+1) - 1$ if p is odd.

Furthermore such a spectrum is unique up to homotopy.

Definition 3.2.3 *$B(k,p)$ is called the Brown-Gitler spectrum [7].*

Mahowald in [38] conjectured that the stable summands of $\Omega^2 S^{2k+1}$ localized at 2 realize the Brown-Gitler spectra. This fact was proved by Brown and Peterson [8].

Theorem 3.2.4 *Localized at* 2*, we have the following homotopy equivalence*

$$\mathcal{C}_2(j)_+ \wedge_{\Sigma_j} (S^{2k-1})^{\wedge j} \simeq \Sigma^{j(2k-1)} B([\tfrac{j}{2}]).$$

Odd primary analogs are proved by R. Cohen [23].

Theorem 3.2.5 *Localized at an odd prime p, we have the following homotopy equivalence when $j = mp + r$ for $p > r > 0$:*

$$\mathcal{C}_2(pj)_+ \wedge_{\Sigma_{pj}} (S^{2k-1})^{\wedge pj} \simeq \Sigma^{j(2pk-2)} B(m).$$

There is another way of describing the summands of the stable splitting of $\Omega^n S^{n+k}$ closely related to the above theorems.

Consider the vector bundle

$$p_{n,j} : \mathcal{C}_n(j) \times_{\Sigma_j} \mathbf{R}^j \longrightarrow \mathcal{C}_n(j)/\Sigma_j.$$

It is easy to see that the Thom complex of $p_{n,j}$, $T(p_{n,j})$, is $\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} S^j$ which is a stable summand of $\Omega^n S^{n+1}$. More generally, we have

$$T(\underbrace{p_{n,j} \oplus \cdots \oplus p_{n,j}}_{k}) = \mathcal{C}_n(j)_+ \wedge_{\Sigma_j} S^{jk}.$$

It is worthwhile to note that the infinite families in the stable homotopy groups found by Mahowald and R. Cohen comes from the triviality of $p_{2,j} \oplus p_{2,j}$ proved by F. Cohen, Mahowald and Milgram [14]. The order of $p_{n,j}$ in general was studied in [13].

Mahowald's construction of infinite families in the stable homotopy groups of spheres is one of the most important applications of $\Omega^2 S^{2k+1}$ in stable homotopy theory. Note that the classical applications of the James splitting of ΩS^{2k+1} live in the unstable world: For example, the mod p Hopf invariant

$$H_p : \Omega S^{2n+1} \longrightarrow \Omega S^{2np+1}$$

is defined to be the adjoint to the following composition

$$\Sigma\Omega S^{2n+1} \simeq \Sigma \left(\bigvee_j (S^{2n})^{\wedge j} \right) \longrightarrow \Sigma (S^{2n})^{\wedge p} = S^{2np+1}.$$

In order to get an unstable application of the splitting of $\Omega^2 S^{2k+1}$, for example a "secondary Hopf invariant", we need to desuspend the

Snaith splitting. However, as stated in [12], $\Omega^2\Sigma^2 X$ does not split in finitely many suspensions. This is one of the crucial differences between $\Omega\Sigma X$ and $\Omega^n\Sigma^n X$ for $n \geq 2$.

This is the difficulty in applying $\Omega^n\Sigma^n X$ in unstable homotopy theory when $n \geq 2$. But F. Cohen, May and Taylor [15] proved that each piece can be split off in finitely many suspensions.

Theorem 3.2.6 *Fix a positive integer k, then there exists a positive integer L (depending on k) and a map*

$$\Sigma^L C_n(X) \longrightarrow \Sigma^L \mathcal{C}_n(k)_+ \wedge_{\Sigma_k} X^{\wedge k}$$

which is equivalent to the composition of the Snaith splitting and the projection on the k-th component after suspending infinitely many times.

Furthermore F. Cohen proved that L can be taken to be $2k$ when $n = 2$ [12]. This allowed him to construct a map

$$\sigma_n : W(n) \longrightarrow \Omega^{2p} W(n+1)$$

which can be considered to be the "secondary double-suspension", where $W(n)$ is the homotopy theoretic fiber of the double suspension map

$$E^2 : S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}.$$

σ_n played an essential role in the work of Mahowald and Thompson [39, 48] where they determined the unstable v_1-periodic homotopy groups of spheres.

B. Gray began a systematic study in this direction in [28]. He introduced the notion of EHP spectra which is an unstable way of studying Toda-Smith spectra $V(n)$ in the fashion of Cohen-Moore-Neisendorfer [18, 19, 20]. In order his program to be accomplished, we need to find various unstable maps between loop spaces. As in the case of the classical EHP sequence, those maps could be constructed by using unstable splitting of loop spaces. In fact, Gray is very close to proving the existence of an EHP spectrum for $V(0)$. His construction would be completed, if we could prove $\Omega^2\Sigma^2 X$ localized at an odd prime splits after suspending twice, which is conjectured in [28]. However nothing is known about localization of $\Omega^n\Sigma^n X$.

4 Constructing Maps and Spaces

As we have seen in the previous section, little cube model played an important role in the construction and applications of the stable splittings of iterated loop spaces.

It is often helpful to have extra geometric information even though what we need is abstract homotopy-theoretic results.

As is suggested by Quillen's work on closed model category, homotopy theory (stable or unstable) is fairly formal. It can be axiomatized nicely. But we need concrete models for practical applications.

Little cubes can be used to construct important spaces and maps. The followings are some of the examples.

4.1 Fibrations

May proved the Approximation theorem (Theorem 3.1.1) by constructing a little cube model for the path-loop fibration:

$$\Omega^n\Sigma^n X \longrightarrow P\Omega^{n-1}\Sigma^n X \longrightarrow \Omega^{n-1}\Sigma^n X.$$

Definition 4.1.1 *For a pair of pointed space (X, A), define $E_n(X, A)$ to be the subspace of $C_n(X)$ consisting of elements $(c_1, \cdots, c_j; x_1, \cdots, x_j)$ satisfying the following properties: if $x_{i_1}, \cdots, x_{i_\ell} \notin A$ then*

$$(pr_{n-1}(c_{i_1}), \cdots, pr_{n-1}(c_{i_\ell})) \in \mathcal{C}_{n-1}(\ell),$$

where pr_{n-1} is the projection onto the last $(n-1)$-coordinates.

Theorem 4.1.2 (May) *1. If X is a path-connected pointed space with nondegenerate basepoint, we have a quasifibration*

$$C_n(X) \longrightarrow E_n(CX, X) \longrightarrow C_{n-1}(\Sigma X) \tag{2}$$

and natural maps

$$\begin{aligned} \alpha &: C_n(X) \longrightarrow \Omega^n\Sigma^n X \\ \tilde{\alpha} &: E_n(CX, X) \longrightarrow P\Omega^{n-1}\Sigma^n X \end{aligned}$$

making the following diagram commutative up to homotopy

$$\begin{array}{ccccc} C_n(X) & \longrightarrow & E_n(CX, X) & \longrightarrow & C_{n-1}(\Sigma X) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \alpha} \\ \Omega^n\Sigma^n X & \longrightarrow & P\Omega^{n-1}\Sigma^n X & \longrightarrow & \Omega^{n-1}\Sigma^n X. \end{array}$$

2. *$E_n(CX, X)$ is contractible. Therefore α is a weak homotopy equivalence.*

Since path-loop fibration is a principal fibration with contractible total space, principal fibrations with fiber $\Omega^n\Sigma^n X$ are pull-back of the path-loop fibration. Thus it is natural to expect to obtain a little cube model of such a fibration by pulling back May's quasifibration (2). Unfortunately, however, taking a pull-back in general does not preserve quasifibration.

The quasifibration (2) is not just a quasifibration. The base space is filtered and the quasifibration satisfies a kind of local-triviality on each successive difference of the filtration.

Thanks to this additional structure, a pull-back of (2) has a chance to be a quasifibration. In fact, S.-C. Wong proved the following [49]:

Definition 4.1.3 *For a pointed space X, define $W(k, n, \Sigma X)$ to be the homotopy fiber of the Freudenthal suspension map:*

$$\Omega^{k-1}E^n : \Omega^{k-1}\Sigma^k X \longrightarrow \Omega^{n+k-1}\Sigma^{n+k}X.$$

Namely $W(k, n, \Sigma X)$ is defined by the following pull-back diagram:

$$\begin{array}{ccc} W(k,n,\Sigma X) & \longrightarrow & P\Omega^{n+k-1}\Sigma^{n+k}X \\ \downarrow & & \downarrow \\ \Omega^{k-1}\Sigma^k X & \xrightarrow{\Omega^{k-1}E^n} & \Omega^{n+k-1}\Sigma^{n+k}X. \end{array}$$

Definition 4.1.4 *For a pair of pointed spaces (X, A), define $\xi(k, n; X, A)$ by the following pull-back diagram:*

$$\begin{array}{ccc} \xi(k,n;X,A) & \longrightarrow & E_{n+k-1}(X,A) \\ {\scriptstyle \pi(k,n)}\downarrow & & \downarrow{\scriptstyle \pi_{n+k}} \\ C_{k-1}(X/A) & \xrightarrow{\sigma^n} & C_{n+k-1}(X/A). \end{array}$$

Theorem 4.1.5 (Wong) *For $k \geq 1, n \geq 0$ and a strong NDR pair (X, A),*

$$\pi(k, n) : \xi(k, n; X, A) \longrightarrow C_{k-1}(X/A)$$

is a quasifibration with fiber $C_{n+k}(A)$. Thus we have a weak homotopy equivalence

$$\xi(k, n; X, A) \simeq W(k, n; X/A).$$

Wong used $\xi(k,n;X,A)$ to prove a stable splitting of $W(k,n;X/A)$. Since the secondary-suspension σ_n was constructed by splitting off the p-adic piece of $\Omega^2 S^{2n+1}$ after the $2p$-fold suspension, it would be interesting if we could desuspend Wong's stable splitting to construct "tertiary suspension map".

Wong's idea can be also applied to the following case.

Since $C_n(X)$ has a structure of a monoid by concatenation and the equivalence

$$C_n(X) \simeq \Omega^n \Sigma^n X$$

is an equivalence of Hopf spaces, the p-th power map

$$p\times : \Omega^n \Sigma^n X \longrightarrow \Omega^n \Sigma^n X$$

is equivalent to the p-th power map on $C_n(X)$. Thus we have a diagram

$$\begin{array}{ccc} ? & \longrightarrow & E_n(CX,X) \\ \downarrow & & \downarrow{\scriptstyle \pi_n} \\ C_n(\Sigma X) & \xrightarrow{p\times} & C_n(\Sigma X). \end{array}$$

in which all maps are defined in terms of little cubes.

The homotopy fiber of $p\times : \Omega^n\Sigma^{n+1}X \longrightarrow \Omega^n\Sigma^{n+1}X$ is the mapping space from the mod p Moore space $\mathrm{Map}_*(P^{n+1}(p), \Sigma^{n+1}X)$. Following Wong's idea K. Iwama recently proved the following [29].

Theorem 4.1.6 *Let X be a pointed space with $(X,*)$ a strong NDR pair. Define $E_n(p)(CX,X)$ by the following pull-back diagram*

$$\begin{array}{ccc} E_n(p)(CX,X) & \longrightarrow & E_n(CX,X) \\ {\scriptstyle \pi_n(p)}\downarrow & & \downarrow{\scriptstyle \pi_n} \\ C_n(\Sigma X) & \xrightarrow{p\times} & C_n(\Sigma X). \end{array}$$

Then $\pi_n(p)$ is a quasifibration with fiber $C_{n+1}(X)$. Therefore we have a weak homotopy equivalence

$$E_n(p)(CX,X) \simeq \mathrm{Map}_*(P^{n+1}(p), \Sigma^{n+1}X),$$

where $P^{n+1}(p) = S^n \cup_p e^{n+1}$.

It is not known if $E_n(p)(CX, X)$ has a stable splitting. Although the proof of the above theorem is parallel to that of Wong's, the proof of stable splitting by Wong cannot be applied in this case. The difficulty comes from the fact that the p-th power map on $C_n(\Sigma X)$ does not preserve the filtration.

4.2 Kahn-Priddy Transfer

As the Snaith splitting suggests, little cubes are useful for handling stable maps in concrete ways. Another example of applications of little cube in stable homotopy theory is the construction of transfer by D.S. Kahn and S.B. Priddy.

Transfer is a map going to the wrong direction in the homology or cohomology, classically in group cohomology. It is a well-known fact that they can be realized as stable maps between spaces. One of the examples is constructed by Kahn and Priddy by using the little cube model of $\Omega^\infty\Sigma^\infty X$ [30, 31].

Suppose we are given an N-fold covering space

$$p : E \longrightarrow B.$$

The symmetric group of N-letters acts on each fiber by deck transformation and p can be considered to be a fiber bundle with fiber $\pi_1(B)/\pi_1(E)$ and structure group Σ_N. Since $\mathcal{C}_\infty(N)$ is a contractible space with a free Σ_N-action, the projection

$$\pi : \mathcal{C}_\infty(N) \longrightarrow \mathcal{C}_\infty(N)/\Sigma_N$$

is a universal Σ_N-bundle. Thus we have the following pull-back diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & \mathcal{C}_\infty(N) \times_{\Sigma_N} (\pi_1(B)/\pi_1(E))^N \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\bar{\varphi}} & \mathcal{C}_\infty(N)/\Sigma_N. \end{array}$$

Define

$$\Phi : B \longrightarrow \mathcal{C}_\infty(N) \times_{\Sigma_N} E^N$$

by

$$\Phi(p(x)) = (\varphi(x), x\tau_1, \cdots, x\tau_N)$$

where $\{\tau_1, \cdots, \tau_N\}$ is a choice of coset representatives of $\pi_1(B)/\pi_1(E)$.

Composed with

$$\mathcal{C}_\infty(N) \times_{\Sigma_N} E^N \longrightarrow C_\infty(E)$$

we have

$$B_+ \xrightarrow{\Phi} C_\infty(E_+) \simeq \Omega^\infty \Sigma^\infty(E_+).$$

The Kahn-Priddy transfer is the stable map adjoint to this map:

$$p^! : \Sigma^\infty B_+ \longrightarrow \Sigma^\infty E_+.$$

Let p be a prime number. Kahn and Priddy applied this construction to the covering

$$E\Sigma_p \longrightarrow B\Sigma_p$$

to get the famous Kahn-Priddy theorem which states

$$\pi_*^S(B\Sigma_p) \longrightarrow \pi_*^S(S^0)$$

is surjective on the p-primary components.

4.3 Strong Convergent Cobar Spectral Sequence

Little cubes can be used to construct a spectral sequence converging to the homology of iterated loop spaces:

$$E^2 \cong \mathrm{Cotor}^{h_*(\Omega^{n-1}\Sigma^n X)}(h_*, h_*) \Longrightarrow h_*(\Omega^n \Sigma^n X). \tag{3}$$

The following is a quick review of the construction by the author [46]. Throughout this subsection $h_*(-)$ denotes a multiplicative homology theory.

In order to construct a spectral sequence (3), the most natural idea would be to try to find a filtration on $\Omega^n\Sigma^n X$ with which the E^1-term of the resulting spectral sequence becomes the algebraic cobar construction on the coalgebra $h_*(\Omega^{n-1}\Sigma^n X)$. Thanks to the stable splitting (1), it enough to define a filtration on each $\mathcal{C}_n(j)$, separately.

Suppose we have a filtration $\{F_{-q}\mathcal{C}_n(j)\}$ on each $\mathcal{C}_n(j)$. Define

$$F_{-s}C_n(X) = \bigvee_j F_{-s}\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}.$$

$\{F_{-s}C_n(X)\}$ is a stable filtration for $\Omega^n\Sigma^n X$. The E^1-term of the spectral sequence defined by this filtration is

$$\begin{aligned} E^1_{-s,t} &= h_{-s+t}(F_{-s}C_n(X), F_{-s-1}C_n(X)) \\ &= \bigoplus_j h_{-s+t}(F_{-s}\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}, F_{-s-1}\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}) \\ &= \bigoplus_j \tilde{h}_{-s+t}(F_{-s}\mathcal{C}_n(j)/F_{-s-1}\mathcal{C}_n(j) \wedge_{\Sigma_j} X^{\wedge j}). \end{aligned}$$

On the other hand, the tensor algebra on $\tilde{h}_*(\Omega^{n-1}\Sigma^n X)$ has summands of the following form:

$$\tilde{h}_*((\mathcal{C}_{n-1}(j_1) \times \cdots \times \mathcal{C}_{n-1}(j_s))_+ \wedge_{\Sigma_{j_1}\times\cdots\times\Sigma_{j_s}} (\Sigma X)^{\wedge(j_1+\cdots+j_s)}).$$

Thus what we need is filtrations on $\{\mathcal{C}_n(j)\}$ so that $F_{-s}\mathcal{C}_n(j) - F_{-s-1}\mathcal{C}_n(j)$ becomes s vertically aligned stacks of cubes. In order to define such filtrations, we need auxiliary functions on $\mathcal{C}_n(j)$.

Let

$$pr_1 : \mathcal{C}_n(1) \longrightarrow \mathcal{C}_1(1)$$

be the map induced by the projection onto the first coordinate. It is not difficult to find a function

$$d : \mathcal{C}_n(1) \times \mathcal{C}_n(1) \longrightarrow [0,1]$$

with the following properties.

$$\begin{aligned} d(c,c') = 0 &\iff pr_1(c)(\tfrac{1}{2}) \notin pr_1(c')([0,1]) \text{ or } pr_1(c')(\tfrac{1}{2}) \notin pr_1(c)([0,1]) \\ d(c,c') = 1 &\iff pr_1(c)(\tfrac{1}{2}) = pr_1(c')(\tfrac{1}{2}). \end{aligned}$$

For $\mathbf{c} = (c_1, \cdots, c_j) \in \mathcal{C}_n(j)$, a stack in $\mathbf{c}$ is a subset of $\{c_1, \cdots, c_j\}$. We say that a stack $\{c_i | i \in S\}$ is stable under gravity if and only if $d(c_{i_1}, c_{i_2}) \neq 0$ for any $i_1, i_2 \in S$.

With these terminologies we define a filtration on little cubes as follows.

Definition 4.3.1 *Let $F_0\mathcal{C}_n(j) = F_{-1}\mathcal{C}_n(j) = \mathcal{C}_n(j)$. For $q > 0$, $\mathbf{c} = (c_1, \cdots, c_j) \in F_{-s-1}\mathcal{C}_n(j)$ if and only if, for any partition $\{1, \cdots, j\} = S_1 \coprod \cdots \coprod S_q$ with $S_1 \neq \emptyset, \cdots, S_q \neq \emptyset$, at least one of the stacks corresponding to $S_1, \cdots, S_q$ is not stable under gravity.*

This filtration is called the gravity filtration. The spectral sequence induced by this filtration is called the gravity spectral sequence.

On the contrary to our intension, it is not known whether the E^1-term of the spectral sequence defined by this filtration is isomorphic to the cobar construction. However the following fact proved in [46] is enough to identify the E^2-term with Cotor.

Theorem 4.3.2 *There exists a stable filtration on $E_n(CX, X)$ with the following isomorphism of differential graded h_*-modules, if $h_*(\Omega^{n-1}\Sigma^n X)$ is flat over h_*:*

$$E^1(E_n) \cong E^1(C_n) \otimes_{h_*} h_*(C_{n-1}(\Sigma X)),$$

where $\{E^r(E_n)\}$ is the spectral sequence induced by the filtration on $E_n(CX, X)$ and $\{E^r(C_n)\}$ is the spectral sequence induced by the gravity filtration on $C_n(X)$.

Furthermore $(E^1(E_n), d^1)$ is acyclic.

Corollary 4.3.3 *If $h_*(\Omega^{n-1}\Sigma^n X)$ is flat over h_*, we have the following isomorphism*

$$E^2(C_n) \cong \mathrm{Cotor}^{h_*(\Omega^{n-1}\Sigma^n X)}(h_*, h_*).$$

For a pointed space X, if $h_*(X)$ is a flat h_*-module, we have a spectral sequence, so-called the classical Eilenberg-Moore spectral sequence with

$$E^2 \cong \mathrm{Cotor}^{h_*(X)}(h_*, h_*).$$

The E^∞-term of this spectral sequence is not in general directly related to $h_*(\Omega X)$. However, the gravity spectral sequence does calculate $h_*(\Omega^n\Sigma^n X)$, since it is a direct sum of the spectral sequences defined by finite filtrations.

Theorem 4.3.4 *The gravity spectral sequence converges to $h_*(\Omega^n\Sigma^n X)$.*

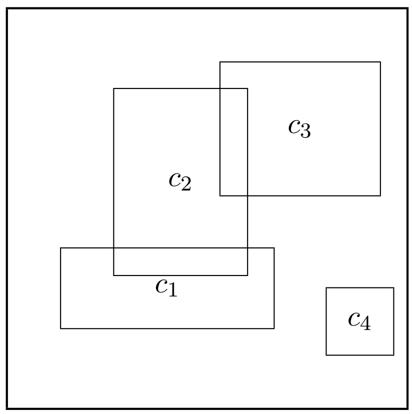
Although the gravity spectral sequence has this important property, it also has a disadvantage: the E^1-term is mysterious. In order to use the gravity spectral sequence for practical computations, it is important to find good generators in the E^2-term. In the case of the classical Eilenberg-Moore spectral sequence, we have the Dyer-Lashof operations which are defined in terms of the cobar construction. The following fact helps us to find generators in the E^2-term of the gravity spectral sequence.

Theorem 4.3.5 ([47]) *If $h_*(\Omega^{n-1}\Sigma^n X)$ is flat over h_*, the gravity spectral sequence is isomorphic to the classical Eilenberg-Moore spectral sequence from the E^2-term on.*

4.4 Little Cubes with Overlappings Allowed

We have been considering little cubes disjoint from each other, so far. If we remove this disjointness condition, we obtain loop spaces of different types.

Definition 4.4.1 *Define $\mathcal{D}_n(j) = \mathcal{C}_n(1)^j$. This is the space of j little n-cubes with overlappings allowed.*



The symmetric group of j-letters acts on $\mathcal{D}_n(j)$ by renumbering of cubes and the definition of $C_n(X)$ can be applied without modification to get a functor $D_n(X)$.

Since the little cube operad $\mathcal{C}_n$ acts on $D_n(X)$ in a natural way, $D_n(X)$ is an n-fold loop space by the recognition principle of May. In fact, we can easily see that this is equivalent to a classical construction due to Dold and Thom, i.e. infinite symmetric product, and thus it has a structure of an infinite loop space.

Definition 4.4.2 *For a pointed space X and a nonnegative integer i, define*

$$\mathrm{SP}^i(X) = \left(\coprod_{j=0}^{i} X^j/\Sigma_j \right) / \sim$$

where the equivalence relation "$\sim$" is generated by the relation removing basepoint.

$\mathrm{SP}^i(X)$ *is called the i-th symmetric product of X. When $i = \infty$, it is called the infinite symmetric product.*

Since there is no restriction on the movements of cubes, $\mathcal{D}_n(j)$ is Σ_j-equivariantly contractible. Thus we have

Proposition 4.4.3 *For any pointed space X, $D_n(X)$ is homotopy equivalent to* $\mathrm{SP}^\infty(X)$*.*

The following is a classical theorem of Dold and Thom [25],

Theorem 4.4.4 *For a path-connected pointed CW-complex X, we have the following natural isomorphism*

$$\pi_n(\mathrm{SP}^\infty(X)) \cong \widetilde{H}_n(X)$$

As an immediate corollary, we have

Theorem 4.4.5 *For a pointed connected CW-complex X of finite type, we have the following homotopy equivalence*

$$\mathrm{SP}^\infty(X) \simeq \prod_{n=0}^{\infty} K(\pi_n(X), n).$$

$C_n(X)$ and $D_n(X)$ are two extreme cases: cubes are disjoint from each other in $\mathcal{C}_n(j)$, while any kinds of overlapping are allowed in $\mathcal{D}_n(j)$. $\pi_*(C_\infty(X))$ is the stable homotopy groups of X, while $\pi_*(D_\infty(X))$ is the (reduced) integral homology groups of X.

Let us consider the following intermediate objects.

Definition 4.4.6 *Let $\mathcal{D}_n^i(j)$ be the subspace of $\mathcal{D}_n(j)$, consisting of j little cube $(c_1, \cdots, c_j)$ satisfying the following property: each $t \in I^n$ is contained in the interior of the image of at most i cubes.*

$\mathcal{D}_n^i(j)$ inherits the action of Σ_j. For a pointed space X, define $D_n^i(X)$ in the same way as $C_n(X)$ is defined.

Now we have successive inclusions

$$C_n(X) = D_n^0(X) \subset D_n^1(X) \subset \cdots \subset D_n^\infty(X) = D_n(X).$$

The commutativity of the following diagram

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\subset} & D_n(X) \\ & \nwarrow \quad \nearrow & \\ & X & \end{array}$$

and the fact that the induced map on homotopy groups

$$\pi_*(X) \longrightarrow \pi_*(D_n(X)) \cong \pi_*(\mathrm{SP}^\infty(X)) \cong \widetilde{H}_*(X)$$

coincides with the Hurewicz homomorphism implies that the Hurewicz homomorphism factors through $\pi_*(D_n^i(X))$

$$\begin{array}{ccc} \pi_*(D_n^i(X)) & \xrightarrow{\ \subset\ } & \widetilde{H}_*(X) \\ & \nwarrow \quad \nearrow & \\ & \pi_*(X). & \end{array}$$

F. Kato determined the homotopy type of $D_n^i(X)$ [33].

Theorem 4.4.7 *For a path-connected pointed space X with $(X, *)$ a strong NDR pair, we have the following weak homotopy equivalence*

$$D_n^i(X) \simeq \Omega^n \mathrm{SP}^i \Sigma^n X.$$

The idea of the proof is to extend May's construction of the quasifibration (2).

Since

$$\mathcal{D}_0^i(j) = \begin{cases} * & \text{if } j \leq i \\ \emptyset & \text{if } j > i \end{cases}$$

we have

$$D_0^i(X) = \mathrm{SP}^i(X).$$

Once we have a quasifibration

$$D_n^i(X) \longrightarrow E \longrightarrow D_{n-1}^i(\Sigma X)$$

with E contractible, we have the desired weak homotopy equivalence

$$D_n^i(X) \simeq \Omega D_{n-1}^i(\Sigma X) \simeq \cdots \simeq \Omega^n D_0^i(\Sigma^n X) = \Omega^n \mathrm{SP}^i \Sigma^n X.$$

Definition 4.4.8 *For a pair of pointed spaces (X, A), define $E_n^i(X, A)$ to be the subspace of $C_n^i(X)$ consisting of elements $(c_1, \cdots, c_j; x_1, \cdots, x_j)$ satisfying the following properties: if $x_{i_1}, \cdots, x_{i_\ell} \notin A$ then*

$$(pr_{n-1}(c_{i_1}), \cdots, pr_{n-1}(c_{i_\ell})) \in \mathcal{C}_{n-1}^i(\ell).$$

Proposition 4.4.9 *For a pointed space X with $(X, *)$ a strong NDR pair, the projection*

$$E_n^i(CX, X) \longrightarrow D_{n-1}^i(\Sigma X)$$

is a quasifibration with fiber $D_n^i(X)$.

Furthermore $E_n^i(CX, X)$ is contractible if X is path-connected.

Sadok Kallel [32] independently studied an analogous construction $C^i(\mathbf{R}^n; X)$ by using (labelled) configuration space of points in $\mathbf{R}^n$, instead of little cubes, and proved a homotopy equivalence

$$C^i(\mathbf{R}^n; X) \simeq \Omega^n \mathrm{SP}^i \Sigma^n X$$

for connected CW complexes together with a “delooped version” of this homotopy equivalence.

We should point out that most of the constructions using little n-cubes can be also done by using configuration space of points in $\mathbf{R}^n$. One of the advantages of using configuration space of points rather than little cubes is that $\mathbf{R}^n$ can be replaced with any space M. Kallel also studied $C^d(M; X)$ where M is a stably parallelizable smooth manifold with nonempty boundary.

5 Problems

We conclude this article with open problems related to little cubes.

1. Find a little cube model of the fiber of the secondary suspension

$$\sigma_n : W(n) \longrightarrow \Omega^{2p} W(n+1).$$

2. Find a stable splitting of $\mathrm{Map}_*(P^n(p), \Sigma^n X)$.

3. Prove that, localized away from 2, $\Omega^2 \Sigma^2 X$ splits into a wedge of the (localized) Snaith summands after suspending twice.

4. For each prime p, find a combinatorial model for $\Omega^n \Sigma^n X$ localized at or away from p.

5. Compute the homology of $\Omega^n \mathrm{SP}^i \Sigma^n X$.

Acknowledgment

The author is very grateful to Jesus Gonzalez, CINVESTAV, for inviting him to write this article for Morfismos, which gave him a good chance to review the development of the theory of little cubes. He would also like to thank the referees for informing him of Kallel's work and for pointing out ambiguous assumptions in some theorems.

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