

# Bayesian procedures for pricing contingent claims: Prior information on volatility \*

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## Abstract

This paper develops a Bayesian model for pricing derivative securities with prior information on volatility. Prior information is given in terms of expected values of levels and rates of precision: the inverse of volatility. We provide several approximate formulas, for valuing European call options, on the basis of asymptotic and polynomial approximations of Bessel functions.

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## 1 Introduction

When studying market volatility, the standard procedure is to analyze data. For example, we may explore plots and frequency histograms, or even examine how observations were collected. However, there is another approach to study market volatility before data is observed, which is based on previous practical experience and understanding, the Bayesian approach. In such a case, the parameters of a sampling model are regarded as random variables, and all judgements are made in terms of the degree of belief on potential values of the parameters. In this framework, a prior distribution is used to describe initial knowledge of the possible values of the parameters of a sampling model. For example, we may feel, based on earlier experience, that our degree of belief

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about the values of a given parameter may be expressed by a specific probability distribution, which describes initial knowledge.

In pricing contingent claims it is of particular interest to draw inferences about unknown volatility or uncertain volatility parameters of the underlying asset on the basis of prior information. Considering initial information before data is observed is not just a sophisticated extension but an essential issue to be taken into account for the theory and practice of derivatives. In this paper, we present a new Bayesian method to price derivative securities when there is prior information on uncertain and changing volatility. In our proposal, investors are rational in the sense that they use efficiently prior information by choosing a prior distribution that maximizes logarithmic utility among all admissible distributions describing available information. After all, the core of finance theory (mathematical or empirical) is the study of the rational behavior of investors in an uncertain environment. A study for the behavior of rational agents in the Mexican case can be seen in Venegas-Martínez [20].

This paper is organized as follows. In the next section, we mention some of the limitations of the stochastic volatility approach, and discuss the need of considering prior information in pricing derivatives. In section 3, we review the Bayesian inference framework and its relationship with information theory. In section 4, we develop a Bayesian model to price derivative securities and exploit its relationship with Bessel functions. In sections 5 and 6, we examine some asymptotic and polynomial approximations of the basic Bayesian valuation problem. Through section 7, we carry out a comparison of our approach with other models available in the literature. Finally, In section 8, we draw conclusions, acknowledge limitations, and make suggestions for further research.

## **2 Limitations of the stochastic volatility approach**

The most common set-up of the stochastic volatility model consists in a geometric Brownian motion correlated with a mean-reverting Orstein-Uhlenbeck process. This approach for pricing derivatives has been widely studied with a remarkable theoretical progress; see, for instance, Ball and Roma [3], Heston [10], Renault and Touzi [18], Stein and Stein

[19], Wiggins [22], and Avellaneda *et al.* [2]. In particular, the stochastic volatility models allow us to reproduce in a more realistic way asset returns, specially in the presence of fat tails (Wilmott [23]), asymmetry in the distribution (Fouque, Papanicolaou, and Sircar [7]), and the smile effect (Hull and White [11]). However, there is a set of empirical regularities (or stylized facts) that still need to be explained. In particular, the existing models do not explain how investors, ranging from non corporate individual to large trading institution, choose the best patterns of investment (rational behavior) if there is prior information on volatility, and its implications when valuing derivatives.

### 3 The Bayesian approach to price derivative securities

In the real world, volatility is neither constant nor directly observed. Hence, it is natural to think of volatility as a non-negative random variable with some initial knowledge coming from practical experience and understanding before data is observed. This is just the Bayesian way of thinking about prior information. Under this approach, prior information is described in terms of a probability distribution (subjective beliefs) of the potential values of volatility. It is common in Bayesian inference, instead of studying volatility,  $\sigma > 0$ , to study precision, which is defined as the inverse of the variance,  $h = \sigma^{-2}$ ; see, for instance, Leonard and Hsu [12], and Berger [4]. Thus, the lower the variance, the higher the precision. More precisely, from the Bayesian point of view, we have a distribution,  $P_h$ ,  $h > 0$ , describing prior information. We shall assume that  $P_h$  is absolutely continuous with respect to the Lebesgue measure  $\nu$ , so that the Radon-Nykodim derivative provides a prior density,  $\pi(h)$ , *i.e.*,  $dP_h / \nu(h) = \pi(h)$  for all  $h > 0$ . Then, we may write

$$P_h\{h \in A\} = \int_A \pi(h) d\nu(h)$$

for all Borel sets  $A$ .

#### 3.1 Maximum entropy priors

There are several well-known methods reported in the Bayesian literature to construct densities that incorporate prior information by max-

imizing a criterion functional subject to a set of constraints in terms of expected values. Some of such methods are: non-informative priors (Jeffreys [14]); maximal data information priors (Zellner [24]); maximum entropy priors (Jaynes [13]); minimum cross-entropy priors, also known as relative entropy priors (Kullback [16]); reference priors (Good [8] and Bernardo [5]); and controlled priors (Venegas-Martínez *et al.* [21]). We shall specialize in this paper in Jaynes' maximum entropy for pragmatic and theoretical reasons that will appear later.

Let us suppose that there is initial information on volatility in terms of expected values, say  $\int a_k(h)\pi(h)I_{\{h>0\}}d\nu(h) = \bar{a}_k$ ,  $k = 0, 1, 2, \dots, N$ , where the functions  $a_k(h)$  are Lebesgue-mesurable known functions and all the constants  $\bar{a}_k$  are known, as well. The maximum entropy principle states that from all densities satisfying the given information (constraints) we should choose the one that maximizes

$$H[\pi(\theta)] = - \int_{h>0} \ln[\pi(h)]\pi(h)d\nu(h).$$

We define  $a_0(h) \equiv 1$  and  $\bar{a}_0 = 1$  to ensure that the solution is indeed a proper density. Hence, we are interested in finding  $\pi(h)$  that solves the following variational problem:

$$\max_{\pi} H[\pi(\theta)] = - \int_{h>0} \ln[\pi(h)]\pi(h)d\nu(h),$$

$$\text{subject to } \mathcal{C} : \int a_k(h)\pi(h)I_{\{h>0\}}d\nu(h) = \bar{a}_k, \quad k = 0, 1, 2, \dots, N.$$

In the sequel, we shall assume that the set of the constraints,  $\mathcal{C}$ , form a convex and compact set on  $\pi$ . Since  $H[\pi(h)]$  is strictly concave in  $\pi(h)$ , the solution exists and is unique. In such a case, the necessary condition for  $\pi(h)$  to be a maximum, is also sufficient. By using standard necessary conditions derived from calculus of variations (see, for instance, Chiang [6]), we found that if  $\pi(h)$  is optimal, then

$$(1) \quad \pi(h) = e^{1+\lambda_0} \exp \left\{ \sum_{k=1}^N \lambda_k a_k(h) \right\},$$

where  $\lambda_k$ ,  $k = 0, 1, 2, \dots, N$ , are the Lagrange multipliers associated with the constraints  $\mathcal{C}$ .

### 3.2 Relative entropy

Another useful inference method to estimate an unknown probability density,  $\pi(h)$ , when there is an initial estimate  $p(h)$  of  $\pi(h)$ , and information about precision  $h$  in terms of expectations, is based on determining  $\pi(h)$  that solves the following variational problem:

$$\min_{\pi} \int_{h>0} \pi(h) \ln \frac{\pi(h)}{p(h)} d\nu(h),$$

$$\begin{aligned} \text{subject to: } \quad & \int \pi(h) I_{\{h>0\}} d\nu(h) = 1, \\ & \int a_k(h) \pi(h) I_{\{h>0\}} d\nu(h) = \bar{a}_k, \quad k = 1, 2, \dots, N. \end{aligned}$$

The quantity  $\int_{h>0} \pi(h) \ln(\pi(h)/p(h)) d\nu(h)$  is called the relative entropy between  $\pi(h)$  and  $p(h)$ , and satisfies a set of axioms of consistency: uniqueness of the final estimate; invariance under one-to-one coordinate transformations; system independence; and subset independence. In this case, if  $\pi(h)$  is optimal, we have that

$$\pi(h) = p(h) e^{1+\lambda_0} \exp \left\{ \sum_{k=1}^N \lambda_k a_k(h) \right\},$$

where  $\lambda_k$ ,  $k = 0, 1, 2, \dots, N$ , are the Lagrange multipliers associated with the constraints. Observe that when the initial estimate is a uniform density, then relative entropy becomes entropy, as defined in section 3.1. Finally, it is important to mention the work of Avellaneda, Levy, and Parás [1] on derivative securities when modeling potential volatility values occurring within an open interval using relative entropy.

### 3.3 Examples of priors on precision

Suppose that prior information on precision is given in terms of expected values of levels and rates. That is, prior knowledge is expressed as:

$$(2) \quad \int_{h>0} h \pi(h) d\nu(h) = \frac{\beta}{\alpha},$$

and

$$(3) \quad \int_{h>0} \ln(h) \pi(h) d\nu(h) = \psi(\alpha) - \ln(\beta),$$

where  $\alpha > 0$ ,  $\beta > 0$ ,  $\psi(\alpha) = d\Gamma(\alpha)/d\alpha$ , and  $\Gamma(\cdot)$  is the Gamma function. Notice that for given expected values on levels and rates, equations (2) and (3) become a nonlinear system in the variables  $\alpha$  and  $\beta$ . Since entropy is strictly concave and the Gamma distribution is the unique distribution that satisfies (2) and (3), we find that

$$(4) \quad \pi(h|\alpha, \beta) = \frac{h^{\alpha-1}\beta^\alpha e^{-\beta h}}{\Gamma(\alpha)}, \quad h > 0, \alpha > 0, \quad \text{and} \quad \beta > 0,$$

solves the maximum entropy problem. Another priors of interest, after some changes of variable, could be:

$$(5) \quad \pi\left(\frac{1}{h}\middle|\alpha, \beta\right) = \frac{h^{\alpha+1}\beta^\alpha e^{-\beta h}}{\Gamma(\alpha)}, \quad h > 0, \alpha > 0, \quad \text{and} \quad \beta > 0,$$

$$(6) \quad \pi\left(\frac{1}{\sqrt{h}}\middle|\alpha, \beta\right) = \frac{2h^{\alpha+\frac{1}{2}}\beta^\alpha e^{-\beta h}}{\Gamma(\alpha)}, \quad h > 0, \alpha > 0, \quad \text{and} \quad \beta > 0,$$

and

$$\pi\left(\ln\left(\frac{1}{h}\right)\middle|\alpha, \beta\right) = \frac{\beta^\alpha e^{-\beta e^{-\ln(1/h)} - \ln(1/h)}}{\Gamma(\alpha)},$$

$h > 0$ ,  $\alpha > 0$ , and,  $\beta > 0$ , which stand, respectively, for prior distributions of  $\sigma^2$ ,  $\sigma$ , and  $\ln(\sigma^2)$ . In any case, the best choice should reflect what has been learned from previous practical experience.

## 4 Statement of the basic Bayesian valuation problem

Let us consider a Wiener process  $(W_t)_{t \geq 0}$  defined on some fixed filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , and a European call option on an underlying asset whose price at time  $t$ ,  $S_t$ , is driven by a geometric Brownian motion accordingly to

$$dS_t = rS_t dt + h^{-1/2}S_t dW_t,$$

that is,  $(W_t)_{t \geq 0}$  is defined on a risk neutral probability measure  $\mathbb{P}$ . Notice that the stochastic differential equation driving the price of the

underlying asset depends only on the risk-free rate of interest. The drift is independent of risk preferences about the expected return on the asset. In this case, investors do not require a premium as long as volatility remains constant. Girsanov's theorem can be used to remove a drift with risk preferences by providing an equivalent risk neutral probability measure (see, for instance, Fouque, Papanicolaou, and Sircar [7]). The option is issued at  $t_0 = 0$  and matures at  $T > 0$  with strike price  $X$ . Under the Bayesian framework, we have that the price, at time  $t_0 = 0$ , of the contingent claim when there is prior information on volatility, as expressed in (4), is given by:

$$\begin{aligned} c(S_0, T, X, r|\alpha, \beta) &= e^{-rT} \mathbf{E}^{(\pi)} \{ \mathbf{E} [\max(S_T - X, 0) | S_0] \} \\ (7) \quad &= e^{-rT} \int_{h>0} \left\{ \int_{s>X} (s - X) f_{S_T|S_0}(s) ds \right\} \pi(h) d\nu(h), \end{aligned}$$

where the conditional density of  $S_T$  given  $S_0$  satisfies

$$f_{S_T|S_0}(s) = \frac{h^{1/2}}{s\sqrt{2\pi T}} \exp \left\{ -\frac{h}{2T} \left( G(s) + \frac{T}{2h} \right)^2 \right\},$$

and

$$G(s) = \ln \left( \frac{s}{S_0 e^{-rT}} \right).$$

If we assume that the required conditions to apply Fubini's theorem are satisfied, so we can guarantee that integrals can be interchanged, then (7) becomes

$$(8) \quad c = \frac{e^{-rT} \beta^\alpha}{\sqrt{2\pi T} \Gamma(\alpha)} \int_{s>X} \left( 1 - \frac{X}{s} \right) \mathcal{I}(s|\alpha, \beta) ds,$$

where

$$(9) \quad \mathcal{I}(s|\alpha, \beta) = \int_{h>0} \exp \left\{ -\frac{h}{2T} \left( G(s) + \frac{T}{2h} \right)^2 \right\} h^{\alpha-1/2} e^{-\beta h} d\nu(h).$$

Notice now that (9) can be, in turn, rewritten as

$$(10) \quad \mathcal{I}(s|\alpha, \beta) = \exp \left\{ \frac{-G(s)}{2} \right\} \int_{h>0} \exp \left\{ -A(s)h - \frac{B}{h} \right\} h^{\delta-1} d\nu(h),$$

where

$$A(s) = \left( \frac{G(s)^2}{2T} + \beta \right) > 0, \quad B = \frac{T}{8} > 0, \quad \text{and} \quad \delta = \alpha + \frac{1}{2} > 0.$$

The integral in (10) satisfies (see, for instance, Gradshteyn and Ryzhik [9])

$$(11) \quad \int_{h>0} \exp \left\{ -A(s)h - \frac{B}{h} \right\} h^{\delta-1} d\nu(h) = 2 \left( \frac{B}{A(s)} \right)^{\frac{\delta}{2}} K_{\delta} \left( 2\sqrt{BA(s)} \right),$$

where  $K_{\delta}(x)$ ,  $x = 2\sqrt{BA(s)}$ , is the modified Bessel function of order  $\delta$ , which is solution of the second-order ordinary differential equation (see, for instance, Redheffer [17])

$$(12) \quad y'' + \frac{1}{x}y' - \left( 1 + \frac{\delta^2}{x^2} \right) y = 0, \quad x > 0.$$

We also have that  $K_{\delta}(x)$  is always positive, and  $K_{\delta}(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Equation (11) is of noticeable importance since it says that all the additional information on volatility provided by the prior distribution and the relevant information on the process driving the dynamics of the underlying asset are now contained in  $K_{\delta}$ .

#### 4.1 Constant elasticity of return variance

In this section, we deal with the constant elasticity instantaneous variance case. Let us assume the underlying asset,  $S_t$ , evolves according to

$$dS_t = rS_t dt + h^{-1/2}S_t^{b/2} dW_t,$$

where the elasticity of return variance with respect to the price is defined as  $b - 2$ . If  $b = 2$ , then the elasticity is zero and asset prices are lognormally distributed. In this section, we are concerned with the case  $b < 2$ . After computing the Jacobian for transforming  $W_t \sim \mathcal{N}(0, t)$  into  $S_T$ , we find that the conditional density of  $S_T$  given  $S_t$  satisfies

$$f_{S_T|S_0}(s) = \frac{h}{\delta} D[UV(s)^{1-2b}]^{1/(4-2b)} e^{-h[U+V(s)]} I_{\delta} \left( 2h\sqrt{UV(s)} \right),$$

where



$$\begin{aligned}\delta &= 1/(2-b), \\ D &= \left[ \frac{2r}{(2-b)[e^{r(2-b)T} - 1]} \right]^{1/(2-b)}, \\ U &= (DS_0 e^{rT})^{2-b}, \\ V(s) &= (Ds)^{2-b},\end{aligned}$$

and  $I_\delta(x)$ ,  $x = 2h\sqrt{UV(s)}$ , is the modified Bessel function of the first kind of order  $\delta$ . If we assume that prior distribution is described by a Gamma density, then

$$c = \frac{De^{-rT}\beta^\alpha}{\delta\sqrt{2\pi T}\Gamma(\alpha)} \int_{s>X} (s-X) [UV(s)^{1-2b}]^{1/(4-2b)} \mathcal{J}(s|\alpha, \beta) ds,$$

where

$$\mathcal{J}(s|\alpha, \beta) = \int_{h>0} h^\alpha e^{-h[\beta+U+V(s)]} I_\delta(2h\sqrt{UV(s)}) dv(h)$$

which is related with the non-central chi-square density function. Moreover,

$$I_\delta(2h\sqrt{UV(s)}) = \sum_{k=0}^{\infty} h^{\delta+2k} \frac{[UV(s)]^{k+(\delta/2)}}{\Gamma(k+1)\Gamma(\delta+k+1)}.$$

Hence,

$$\begin{aligned}\mathcal{J}(s|\alpha, \beta) &= \sum_{k=0}^{\infty} \frac{[UV(s)]^{k+(\delta/2)}}{\Gamma(k+1)\Gamma(\delta+k+1)} \int_{h>0} h^{\alpha+\delta+2k} e^{-h[\beta+U+V(s)]} dv(h) \\ &= \frac{[UV(s)]^{\delta/2}}{(\beta+U+V(s))^\alpha} \sum_{k=0}^{\infty} \frac{[UV(s)]^k \Gamma[\alpha+\delta+2k+1]}{\Gamma(k+1)\Gamma(\delta+k+1)}.\end{aligned}$$

In the particular case that there is not prior information, the solution of maximizing  $H[\pi(\theta)]$ , subject only to the normalizing constraint, will lead to an improper uniform prior distribution, say  $\pi(h) \equiv 1$  almost everywhere with respect to  $\nu$ , then if  $z = [V(s)/D]^{1/(2-b)}$ , equivalently  $V(s) = Dz^{2-b}$ , we have

$$\begin{aligned}c &= S_0 \int_{hDX^{2-b}}^{\infty} e^{h(U+Dz^{2-b})} \left( \frac{Dz^{2-b}}{U} \right)^{1/(4-2b)} I_\delta(2h\sqrt{UDz^{2-b}}) dz \\ &+ Xe^{-rT} \int_{hDX^{2-b}}^{\infty} e^{h(U+Dz^{2-b})} \left( \frac{U}{Dz^{2-b}} \right)^{1/(4-2b)} \\ &\cdot I_\delta(2h\sqrt{UDz^{2-b}}) dz,\end{aligned}$$

where the following identity holds

$$\int_{hDX^{2-b}}^{\infty} e^{h(U+Dz^{2-b})} \left[ \left( \frac{Dz^{2-b}}{U} \right)^{1/(4-2b)} + \left( \frac{U}{Dz^{2-b}} \right)^{1/(4-2b)} \right] I_{\delta} \left( 2h\sqrt{UDz^{2-b}} \right) dz = 1.$$

## 5 Asymptotic approximations for the basic Bayesian valuation problem

In this section, we find an asymptotic approximate formula for pricing vanilla contingent claims according to equation (8)-(11). In order to use asymptotic approximations for equation (11), we have to make some assumption on the strike price,  $X$ . Note first, that if the strike price  $X$  is large, then  $x$  is large. In such a case, we may use the following approximation (see, for instance, Gradshteyn and Ryzhik [9]):

$$K_{\delta}(x) \sim \widehat{K}_{\delta}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left( 1 + \frac{4\delta^2 - 1}{8x} \right),$$

which, in practice, performs well. In this case, we have the estimate price

$$\widehat{c} = S_0 \widehat{\mathcal{M}}_1(S_0, T, X, r | \alpha, \beta) - e^{-rT} X \widehat{\mathcal{M}}_2(S_0, T, X, r | \alpha, \beta),$$

where

$$\widehat{\mathcal{M}}_1 = \frac{\sqrt{2}\beta^{\alpha}}{S_0\sqrt{\pi T}\Gamma(\alpha)} \int_X^{\infty} e^{-(\frac{1}{2}G(s)+r)} \left( \frac{T}{8A(s)} \right)^{\frac{\delta}{2}} \widehat{K}_{\delta}(2\sqrt{BA(s)}) ds,$$

and

$$\widehat{\mathcal{M}}_2 = \frac{\sqrt{2}\beta^{\alpha}}{\sqrt{\pi T}\Gamma(\alpha)} \int_X^{\infty} \frac{1}{s} e^{-\frac{1}{2}G(s)} \left( \frac{T}{8A(s)} \right)^{\frac{\delta}{2}} \widehat{K}_{\delta}(2\sqrt{BA(s)}) ds.$$

The integrals  $\widehat{\mathcal{M}}_1$  and  $\widehat{\mathcal{M}}_2$  can be approximated with simple procedures in MATLAB by using a large enough upper limit in the integral. The upper limits of the integrals  $\widehat{\mathcal{M}}_1$  and  $\widehat{\mathcal{M}}_2$  are taken large enough so that the values of  $\widehat{\mathcal{M}}_1$  and  $\widehat{\mathcal{M}}_2$  have no substantial change when larger upper limits are used (within an error of 0.0001). Figure 1 shows the values of  $\widehat{c}$  as a function of  $\alpha$  ( $\delta = \alpha + \frac{1}{2}$ ) and  $\beta$ , with  $S_0 = 42.00$ ,  $X = 41.00$ ,  $r = 0.11$ , and  $T = 0.25$ .

Figure 1. Values of  $\hat{c}$  as a function of  $\alpha$  and  $\beta$ .

## 6 Polynomial approximations for the basic Bayesian valuation problem

Polynomial approximations, for the basic Bayesian valuation problem stated in (8)-(11), can be done only for some numerical values of the parameters. In this case, we apply the Frobenius' method to obtain an approximate polynomial of finite order. Let us consider the particular case  $\alpha = 0.5$ , *i.e.*,  $\delta = 1$ , in equation (9). The following polynomial approximation is based on Frobenius' method of convergent power-series expansion:

$$(13) \quad K_1(x) = \frac{1}{x} \left[ x \ln \left( \frac{x}{2} \right) I_1(x) + \sum_{k=0}^6 a_k \left( \frac{x}{2} \right)^{2k} + \epsilon \right], \quad 0 < x \leq 2,$$

where  $a_0 = 1$ ,  $a_1 = 0.15443144$ ,  $a_2 = -0.67278579$ ,  $a_3 = -0.18156897$ ,  $a_4 = -0.01919402$ ,  $a_5 = -0.0110404$ ,  $a_6 = -0.00004686$ , and

$$I_1(x) = x \left[ \sum_{k=0}^6 b_k \left( \frac{4x}{5} \right)^k + \epsilon \right], \quad 0 < x \leq \frac{15}{4},$$

where  $b_0 = 1/2$ ,  $b_1 = 0.878900594$ ,  $b_2 = 0.51498869$ ,  $b_3 = 0.15084934$ ,  $b_4 = 0.02658733$ ,  $b_5 = 0.00301532$ ,  $b_6 = 0.00032411$ , and  $\epsilon < 8 \times 10^{-9}$ . The complementary polynomial are given by

$$(14) \quad K_1(x) = \frac{1}{\sqrt{x}e^x} \ln \left( \frac{x}{2} \right) I_1(x) + \sum_{k=0}^6 \bar{a}_k \left( \frac{x}{2} \right)^{-2k} + \bar{\epsilon}, \quad x > 2,$$

where  $\bar{a}_0 = 1.25331414$ ,  $\bar{a}_1 = 0.23498619$ ,  $\bar{a}_2 = -0.03655620$ ,  $\bar{a}_3 = 0.01504268$ ,  $\bar{a}_4 = -0.00780353$ ,  $\bar{a}_5 = 0.00325614$ ,  $\bar{a}_6 = -0.00068245$ , and

$$I_1(x) = x \left[ \sum_{k=0}^8 \bar{b}_k \left( \frac{4x}{5} \right)^{-k} + \bar{\epsilon} \right], \quad x > \frac{15}{4},$$

where  $\bar{b}_0 = 39894228$ ,  $\bar{b}_1 = -0.03988024$ ,  $\bar{b}_2 = -0.00362018$ ,  $\bar{b}_3 = 0.00163801$ ,  $\bar{b}_4 = -0.01031555$ ,  $\bar{b}_5 = 0.02282967$ ,  $\bar{b}_6 = -0.02895312$ ,  $\bar{b}_7 = 0.01787654$ ,  $\bar{b}_8 = -0.00420059$ , and  $\bar{\epsilon} < 2.2 \times 10^{-7}$ . It is important to point out that  $K_1(x)$  and  $I_1(x)$  are linearly independent modified Bessel functions, thus they determine a unique solution of Bessel differential equation. If we denote by  $K_1^{(\epsilon)}(x)$  the polynomial approximation in (13) and (14), we get from (8)-(11) the following call option price:

$$c^{(\epsilon)} = S_0 \mathcal{M}_1(S_0, T, X, r | \alpha = 0.5, \beta) - e^{-rT} X \mathcal{M}_2(S_0, T, X, r | \alpha = 0.5, \beta),$$

where

$$\mathcal{M}_1^{(\epsilon)} = \frac{\beta^{\frac{1}{2}}}{2S_0\pi} \int_X^\infty e^{-(\frac{1}{2}G(s)+r)} [A(s)]^{-\frac{1}{2}} K_1^{(\epsilon)} \left( \sqrt{\frac{TA(s)}{2}} \right) ds,$$

and

$$\mathcal{M}_2^{(\epsilon)} = \frac{\beta^{\frac{1}{2}}}{2\pi} \int_X^\infty \frac{1}{s} e^{-\frac{1}{2}G(s)} [A(s)]^{-\frac{1}{2}} K_1^{(\epsilon)} \left( \sqrt{\frac{TA(s)}{2}} \right) ds.$$

As before, integrals  $\mathcal{M}_1^{(\epsilon)}$  and  $\mathcal{M}_2^{(\epsilon)}$  can be approximated by using simple procedures in MATLAB. Figure 2 shows the values of  $c^{(\epsilon)}$  as a function of  $\beta$  with  $\alpha = 0.5$ ,  $S_0 = 42.00$ ,  $X = 41.00$ ,  $r = 0.11$ , and  $T = 0.25$ .

Figure 2. Values of  $\hat{c}^{(\epsilon)}$  as a function of  $\beta$ .

## 7 Comparison with other models available in the literature

In the Mexican case, there is not an exchange for trading options, and the over-the-counter market on options is an incipient market, so data is poor in both quantity and quality. Hence, it is impossible to carry out a reliable empirical analysis to compare market option prices with our theoretical prices. However, we work out an interesting numerical experiment. In this experiment, we compare our prices with two other prices from models available in the literature. In figure 3, the case of the classical Black and Scholes' price, as a function of the strike price, is considered as a benchmark with parameter values  $S_0 = 100$ ,  $T = 0.5$ ,  $r = 0.05$ , and  $\sigma = 0.2$ , and is represented by the solid line. The Korn and Wilmott's [15] price with subjective beliefs on future behavior of stock prices is represented by the dashed line. The parameter values in the Korn and Wilmott's [15] model are  $\mu = 0.1$ ,  $\alpha = 0.33$ ,  $\beta = 3.33$ , and  $\gamma = 0.1$ . Finally, the dotted line shows our price  $\hat{c}^{(\epsilon)}$  with prior information on levels and rates. We examined, in this experiment,

about 800 different combinations of the parameter values  $\alpha$  and  $\beta$ , with parameter values  $\beta = 17$  and  $\alpha = 0.5$ . Notice that option prices with prior information on levels are higher than option prices with only prior information on future prices. As expected, Black and Scholes prices are smaller than option prices with any prior information.

Figure 3. Option values as a function of the strike price.

## 8 Summary and conclusions

Prior information is a subjective issue, that is, different individuals have different initial beliefs. It is difficult to accept that all individuals participating in a specific market can describe their initial knowledge with the same functional form for the prior distribution, and it is still more difficult to recognize as being true that all of such distributions have the same parameters. The existence of a prior distribution is useful to describe initial beliefs in much more complex markets than those in a naive Black-Scholes. In a richer stochastic environment, we have developed a Bayesian procedure to value a European call option when there is prior information on uncertain or changing volatility. In conclusion, the existence of a prior distribution is useful to describe initial beliefs

in much more complex markets than those in a naive Black-Scholes. Needless to say, Monte Carlo methods should be developed and applied in our proposed framework, and that will be our next goal.

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