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Existence of Nash equilibria in discounted nonzero-sum stochastic games with additive structure *

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Abstract

This work considers two-person nonzero-sum dynamic stochastic games when the state and action sets are Borel spaces, with possibly unbounded (immediate) cost functions, and discounted cost criteria. The aim is to prove, under suitable assumptions, the existence of a Nash equilibrium in stationary strategies. One of those assumptions is that the transition law and the cost functions have an additive (or separable) structure.

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1 Introduction

In this work we study two–person nonzero–sum stochastic games when the state and action sets are uncountable Borel spaces. For such a class of games, with either a discounted or an average cost criterion, the existence of stationary Nash equilibria is still an open problem. Here we give a solution to this problem for games with discounted cost criteria in the particular case when the cost functions and the transition law have an additive structure.

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The existence of Nash equilibria in stationary strategies for discounted stochastic games with additive (also known as "separable") structure was first studied by Himmelberg et al. [7], Theorem 1, under the assumption that the action spaces are finite sets and the state space is Borel. They showed that, given a probability distribution p of the initial state, there is a pair of stationary strategies which form a Nash equilibrium p–a.e. (a so-called p–equilibrium). This result was strengthened by Parthasarathy [15], Theorem 4, who showed that such games have a Nash equilibrium in stationary strategies. An extension of Parthasarathy's result in [15] was given in Nowak [12], Theorem 1.1. In the latter work the state set is a measurable space with a countably generated σ–algebra. Nowak also considered compact metric action spaces and bounded reward functions. Our main result, Theorem 4.4, is also an improvement of Parthasarathy's Theorem 4 in [15], because the state space and the action spaces that we consider are Borel spaces and the cost functions are possibly unbounded. Moreover, our approach to prove Theorem 4.4 is different from Nowak's in [12]. Here we follow the approach developed in several works, via a fixed-point theorem for multifunctions, as in Ghosh and Bagchi [5] and Parthasarathy [15], for instance.

Other works dealing with an additive structure include [3], [8], [10], [16]. Examples may be found in [3] and [5]. A good survey of the existing literature for both discounted and average criteria is given in [13].

The work is organized as follows. Section 2 presents standard material on stochastic games. The discounted optimality criteria we are interested in is introduced in section 3. Our assumptions and main result, Theorem 4.4, are stated in section 4. Sections 5 and 6 are devoted to prove our main result. Section 7 presents some concluding remarks.

2 The stochastic game model

In this section we introduce the game model we are interested in. We start with the following remark on terminology and notation —for further details see Bertsekas and Shreve [1], chapter 7, for instance.

Remark 2.1 (a) A Borel subset X of a complete and separable metrizable space is called a *Borel space*, and its Borel σ–algebra is denoted by $\mathcal{B}(X)$. We only deal with Borel spaces, and so "measurable" (for either sets or functions) always means "Borel measurable". Given a Borel space X, we denote by $\mathbb{P}(X)$ the family of all probability measures on X, endowed with the weak topology. In this case, $\mathbb{P}(X)$ is a Borel space. Moreover, if X is compact, then so is $\mathbb{P}(X)$.

(b) Let X and Y be Borel spaces. A measurable function $\phi : Y \to \mathbb{P}(X)$ is called a *stochastic kernel* on X given Y, and we denote by $\mathbb{P}(X|Y)$ the family of all those stochastic kernels. If ϕ is in $\mathbb{P}(X|Y)$, then we write its values either as $\phi(y)(C)$ or as $\phi(C|y)$, for all $y \in Y$ and $C \in \mathcal{B}(X)$.

The game model. We shall consider the two-person nonzero-sum stochastic game model

$$(X, A, B, \mathbb{K}_A, \mathbb{K}_B, Q, c_1, c_2)$$

where X is the *state space*, and A and B are the *action spaces* for players 1 and 2, respectively. These spaces are all assumed to be Borel spaces. The sets $\mathbb{K}_A \in \mathcal{B}(X \times A)$ and $\mathbb{K}_B \in \mathcal{B}(X \times B)$ are the *constraint sets*. That is, for each state $x \in X$ the x–section in $\mathbb{K}_A$, namely

$$A(x) := \{a \in A | (x, a) \in \mathbb{K}_A\},$$

represents the set of admissible actions for player 1 in the state x. Similarly, the x–section in $\mathbb{K}_B$,

$$B(x) := \{b \in B | (x, b) \in \mathbb{K}_B\},$$

stands for the set of admissible actions for player 2 in the state x. Let

$$\mathbb{K} := \{(x, a, b) | x \in X, a \in A(x), b \in B(x)\},$$

which is a Borel subset of $X \times A \times B$ (see Lemma 1.1 in Nowak [11], for instance). Then $Q \in \mathbb{P}(X|\mathbb{K})$ is the game's *transition law*, and, finally, $c_i : \mathbb{K} \to \mathbb{R},\ i = 1, 2$, is a measurable function that represents the one-stage cost function for player i.

The game is played as follows. At each stage (or time) $t = 0, 1, 2, \ldots$, the players 1 and 2 observe the current state $x \in X$ of the system, and then independently choose actions $a \in A(x)$ and $b \in B(x)$, respectively. As a result of this two things happen: (1) player i ($i = 1, 2$) incurs a cost $c_i(x, a, b)$; and (2) the system moves to a new state according to the probability distribution $Q(\cdot|x, a, b)$. Cost accumulates throughout the course of the game, and the goal of each player is to minimize his/her cost.

Strategies. Let $H_0 := X$, and $H_t := \mathbb{K}^t \times X$ for $t = 1, 2, \ldots$. For each t, an element $h_t = (x_0, a_0, b_0, \ldots, x_{t-1}, a_{t-1}, b_{t-1}, x_t)$ of H_t represents a "history" of the game up to time t. A *strategy* for player 1 is then defined as a sequence $\pi^1 = \{\pi_t^1, t = 0, 1, \ldots\}$ of stochastic kernels $\pi_t^1 \in \mathbb{P}(A|H_t)$ such that

$$\pi_t^1(A(x_t)|h_t) = 1 \quad \forall h_t \in H_t,\ t = 0, 1, \ldots .$$

The set of all strategies for player 1 is denoted by Π_1. Let S_1 be the set of stochastic kernels $\phi \in \mathbb{P}(A|X)$ with the property

$$\phi(A(x)|x) = 1 \quad \forall x \in X.$$

A strategy $\pi^1 = (\pi_t^1)$ is called *stationary* if there exists $\phi_1 \in S_1$ such that

$$\pi_t^1(\cdot|h_t) = \phi_1(\cdot|x_t) \quad \forall h_t \in H_t,\ t = 0, 1, \ldots .$$

We shall identify S_1 with the family of all stationary strategies for player 1. Let $\mathbb{F}_1$ be the set of measurable functions $f : X \to A$ such that $f(x)$ is in $A(x)$ for all x. Identifying $f(x)$ with the Dirac measure $\delta_{f(x)}(\cdot)$ concentrated at $f(x)$ we see that

$$\mathbb{F}_1 \subset S_1.$$

A stationary strategy $\phi \in S_1$ for player 1 is said to be *deterministic* if there exists $f \in \mathbb{F}_1$ such that

$$\phi(\cdot|x) = \delta_{f(x)}(\cdot) \qquad \forall x \in X.$$

The sets of strategies Π_2, S_2 and $\mathbb{F}_2$ for player 2 are defined similarly.

Let $(\Omega, \mathcal{F})$ be the canonical measurable space that consists of the sample space $\Omega = (X \times A \times B)^\infty$ and its product σ–algebra $\mathcal{F}$. Then for each pair of strategies $(\pi^1, \pi^2) \in \Pi_1 \times \Pi_2$ and each "initial state" $x \in X$, there exists, by a theorem of C. Ionescu-Tulcea (see, for example, Bertsekas and Shreve [1], pp. 140-141), a unique probability measure $P_x^{\pi^1,\pi^2}$ and a stochastic process $\{(x_t, a_t, b_t), t = 0, 1, \ldots\}$ defined on $(\Omega, \mathcal{F})$ in a canonical way, where x_t, a_t, and b_t represent the state and the actions of players 1 and 2, respectively, at each stage $t = 0, 1, \ldots$. The expectation operator with respect to $P_x^{\pi^1,\pi^2}$ is denoted by $E_x^{\pi^1,\pi^2}$.

3 The discounted cost

We now introduce the optimality criteria we are concerned with. Let α be a number in $(0, 1)$, a so-called "discount factor", to be fixed throughout this work.

Definition 3.1 Let $(\pi^1, \pi^2) \in \Pi_1 \times \Pi_2$ and $x \in X$. The total expected *α–discounted cost* for player $i, i = 1, 2$, when the players use the strategies π^1, π^2, given the initial state $x_0 = x$ is defined as

$$V^i(\pi^1, \pi^2, x) := E_x^{\pi^1,\pi^2}\left[\sum_{t=0}^{\infty} \alpha^t c_i(x_t, a_t, b_t)\right].$$

Definition 3.2 A pair of strategies $(\pi^{*1}, \pi^{*2}) \in \Pi_1 \times \Pi_2$ is called a *Nash equilibrium* for the α–discounted cost criterion if, for each $x \in X$, we have

$$V^1(\pi^{*1}, \pi^{*2}, x) \le V^1(\pi^1, \pi^{*2}, x) \qquad \forall \pi^1 \in \Pi_1,$$
$$V^2(\pi^{*1}, \pi^{*2}, x) \le V^2(\pi^{*1}, \pi^2, x) \qquad \forall \pi^2 \in \Pi_2.$$

We shall establish, under certain assumptions, the existence of a Nash equilibrium in $S_1 \times S_2$.

Before proceeding we give some notation. For a given measurable function
$f : \mathbb{K} \to \mathbb{R}$ and probability measures $\phi \in \mathbb{P}(A(x))$ and $\psi \in \mathbb{P}(B(x))$, let

$$f(x, \phi, \psi) := \int_{A(x)} \int_{B(x)} f(x, a, b)\psi(db)\phi(da),$$

whenever the integrals are well defined. In particular,

$$c_i(x, \phi, \psi) := \int_{A(x)} \int_{B(x)} c_i(x, a, b)\psi(db)\phi(da),$$

$$Q(\cdot|x, \phi, \psi) := \int_{A(x)} \int_{B(x)} Q(\cdot|x, a, b)\psi(db)\phi(da).$$

4 Existence of Nash equilibria

In this section we state our main result, Theorem 4.4, which requires the following assumptions.

Assumption 4.1 (a) For each $x \in X$, the sets $A(x)$ and $B(x)$ are compact.

(b) There exists a measurable function $w : X \to \mathbb{R}$, with $w(\cdot) \ge 1$, such that

$$\int_X w(y)Q(dy|x, \cdot, b) \qquad \text{and} \qquad \int_X w(y)Q(dy|x, a, \cdot)$$

are continuous on the sets $A(x)$ and $B(x)$, respectively.

(c) There exists a constant $1 \leq \beta < 1/\alpha$ such that

$$\int_X w(y)Q(dy|x,a,b) \leq \beta w(x) \qquad \forall (x,a,b) \in \mathbb{K}.$$

The following assumption concerns the game's additive (or "separable") structure.

Assumption 4.2 (a) For $i = 1,2$, there exist measurable functions $c_{i1} : \mathbb{K}_A \to \mathbb{R}$ and $c_{i2} : \mathbb{K}_B \to \mathbb{R}$ such that

$$c_i(x,a,b) = c_{i1}(x,a) + c_{i2}(x,b) \qquad \forall (x,a,b) \in \mathbb{K}.$$

Moreover, the functions $c_{i1}(x,\cdot)$ and $c_{i2}(x,\cdot)$ are continuous on $A(x)$ and $B(x)$, respectively, and

$$\max_{a\in A(x)} |c_{i1}(x,a)| \leq w(x), \quad \max_{b \in B(x)} |c_{i2}(x,b)| \leq w(x) \qquad \forall x \in X,$$

where $w(\cdot)$ is the function in Assumption 4.1.

(b) There exist substochastic kernels $Q_1 \in \mathbb{P}(X|\mathbb{K}_A)$, and $Q_2 \in \mathbb{P}(X|\mathbb{K}_B)$, such that

$$Q(\cdot|x,a,b) = Q_1(\cdot|x,a) + Q_2(\cdot|x,b) \qquad \forall (x,a,b) \in \mathbb{K}.$$

Further, $Q_1(C|x,\cdot)$ and $Q_2(C|x,\cdot)$ are continuous on $A(x)$ and $B(x)$, respectively, for each $C \in \mathcal{B}(X)$.

Now we give our last assumption, in which we use again the function $w(\cdot)$ in Assumption 4.1.

Assumption 4.3 There exists a probability measure $\mu \in \mathbb{P}(X)$ and a density function z on $\mathbb{K} \times X$ such that for each $(x,a,b) \in \mathbb{K}$,

$$Q(C|x,a,b) = \int_C z(x,a,b,y)\mu(dy) \qquad \forall C \in \mathcal{B}(X).$$

Moreover, for each $x \in X$,

$$\lim_{n\to\infty} \int_X |z(x,a_n,b_n,y) - z(x,a_0,b_0,y)| w(y)\mu(dy) = 0$$

whenever $a_n \to a_0$ in $A(x)$ and $b_n \to b_0$ in $B(x)$. It is also assumed that $w \in L^1(\mu)$.

We now state our main result.

Theorem 4.4 Under the Assumptions 4.1, 4.2 and 4.3, there exists a Nash equilibrium in $S_1 \times S_2$.

Theorem 4.4 is proved in Section 6, after some technical preliminaries in Section 5.

5 Technical preliminaries

The w–norm. We denote by $\mathbb{M}(X)$ the family of real–valued measurable functions on X, and by $\mathbb{B}(X)$ the subfamily of bounded functions in $\mathbb{M}(X)$.

If $u \in \mathbb{M}(X)$ we define its *w–norm* as

$$\|u\|_w := \sup_{x \in X} \frac{|u(x)|}{w(x)},$$

and u is *w–bounded* if $\|u\|_w < \infty$. Let $\mathbb{B}_w(X)$ be the Banach space of all w–bounded functions in $\mathbb{M}(X)$. Thus $\mathbb{B}_w(X)$ contains $\mathbb{B}(X)$.

Lemma 5.1 Let $x \in X$, $\pi^1 \in \Pi_1$, and $\pi^2 \in \Pi_2$ be arbitrary. Then

$$|V^i(\pi^1, \pi^2, x)| \leq w(x) \frac{2}{1 - \alpha\beta} \quad i = 1, 2.$$

Proof: Assumption 4.1(c) implies

$$E_x^{\pi^1,\pi^2}[w(x_t)] \leq \beta^t w(x) \qquad \forall t = 0, 1, \ldots .$$

Also, by Assumption 4.2(a) we have, for $i = 1, 2$,

$$|c_i(x_t, a_t, b_t)| \leq 2w(x_t) \qquad \forall t = 0, 1, 2, \ldots .$$

Hence

$$E_x^{\pi^1,\pi^2}|c_i(x_t, a_t, b_t)| \leq 2E_x^{\pi^1,\pi^2}[w(x_t)] \leq 2\beta^t w(x),$$

and it follows that

$$\begin{aligned} |V^i(\pi^1, \pi^2, x)| &\leq \sum_{t=0}^{\infty} \alpha^t E_x^{\pi^1,\pi^2} |c_i(x_t, a_t, b_t)| \\ &\leq \sum_{t=0}^{\infty} \alpha^t 2\beta^t w(x) \\ &= w(x) \frac{2}{1 - \alpha\beta}. \ \Box \end{aligned}$$

Dynamic programming. We next develop the dynamic programming results needed to prove Theorem 4.4.

If one of the players, say player 2, fixes a strategy $\phi_2 \in S_2$, then we have a Markov control process for player 1, with control model

$$(X, A, \mathbb{K}_A, Q_{\phi_2}, c^1_{\phi_2}), \tag{1}$$

where

$$Q_{\phi_2}(\cdot|x,a) = Q(\cdot|x,a,\phi_2(x)),$$

and

$$c^1_{\phi_2}(x,a) = c_1(x,a,\phi_2(x)).$$

Let $V^1_{\phi_2}(\cdot)$ be the α–discounted value function associated to (1). Note that every (deterministic) stationary strategy for player 1 is a (deterministic) stationary policy for (1) and vice versa. Also, the corresponding α–discounted costs coincide. This fact, together with Lemma 5.1 and Theorem 8.3.6(b) in Hernández-Lerma and Lasserre [6], p. 47, implies

$$||V^1_{\phi_2}||_w \leq 2/(1-\alpha\beta) \qquad \forall \phi_2 \in S_2. \tag{2}$$

We say that $\pi^{*1} \in \Pi_1$ is an *optimal response* to ϕ_2 if

$$V^1(\pi^{*1}, \phi_2, x) = \inf_{\pi^1 \in \Pi_1} V^1(\pi^1, \phi_2, x) \quad \forall x \in X.$$

Similar considerations apply for each fixed $\phi_1 \in S_1$.

Lemma 5.2 Let $v : \mathbb{K}_A \to \mathbb{R}$ be a measurable function such that $v(x,\cdot)$ is continuous on $A(x)$ for each $x \in X$. Then there exists $f \in \mathbb{F}_1$ such that, for each $x \in X$,

$$v^*(x) := \min_{a \in A(x)} v(x,a) = v(x, f(x)), \tag{3}$$

and v^* is a measurable function. Moreover, for each $x \in X$,

$$v^*(x) = \min_{\lambda \in \mathbb{P}(A(x))} v(x,\lambda). \tag{4}$$

Proof: The existence of $f \in \mathbb{F}_1$ that satisfies (3) follows, for instance, from Lemma 8.3.8(a) in Hernández–Lerma and Lasserre [6], p. 50. On the other hand, identifying each $a \in A(x)$ with the Dirac measure $\delta_a(\cdot) \in \mathbb{P}(A(x))$ we get

$$v^*(x) = \min_{a \in A(x)} v(x,a) \geq \min_{\lambda \in \mathbb{P}(A(x))} v(x,\lambda). \tag{5}$$

On the other hand, for each $\lambda \in \mathbb{P}(A(x))$ we have

$$v(x,\lambda) = \int_{A(x)} v(x,a)\lambda(da) \geq v^*(x) \qquad \forall x \in X.$$

Hence

$$\min_{\lambda \in \mathbb{P}(A(x))} v(x,\lambda) \geq v^*(x). \tag{6}$$

From (6) and (5) we obtain (4). □

Proposition 5.3 Let $\phi_2 \in S_2$ be fixed. Then:

(a) There exists $f_1 \in \mathbb{F}_1$ such that f_1 is an optimal response of player 1 to ϕ_2.

(b) The function $V^1_{\phi_2}$ is the unique solution in $\mathbb{B}_w(X)$ to the dynamic programming equation

$$V^1_{\phi_2}(x) = \min_{\lambda \in \mathbb{P}(A(x))} \left[c_1(x,\lambda,\phi_2(x)) + \alpha \int_X V^1_{\phi_2}(y) Q(dy|x,\lambda,\phi_2(x)) \right]. \tag{7}$$

(c) If $\phi_1^* \in S_1$ is an optimal response of player 1 to ϕ_2 then, for each $x \in X$,

$$V^1_{\phi_2}(x) = c_1(x,\phi_1^*(x),\phi_2(x)) + \alpha \int_X V^1_{\phi_2}(y) Q(dy|x,\phi_1^*(x),\phi_2(x)). \tag{8}$$

Similar results hold for each fixed $\phi_1 \in S_1$.

Proof: (a),(b). If $f_1 \in \mathbb{F}_1$ is an optimal policy for (1) then f_1 is an optimal response to ϕ_2. This follows as in Theorem 3.1 in Maitra and Parthasarathy [9], p. 295. Hence, to prove (a) it suffices to show the existence of $f_1 \in \mathbb{F}_1$ optimal for (1). To this end, let

$$v(x,a) := c_1(x,a,\phi_2(x)) + \alpha \int_X V^1_{\phi_2}(y) Q(dy|x,a,\phi_2(x)).$$

The function $c_1(x,\cdot,\phi_2(x))$ is continuous on $A(x)$ by Assumption 4.2(a). Also the function $\int_X V^1_{\phi_2}(y)Q(dy|x,\cdot,\phi_2(x))$ is continuous on $A(x)$ by (2) and Lemma 8.3.7 in Hernández–Lerma and Lasserre [6], p. 48. Hence, by Lemma 5.2 there exists $f_1 \in \mathbb{F}_1$ such that for each $x \in X$,

$$v^*(x) = \min_{a \in A(x)} v(x,a) = v(x,f(x)).$$

Therefore, by Theorem 8.3.6 in Hernández–Lerma and Lasserre [6], p. 47, the function f_1 is an optimal policy for (1). This proves (a). The same theorem gives that, for each $x \in X$,

$$V^1_{\phi_2}(x) = v(x, f(x)) = v^*(x). \tag{9}$$

From (9) and (4) we obtain (7). This proves (b).

(c) As in Theorem 3.1 in Maitra and Parthasarathy [9], p. 295, it follows that

$$V^1(\phi_1^*, \phi_2, x) = V^1_{\phi_2}(x) \quad \forall x \in X. \tag{10}$$

On the other hand, as ϕ_1^* is a stationary strategy, it follows as in Remark 8.3.10 in Hernández-Lerma and Lasserre [6], p. 54, that $V^1(\phi_1^*, \phi_2, \cdot)$ is the unique solution in $\mathbb{B}_w(X)$ of the equation

(11)
$$u(x) = c_1(x, \phi_1^*(x), \phi_2(x)) + \alpha \int_X u(y) Q(dy|x, \phi_1^*(x), \phi_2(x)) \quad \forall x \in X.$$

This fact and (10) give (8). □

The proof of Proposition 5.3 also shows that, for each $\phi_2 \in S_2$,

$$\begin{aligned} V^1_{\phi_2}(x) &= \inf_{\phi_1 \in S_1} V^1(\phi_1, \phi_2, x) \\ &= \inf_{\pi^1 \in \Pi_1} V^1(\pi^1, \phi_2, x). \end{aligned} \tag{12}$$

6 Proof of Theorem 4.4

The proof of Theorem 4.4 is based on a standard procedure; see Ghosh and Bagchi [5] or Parthasarathy [15], for example. This procedure consists of two steps:

(i) Topologize S_1 and S_2 so that they become metrizable and compact spaces.
(ii) Show that the multifunction $\tau : S_1 \times S_2 \to 2^{S_1 \times S_2}$ defined by

$$\begin{aligned} \tau(\phi_1, \phi_2) := \{(\phi_1^*, \phi_2^*) | &\phi_1^* \text{ is an optimal response to } \phi_2, \\ &\text{and } \phi_2^* \text{ is an optimal response to } \phi_1\} \end{aligned}$$

has a fixed point. That is, there exists $(\phi_1^*, \phi_2^*) \in S_1 \times S_2$ such that $(\phi_1^*, \phi_2^*) \in \tau(\phi_1^*, \phi_2^*)$. In the latter case, we clearly have that (ϕ_1^*, ϕ_2^*) is a Nash equilibrium.

We next consider step (i). We follow Parthasarathy [15] to topologize S_1 and S_2 with the topology of *relaxed controls* introduced by Warga [17], chapter 4. Let $\mathfrak{B}_1$ be the set of all measurable functions $h : \mathbb{K}_A \to \mathbb{R}$ such that $h(x, \cdot)$ is continuous on $A(x)$ for each $x \in X$, and $\max_{a\in A(x)} |h(x, a)|$ is a μ–integrable function on X. Here μ is the probability measure in Assumption 4.3. Then $\mathfrak{B}_1$ becomes a Banach space if we define the norm of $h \in \mathfrak{B}_1$ as

$$\|h\| := \int_X \max_{a\in A(x)} |h(x, a)|\mu(dx).$$

We identify two stationary strategies $\phi, \psi \in S_1$ if $\phi(x) = \psi(x)$ μ–a.e. Further, we will identify each $\phi \in S_1$ with the bounded linear functional $\Lambda_\phi : \mathfrak{B}_1 \to \mathbb{R}$ given by

$$\begin{aligned}\Lambda_\phi(h) &:= \int_X \int_A h(x, a)\phi(da|x)\mu(dx)\\ &= \int_X h(x, \phi(x))\mu(dx) \qquad \forall h \in \mathfrak{B}_1.\end{aligned}$$

In this way we can view S_1 as a subset of $\mathfrak{B}_1^*$. Equip S_1 with the weak* topology of $\mathfrak{B}_1^*$. So a sequence (ϕ_n) in S_1 converges to $\phi \in S_1$ if and only if

$$\Lambda_{\phi_n}(h) \to \Lambda_\phi(h) \qquad \forall h \in \mathfrak{B}_1,$$

or, more explicitly, for each $h \in \mathfrak{B}_1$

$$\int_X \int_A h(x, a)\phi_n(da|x)\mu(dx) \to \int_X \int_A h(x, a)\phi(da|x)\mu(dx)$$

which is the same as

$$\int_X h(x, \phi_n(x))\mu(dx) \to \int_X h(x, \phi(x))\mu(dx) \qquad \forall h \in \mathfrak{B}_1. \tag{1}$$

Thus, it follows from Theorem IV.3.11 in Warga [17], p. 287, that S_1 is metrizable and compact. Similarly, we define $\mathfrak{B}_2$ to obtain that S_2 is metrizable and compact. In the remainder of this work, convergence in S_1 or in S_2 is understood with respect to the topology just described.

Before proceeding with step (ii), we establish a useful lemma.

Lemma 6.1 Suppose that $(\phi_{1n}, \phi_{2n}) \to (\phi_1, \phi_2)$ in $S_1 \times S_2$. Let $h : \mathbb{K} \to \mathbb{R}$ be such that

$$h(x,a,b) = h_1(x,a) + h_2(x,b) \qquad \forall (x,a,b) \in \mathbb{K},$$

for functions $h_1 \in \mathfrak{B}_1$, $h_2 \in \mathfrak{B}_2$ such that

$$\max_{a \in A(x)} h_1(x,a),\ \max_{b \in B(x)} h_2(x,b) \in \mathbb{B}_w(X).$$

Then, as $n \to \infty$,

$$h(x, \phi_{1n}(x), \phi_{2n}(x)) \to h(x, \phi_1(x), \phi_2(x)) \qquad \mu\text{–a.e.}$$

Proof: Choose an arbitrary $f \in L^1(\mu)$. It is clear that

$$f(\cdot)\frac{h_1(\cdot,\cdot)}{w(\cdot)} \in \mathfrak{B}_1 \text{ and } f(\cdot)\frac{h_2(\cdot,\cdot)}{w(\cdot)} \in \mathfrak{B}_2.$$

Therefore, by (1),

$$\int_X f(x)\frac{h_1(x,\phi_{1n}(x))}{w(x)}\mu(dx) \to \int_X f(x)\frac{h_1(x,\phi_1(x))}{w(x)}\mu(dx)$$

and, similarly,

$$\int_X f(x)\frac{h_2(x,\phi_{2n}(x))}{w(x)}\mu(dx) \to \int_X f(x)\frac{h_2(x,\phi_2(x))}{w(x)}\mu(dx).$$

If we add these two expressions we obtain

$$\int_X f(x)\frac{h(x,\phi_{1n}(x),\phi_{2n}(x))}{w(x)}\mu(dx) \to \int_X f(x)\frac{h(x,\phi_1(x),\phi_2(x))}{w(x)}\mu(dx).$$

Therefore, by a Riesz's representation theorem (see, for example, Folland [4], p. 182), the desired conclusion follows. □

To carry out step (ii) we next show that the multifunction τ is upper-semicontinuous and then we will apply Fan's fixed-point theorem (see Theorem 1 in Fan [2]). To prove that τ is upper-semicontinuous, suppose that

$$(\phi_{1n}, \phi_{2n}) \to (\phi_1, \phi_2) \text{ in } S_1 \times S_2, \tag{2}$$

and that

$$(3) \qquad (\phi_{1n}^*, \phi_{2n}^*) \to (\phi_1^*, \phi_2^*) \text{ in } S_1 \times S_2$$

where, for each $n \in \mathbb{N}$,

$$(4) \qquad (\phi_{1n}^*, \phi_{2n}^*) \in \tau(\phi_{1n}, \phi_{2n});$$

then we have to show that

$$(5) \qquad (\phi_1^*, \phi_2^*) \in \tau(\phi_1, \phi_2).$$

To this end, we only prove that ϕ_1^* is an optimal response to ϕ_2. The proof that ϕ_2^* is an optimal response to ϕ_1 is similar. As the proof is a little bit long, we will split it into several results, the most important being Proposition 6.7 and Proposition 6.10.

Lemma 6.2 The following holds μ–a.e. as $n \to \infty$:

$$c_1(x, \phi_{1n}^*(x), \phi_{2n}(x)) \to c_1(x, \phi_1^*(x), \phi_2(x)).$$

Proof: This result is a consequence of Assumption 4.2 and Lemma 6.1. □

For notational ease, let

$$u_n(\cdot) := V_{\phi_{2n}}^1(\cdot) \text{ and } \tilde{u}_n(\cdot) := \frac{u_n(\cdot)}{w(\cdot)},$$

and define $m := 2/(1 - \alpha\beta)$. By (2) we have

$$|\tilde{u}_n(\cdot)| \leq m \qquad \forall n \in \mathbb{N}.$$

Lemma 6.3 There exists a subsequence (u_{n_k}) of (u_n) and a function $u_0 \in \mathbb{B}_w(X)$ such that (u_{n_k}) converges μ–a.e. to u_0. Therefore, without loss of generality we may assume that (u_n) converges μ–a.e. to u_0.

Proof: Identify two functions in $\mathbb{M}(X)$ if they are equal μ–a.e., and define

$$\mathfrak{U} := \{u \in \mathbb{M}(X) : |u(\cdot)| \leq m \;\; \mu\text{–a.e.}\} \subset L^\infty(\mu).$$

By a Riesz's representation theorem (see, for example, Folland [4], p. 182), we can view $\mathfrak{U}$ as a subset of $[L^1(\mu)]^*$. More precisely, we may identify $\mathfrak{U}$ with the set

$$\mathfrak{U} = \{u \in [L^1(\mu)]^* : ||u|| \leq m\},$$

where $||u||$ is the norm of u in $[L^1(\mu)]^*$. Hence, by Alaoglu's Theorem (see, for example, Folland [4], p. 162), $\mathfrak{U}$ is a metrizable and compact subset of $[L^1(\mu)]^*$ in the corresponding weak* topology. Then, as $(\tilde{u}_n)$ is a sequence in $\mathfrak{U}$, there exists a subsequence $(\tilde{u}_{n_k})$ of $(\tilde{u}_n)$ and a function $\tilde{u}_0 \in \mathfrak{U}$ such that $(\tilde{u}_{n_k})$ converges to $\tilde{u}_0$ in the weak* sense of $[L^1(\mu)]^*$. This implies that $(\tilde{u}_{n_k})$ converges to $\tilde{u}_0$ in $L^\infty(\mu)$. Then, letting $u_0(\cdot) := \tilde{u}_0(\cdot)w(\cdot)$ we have $u_0 \in \mathbb{B}_w(X)$ and (u_{n_k}) converges μ–a.e. to u_0. □

Lemma 6.4 For each $x \in X$ it holds that, as $n \to \infty$,

$$(6) \qquad \max_{\lambda\in\mathbb{P}(A(x)),\gamma\in\mathbb{P}(B(x))} \left| \int_X [u_n(y) - u_0(y)]Q(dy|x,\lambda,\gamma) \right| \to 0.$$

Proof: [Nowak [14], pp. 413-414]. Clearly, (6) will follow if we prove that for each $x \in X$

$$(7) \qquad f_n(x) := \max_{a\in A(x),b\in B(x)} \left| \int_X [u_n(y) - u_0(y)]Q(dy|x,a,b) \right| \to 0.$$

Before doing this, note that, for $n \in \mathbb{N}$, we can write "max" in (6) because
$\left|\int_X [u_n(y) - u_0(y)]Q(dy|x,\cdot,\cdot)\right|$ is continuous on the compact set $\mathbb{P}(A(x)) \times \mathbb{P}(B(x))$ (see Hernández–Lerma and Lasserre [6], p. 48, for the continuity, and Remark 2.1(a) for the compactness of $\mathbb{P}(A(x)) \times \mathbb{P}(B(x))$). Now pick an arbitrary $x \in X$, and let $f_n(x)$ be as in (7). For each $n \in \mathbb{N}$, let (a_n, b_n) be a point in $A(x) \times B(x)$ at which the maximum in (7) is attained; such a point exists by Assumption 4.1(a). By the latter assumption we may assume without loss of generality that $a_n \to a_0 \in A(x)$ and $b_n \to b_0 \in B(x)$.

We obviously have

$$\begin{aligned} f_n(x) &= \left| \int_X [u_n(y) - u_0(y)]Q(dy|x,a_n,b_n) \right| \\ &= \left| \int_X [u_n(y) - u_0(y)]Q(dy|x,a_0,b_0) + \int_X u_n(y)Q(dy|x,a_n,b_n) \right. \\ &\quad - \int_X u_n(y)Q(dy|x,a_0,b_0) - \int_X u_0(y)Q(dy|x,a_n,b_n) \\ &\quad \left. + \int_X u_0(y)Q(dy|x,a_0,b_0) \right| \\ &\leq g_n(x) + h_n(x) + k_n(x), \end{aligned}$$

where

$$
\begin{aligned}
g_n(x) &:= \left| \int_X [u_n(y) - u_0(y)] Q(dy|x, a_0, b_0) \right|, \\
h_n(x) &:= \left| \int_X u_n(y) Q(dy|x, a_n, b_n) - \int_X u_n(y) Q(dy|x, a_0, b_0) \right|, \\
k_n(x) &:= \left| \int_X u_0(y) Q(dy|x, a_n, b_n) - \int_X u_0(y) Q(dy|x, a_0, b_0) \right|.
\end{aligned}
$$

Assumptions 4.3 and 4.1(c) imply that

$$
\int_X w(y) z(x, a_0, b_0, y) \mu(dy) = \int_X w(y) Q(dy|x, a_0, b_0) \leq \beta w(x).
$$

Hence, $w(\cdot)z(x, a_0, b_0, \cdot)$ is in $L^1(\mu)$. Therefore, by Assumption 4.3 and because we may assume that $(\tilde{u}_n)$ converges to $\tilde{u}_0$ in the weak* sense of $[L^1(\mu)]^*$ (see the proof of Lemma 6.3), we have

$$
\begin{aligned}
g_n(x) &= \left| \int_X [\tilde{u}_n(y) - \tilde{u}_0(y)] w(y) Q(dy|x, a_0, b_0) \right| \\
&= \left| \int_X [\tilde{u}_n(y) - \tilde{u}_0(y)] w(y) z(x, a_0, b_0, y) \mu(dy) \right| \to 0
\end{aligned}
$$

as $n \to \infty$. Next, we have

$$
\begin{aligned}
h_n(x) &= \left| \int_X \tilde{u}_n(y) w(y) z(x, a_n, b_n, y) \mu(dy) \right. \\
&\qquad \left. - \int_X \tilde{u}_n(y) w(y) z(x, a_0, b_0, y) \mu(dy) \right| \\
&\leq m \int_X |z(x, a_n, b_n, y) - z(x, a_0, b_0, y)| w(y) \mu(dy),
\end{aligned}
$$

and so $h_n(x) \to 0$ by Assumption 4.3. Since $|\tilde{u}_0(\cdot)| \leq m$, we conclude in the same manner that $k_n(x) \to 0$. Thus we have proved that $f_n(x) \to 0$ as $n \to \infty$, i.e. (7) holds. □

Lemma 6.5 For each fixed $u \in \mathbb{B}_w(X)$ the following holds μ–a.e.:

$$
\int_X u(y) Q(dy|x, \phi_{1n}^*(x), \phi_{2n}(x)) \to \int_X u(y) Q(dy|x, \phi_1^*(x), \phi_2(x)).
$$

Proof: Let $u \in \mathbb{B}_w(X)$. We first show that $\int_X u(y)Q_1(dy|\cdot,\cdot) \in \mathfrak{B}_1$. This function is measurable on $\mathbb{K}_A$ (see, for example, Bertsekas and Shreve [1], p. 144), and continuous on $A(x)$ for each x (see Hernández–Lerma and Lasserre [6], p. 48). Also, for $x \in X$ and $a \in A(x)$,

$$\begin{aligned}
\left|\int_X u(y)Q_1(dy|x,a)\right| &\le \int_X |u(y)|Q_1(dy|x,a) \\
&\le ||u||_w \int_X w(y)Q_1(dy|x,a) \\
&\le ||u||_w \Big[\int_X w(y)Q_1(dy|x,a) \\
&\quad + \int_X w(y)Q_2(dy|x,b)\Big] \\
&= ||u||_w \int_X w(y)Q(dy|x,a,b) \qquad \forall b \in B(x) \\
&\le ||u||_w \beta w(x) \qquad \text{by Assumption 4.1(c)},
\end{aligned}$$

and so

$$\max_{a\in A(x)} \left|\int_X u(y)Q_1(dy|x,a)\right| \le \beta||u||_w w(x).$$

Therefore, as $w(\cdot) \in L^1(\mu)$, we get

$$\int_X u(y)Q_1(dy|\cdot,\cdot) \in \mathfrak{B}_1 \quad \text{and} \quad \max_{a\in A(x)}\left[\int_X u(y)Q_1(dy|x,a)\right] \in \mathbb{B}_w(X).$$

Similarly

$$\int_X u(y)Q_2(dy|\cdot,\cdot) \in \mathfrak{B}_2 \quad \text{and} \quad \max_{b\in B(x)}\left[\int_X u(y)Q_2(dy|x,b)\right] \in \mathbb{B}_w(X).$$

Thus, by Lemma 6.1, the proof is complete. □

Lemma 6.6 For μ–almost all $x \in X$ we have

(8)
$$\lim_{n\to\infty}\int_X u_n(y)Q(dy|x,\phi^*_{1n}(x),\phi_{2n}(x)) = \int_X u_0(y)Q(dy|x,\phi^*_1(x),\phi_2(x)).$$

Proof: For any $x \in X$ we have

$$\left|\int_X u_n(y)Q(dy|x,\phi^*_{1n}(x),\phi_{2n}(x)) - \int_X u_0(y)Q(dy|x,\phi^*_1(x),\phi_2(x))\right|$$
$$\le f_n(x) + g_n(x) + h_n(x),$$

where

$$\begin{aligned} f_n(x) &:= \left| \int_X [u_n(y) - u_0(y)] Q(dy|x, \phi_{1n}^*(x), \phi_{2n}(x)) \right|, \\ g_n(x) &:= \left| \int_X [u_n(y) - u_0(y)] Q(dy|x, \phi_1^*(x), \phi_2(x)) \right|, \\ h_n(x) &:= \left| \int_X u_0(y) Q(dy|x, \phi_{1n}^*(x), \phi_{2n}(x)) \right. \\ &\quad \left. - \int_X u_n(y) Q(dy|x, \phi_1^*(x), \phi_2(x)) \right|. \end{aligned}$$

By Lemma 6.4, both $f_n(x)$ and $g_n(x)$ tend to zero as $n \to \infty$. It remains to show that

$$h_n(x) \to 0 \quad \mu\text{–a.e. as} \quad n \to \infty. \tag{9}$$

Lemma 6.5 implies that the following holds μ–a.e.:

$$\int_X u_0(y) Q(dy|x, \phi_{1n}^*(x), \phi_{2n}(x)) \to \int_X u_0(y) Q(dy|x, \phi_1^*(x), \phi_2(x)). \tag{10}$$

Also, the fact that $|u_n(\cdot)| = |\tilde{u}_n(\cdot) w(\cdot)| \leq m w(\cdot)$, together with Assumption 4.1(c), Lemma 6.3 and Lebesgue's Dominated Convergence Theorem imply that, for each $x \in X$,

$$\int_X u_n(y) Q(dy|x, \phi_1^*(x), \phi_2(x)) \to \int_X u_0(y) Q(dy|x, \phi_1^*(x), \phi_2(x)). \tag{11}$$

Thus, (10) and (11) imply (9). □

The preceding lemmas are summarized in the following proposition.

Proposition 6.7 The following equality holds μ–a.e.:

$$u_0(x) = c_1(x, \phi_1^*(x), \phi_2(x)) + \alpha \int_X u_0(y) Q(dy|x, \phi_1^*(x), \phi_2(x)). \tag{12}$$

Proof: By Proposition 5.3(c) and (4) we have, for each $n \in \mathbb{N}$,

$$u_n(x) = c_1(x, \phi_{1n}^*(x), \phi_{2n}(x)) + \alpha \int_X u_n(y) Q(dy|x, \phi_{1n}^*(x), \phi_{2n}(x)). \tag{13}$$

Then, letting $n \to \infty$ in (13), by Lemmas 6.2, 6.3 and 6.6 we obtain (12) μ–a.e. □

Similar arguments yield the following result.

Lemma 6.8 There exists a set $C \in \mathcal{B}(X)$ such that $\mu(C) = 1$, and for each $x \in C$ and $\lambda \in \mathbb{P}(A(x))$ we have

$$\begin{aligned} c_1(x, \lambda, \phi_{2n}(x)) &\to c_1(x, \lambda, \phi_2(x)), \\ \int_X u_n(y) Q(dy|x, \lambda, \phi_{2n}(x)) &\to \int_X u_0(y) Q(dy|x, \lambda, \phi_2(x)). \end{aligned}$$

We also need the following simple fact.

Lemma 6.9 Let $f_n, f : \mathfrak{X} \to \mathbb{R}$ be given functions, with f continuous, where $\mathfrak{X}$ is a compact metrizable space. If $f_n \to f$ pointwise, then

$$\limsup_{n\to\infty} [\inf_x f_n(x)] \le \min_x f(x).$$

Proof: Let $x^* \in \mathfrak{X}$ be such that

$$f(x^*) = \min_x f(x).$$

Thus $f_n(x^*) \to f(x^*)$ implies

$$\limsup_{n\to\infty} [\inf_x f_n(x)] \le \lim_{n\to\infty} f_n(x^*) = \min_x f(x). \ \square$$

Proposition 6.10 The following equality is satisfied μ–a.e.:

$$(14) \quad u_0(x) = \min_{\lambda\in\mathbb{P}(A(x))} \left[c_1(x, \lambda, \phi_2(x)) + \alpha \int_X u_0(y) Q(dy|x, \lambda, \phi_2(x)) \right].$$

Proof: Choose an arbitrary $x \in C$, where C is the set in Lemma 6.8. Define for $\lambda \in \mathbb{P}(A(x))$ and $n \in \mathbb{N}$

$$f_n(\lambda) := c_1(x, \lambda, \phi_{2n}(x)) + \alpha \int_X u_n(y) Q(dy|x, \lambda, \phi_{2n}(x)).$$

The function f_n is continuous on $\mathbb{P}(A(x))$ because so are the functions $c_1(x, \cdot, \phi_{2n}(x))$ and $\int_X u_n(y) Q(dy|x, \cdot, \phi_{2n}(x))$, by Assumption 4.2 and

Lemma 8.3.7 in Hernández-Lerma and Lasserre [6], p. 48, respectively. By the same argument we have that the function

$$f(\lambda) := c_1(x, \lambda, \phi_2(x)) + \alpha \int_X u_0(y) Q(dy|x, \lambda, \phi_2(x)),$$

is continuous on $\mathbb{P}(A(x))$. We also have, by Lemma 6.8, that $f_n \to f$ pointwise. Proposition 5.3(b) imply that

$$u_n(x) = \min_{\lambda \in \mathbb{P}(A(x))} f_n(\lambda).$$

Therefore, because $\mathbb{P}(A(x))$ is compact (see Remark 2.1(a)), we deduce from Lemmas 6.3 and 6.9 that

$$\begin{aligned} u_0(x) &= \lim_{n\to\infty} u_n(x) \\ &= \lim_{n\to\infty} \left[\min_{\lambda \in \mathbb{P}(A(x))} f_n(\lambda) \right] \\ &\leq \min_{\lambda \in \mathbb{P}(A(x))} f(\lambda) \\ &= \min_{\lambda \in \mathbb{P}(A(x))} \left[c_1(x, \lambda, \phi_2(x)) + \alpha \int_X u_0(y) Q(dy|x, \lambda, \phi_2(x)) \right]. \end{aligned}$$

Thus, since $x \in C$ was arbitrary, the following holds μ–a.e.:

$$u_0(x) \leq \min_{\lambda \in \mathbb{P}(A(x))} \left[c_1(x, \lambda, \phi_2(x)) + \alpha \int_X u_0(y) Q(dy|x, \lambda, \phi_2(x)) \right].$$

The reverse inequality follows from Proposition 6.7. □

To conclude the proof of Theorem 4.4 we need to eliminate the qualifier "μ–a.e." in Proposition 6.7 and Proposition 6.10.

Let $D \in \mathcal{B}(X)$ be such that $\mu(D) = 1$ and such that (12) and (14) are satisfied in D. Define $v_0 : X \to \mathbb{R}$ such that $v_0(x) := u_0(x)$ for $x \in D$, whereas for $x \in D^c$ (the complement of D)

$$v_0(x) := \min_{\lambda \in \mathbb{P}(A(x))} \left[c_1(x, \lambda, \phi_2(x)) + \alpha \int_X u_0(y) Q(dy|x, \lambda, \phi_2(x)) \right].$$

Clearly $v_0 \in \mathbb{B}_w(X)$. Since $\mu(D^c) = 0$, by Assumption 4.3 we have

$$Q(D^c|x, a, b) = \int_{D^c} z(x, a, b, y) \mu(dy) = 0 \quad \forall (x, a, b) \in \mathbb{K}.$$

Therefore, for each $(x,a,b)\in\mathbb{K}$,

$$\int_X v_0(y)Q(dy|x,a,b)=\int_X u_0(y)Q(dy|x,a,b). \tag{15}$$

Thus, from (14) and (15) we have, for all $x\in X$,

$$v_0(x)=\min_{\lambda\in\mathbb{P}(A(x))}\left[c_1(x,\lambda,\phi_2(x))+\alpha\int_X v_0(y)Q(dy|x,\lambda,\phi_2(x))\right]. \tag{16}$$

From Proposition 5.3(b) and (16) we obtain

$$v_0(x)=V^1_{\phi_2}(x)\qquad\forall x\in X. \tag{17}$$

Then, it follows from Proposition 5.3(a),(c) that there exists $f_1\in\mathbb{F}_1$ such that

$$v_0(x)=c_1(x,f_1(x),\phi_2(x))+\alpha\int_X v_0(y)Q(dy|x,f_1(x),\phi_2(x))\quad\forall x\in X. \tag{18}$$

Define

$$\psi_1^*(x):=\begin{cases}\phi_1^*(x) & \text{if } x\in D,\\ f_1(x) & \text{if } x\in D^c.\end{cases}$$

Clearly $\psi_1^*(x)=\phi_1^*(x)$ μ–a.e. and $\psi_1^*\in S_1$. By Proposition 6.7, (15) and (18) we have

$$v_0(x)=c_1(x,\psi_1^*(x),\phi_2(x))+\alpha\int_X v_0(y)Q(dy|x,\psi_1^*(x),\phi_2(x))\quad\forall x\in X. \tag{19}$$

As we are identifying two elements in S_1 if they are equal μ–a.e. then all of the results in this section that involve ϕ_1^* hold if we replace ϕ_1^* by ψ_1^*. Hence, by (19) we may assume without loss of generality that

$$v_0(x)=c_1(x,\phi_1^*(x),\phi_2(x))+\alpha\int_X v_0(y)Q(dy|x,\phi_1^*(x),\phi_2(x))\quad\forall x\in X. \tag{20}$$

Therefore (as in (11)), from (20) we get

$$v_0(x)=V^1(\phi_1^*,\phi_2,x)\qquad\forall x\in X. \tag{21}$$

From (17) and (21) we obtain

$$V^1_{\phi_2}(x)=V^1(\phi_1^*,\phi_2,x)\quad\forall x\in X. \tag{22}$$

Thus, by (12), ϕ_1^* *is an optimal response to* ϕ_2. In a similar way it can be seen that ϕ_2^* is an optimal response to ϕ_1. This establishes (5), and so the multifunction τ is upper-semicontinuous. Finally, by Fan's fixed-point theorem (see Theorem 1 in Fan [2]), we conclude that there exists a Nash equilibrium in stationary strategies. This completes the proof of Theorem 4.4. □

7 Concluding remarks

In this work we have imposed an additive structure on the cost functions and the transition law to establish the existence of a Nash equilibrium in stationary strategies. An interesting and challenging open problem is to establish a similar existence result without such an additive structure.

We have dealt with a two-person game for notational convenience. The result can easily be extended to an N–person game, for any finite $N > 2$.

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