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When does a manifold admit a metric with positive scalar curvature? *

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Abstract

The scalar curvature is the weakest geometric invariant in a Riemannian manifold. M. Gromov, B. Lawson Jr. and J. Rosenberg conjectured that a Riemannian manifold admits a metric with positive scalar curvature if and only if certain topological invariant called $\hat{A}$-genus vanishes. This is known as the Gromov-Lawson-Rosenberg conjecture. In this article we explain this conjecture and give a brief survey of some results related to it.

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1 Introduction

Riemannian Geometry is devoted to the study of Riemannian manifolds (M^n, g), that is, differentiable manifolds M^m endowed with a Riemannian metric g. Since the manifold M^n is also a topological manifold, one of the most important problems in Riemannian Geometry is to study which constrains imposes the topology of M^n on the geometry given by the Riemannian metric g. More specifically, one would like to study the relation between some topological invariants of the underlying manifold M^n with the curvature of the Riemannian manifold (M^n, g). In the present paper we shall only consider closed manifolds, i.e., compact manifolds without boundary.

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The curvature tensor can be viewed as a quadratic form Q in the double exterior product of the tangent bundle of M, $\bigwedge^2 T(M)$, the positive definiteness of Q is one of the strongest positivity conditions, for example all compact symmetric spaces have $Q \geq 0$ while $Q > 0$ distinguishes spheres and real projective spaces. The restriction of Q to bivectors in $\bigwedge^2 T(M)$ is the sectional curvature K and twice the sum of the sectional curvatures over all two planes in a tangent space to a point give us the scalar curvature s (see [2]) therefore, we have the following implications:

$$Q > 0 \Rightarrow K > 0 \Rightarrow s > 0.$$

In a Riemannian manifold (M^n, g) the scalar curvature can be built in a certain way out of the first and second derivatives of g, so we can recover s from the metric g. Hence it is natural to ask:

- Given a Riemannian manifold $M = (M^n, g)$. When does M admit a metric with $s > 0$ or $s = 0$ or $s < 0$?

For the case $s \leq 0$, this condition has no topological effect on M by a theorem of Kasdan and Warner [12, 13] which claims the existence of a metric $s \leq 0$ on every manifold of dimension $n \geq 3$ and a theorem of Lohkamp [18] that states that the space $\mathcal{R}^-(M)$ of negative scalar curvature metrics on M is contractible for every closed manifold M^n of dimension $n \geq 3$.

Going back to the case $s > 0$, there are two obvious questions:

1. How can I construct a manifold with a metric with positive scalar curvature?

2. How can I decide if a manifold admit a metric with positive scalar curvature?

For the existence of a metric with positive scalar curvature, one can prove that if a manifold M has a metric with positive scalar curvature then $M \times N$ also has a metric with positive scalar curvature since we can shrink the product metric by a positive factor at every point and then using the fact that both manifolds are compact find a common factor, there are also generalizations (for vector bundles) of this technique, see [33] for details.

Concerning the other question, the way we decide if a metric has positive scalar curvature is using obstructions:

1. **Index Obstructions.** This method is based on the "Bochner-Lichnerowicz-Weitzenbrock formula" which gives a relation between the scalar curvature and the "Dirac operator" (see 2.2) defined by Atiyah-Singer on any Riemannian manifold with a spin structure (see 1.1).

2. **Minimal hypersurface method.** Schoen and Yau proved that if M is a manifold of dimension n with positive scalar curvature then any stable minimal hypersurface N (i.e. N is a local minimum of the area functional) also admits positive scalar curvature.

3. **Seiberg-Witten invariants.** This is an invariant for 4-dimensional manifolds which vanishes if the manifold admits a metric with positive scalar curvature, see [34] for details.

In the present article we shall focus on the Index Obstruction method. For further details on these methods we recommend the survey of Stolz [33].

Let us start considering the dimension of the manifold $n = 2$, in this case, the scalar curvature coincides with the Gaussian curvature and the Gauss-Bonnet formula relates it to the Euler-Poincaré characteristic $\chi(M)$, which is a topological invariant of the 2-manifold M:

$$\chi(M) = (4\pi)^{-1} \int_M s(x) dvol(x).$$

Thus if a 2 dimensional manifold M admits a metric of positive scalar curvature, then $\chi(M) > 0$ and by the classification theorem of 2-manifolds, this implies that $M = S^2$ or $M = \mathbb{R}P^2$ and indeed, these manifolds do admit metrics of positive scalar curvature. Thus $\chi(M) > 0$ if and only if M admits a metric of positive scalar curvature.

The situation is very different in higher dimensions. In dimension $n = 3$ work of Shoen and Yau [29] with the Thurston conjecture [35] (perhaps soon established by Perelman [20, 19]) yields a complete classification of 3-manifolds with positive scalar curvature. For $n = 4$, see comment about Seiberg–Witten invariants above.

We shall concentrate henceforth on the case $n \geq 5$. If one deforms (cut and paste) the manifold, one obtains a manifold that will have a metric of positive scalar curvature, the two common methods are "surgery" and "attaching handles" which are related. Let M be a manifold with boundary ∂M, we recall that a "handle" is the product of two discs $D^k \times D^{n-k}$, the boundary of this "handle" consist of two parts

$S^{k-1}\times D^{n-k}$ and $D^k\times S^{n-k-1}$, given an embedding of $S^{k-1}\times D^{n-k}$ into the boundary ∂M of an n-dimensional manifold M, we can construct a new manifold

$$\tilde{M} = M \bigcup_{S^{k-1}\times D^{n-k}} D^k \times D^{n-k}$$

by taking the disjoint union of M and the "handle" $D^k \times D^{n-k}$ and identifying the points in $S^{k-1} \times D^{n-k}$ with their image in ∂M. We say that $\tilde{M}$ is obtained by attaching a k-handle to M, or $\tilde{M}$ is obtained by surgery (i.e. by removing $S^k\times D^{n-k}$ and replacing it by $D^{k+1}\times S^{n-k-1}$). It is natural to ask whether a metric of positive scalar curvature in M can be extended to a metric of positive scalar curvature in $\tilde{M}$, here the metrics we have in mind are product metrics near the boundary (i.e. a neighborhood of ∂M is isometric to the product of ∂M with an interval). Gromov-Lawson [9] and Shoen-Yau [29] showed (independently) that if M admits a metric of positive scalar curvature, and $n-k$ (the codimension of the surgery/handle) is greater than 2, then $\tilde{M}$ also admits such a metric. It is worth giving some of the flavor involved. Let S^k be an embedded k dimensional sphere in M with trivial normal bundle ν. This means that a tubular neighborhood of S^k has the form $S^k\times D^{m-k}$ and associated boundary $S^k\times S^{m-k-1}$. Shrink the size of the tubular neighborhood. It is possible to deform the original metric on M to a metric which is greater than 0 in a neighborhood the boundary $S^k \times S^{m-k-1}$ in such a way that the new metric still has positive scalar curvature. It is at this point that the assumption that $m-k\geq 3$ is crucial to ensure that the standard metric of the fiber spheres S^{m-k-1} has positive scalar curvature and this dominates as the size of these spheres is shrunk by taking an adiabatic limit. The surgery can be performed; one cuts out the $S^k \times \mathrm{int}D^{m-k}$ and glues in a $D^{k+1} \times S^{m-k-1}$ and preserves the positivity of the scalar curvature, later Gajer [5] extend the result to

Theorem 1.1 *Let M be a manifold with boundary and let g be a metric of positive scalar curvature on M. Assume that $\tilde{M}$ is obtained from M by attaching a handle of codimension ≥ 3. Then g extends to a metric of positive scalar curvature in $\tilde{M}$*

Gromov and Lawson [9] made the important observation that if a manifold M belongs to certain class of manifolds, called spin manifolds, whether it admits a metric of positive scalar curvature depends only on the bordism class of M in a suitable bordism group called $MSpin_n(B\pi)$

with π be the fundamental group of the manifold M. Recall that two manifolds M and N of dimension n are bordant if there exists a manifold W of dimension $n+1$ such that ∂W is the disjoint union $M \bigsqcup N$. For the group $MSpin_n(B\pi)$ the extra structure we need is called spin structure which we explain in the next section.

1.1 Clifford Algebras and Spin Structures

Let $\mathit{Cliff}^{\pm}(n)$ denote the real Clifford algebra on $\mathbb{R}^n$. This is the universal unital algebra generated by $\mathbb{R}^n$ subject to the Clifford commutation relations

$$v * w + w * v = \pm(v, w)1.$$

Let $\mathit{Cliff}^c(n) := \mathit{Cliff}^-(n) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification. Note that $\mathit{Cliff}^-(n) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathit{Cliff}^+(n) \otimes_{\mathbb{R}} \mathbb{C}$ are isomorphic. Let $Pin^{\pm}(n) \subset \mathit{Cliff}^{\pm}(n)$ be the multiplicative subgroup generated by the unit sphere of $\mathbb{R}^n$; i.e.

$$Pin^{\pm}(n) = \{x = v_1 * ... * v_k : |v_i| = 1 \quad \text{for some} \quad k\}.$$

Define the following groups and representations

- Let $Pin^c(n) := Pin^-(n) \times_{\mathbb{Z}_2} S^1$ where we identify (g, λ) and $(-g, -\lambda)$,
- $\det : Pin^c(n) \to S^1$ by $\det(g, \lambda) = \lambda^2$,
- $\chi : Pin^{\pm}(n) \to \mathbb{Z}_2$ by $\chi(v_1 * ... * v_k) = (-1)^k$, and
- $\Psi : Pin^{\pm}(n) \to O(n)$ by $\Psi(x) : w \mapsto \chi(x) x * w * x^{-1}$.
- $Spin(n) = \ker(\chi) \cap Pin^-(n) \approx \ker(\chi) \cap Pin^+(n)$, and
- $Spin^c(n) = Spin(n) \times_{\mathbb{Z}_2} S^1$.

Let $n \geq 3$. Then Ψ defines a surjective group homomorphism from $Spin(n)$ to the orthogonal group $SO(n)$. Since $Spin(n)$ is connected we have that $\pi_1(SO(n)) = \mathbb{Z}_2$, and $\ker(\Psi) = \{\pm 1\} \subset Spin(n)$, we have $Spin(n)$ is the universal covering group of $SO(n)$.

Note that Ψ defines a surjective group homomorphism from $Pin^{\pm}(n)$ to the orthogonal group $O(n)$; this exhibits $Pin^{\pm}(n)$ as a universal covering groups of $O(n)$. Since $O(n)$ is not connected, the universal cover is not uniquely defined as a group, one must decide how to multiply the arc components and $Pin^{\pm}(n)$ are the two possible universal covering groups. We extend χ and Ψ to $Pin^c(n)$ by defining

$$\chi(x,\lambda)=\chi(x) \quad \text{and} \quad \Psi(x,\lambda)=\Psi(x).$$

Let ξ be a real vector bundle of dimension k with an inner product. We say that ξ admits a *pin*$^{\pm}$ or a *pin*^{c} structure if we can lift the transition functions of ξ from the orthogonal group $O(k)$ to the group $Pin^{\pm}(k)$ or $Pin^c(k)$. We say that ξ admits a *spin* or a *spin*^{c} structure if ξ is orientable and if we can lift the transition functions to $Spin(k)$ or $Spin^c(k)$. We say that a manifold M admits such a structure if the tangent bundle $T(M)$ admits this structure.

This condition can be expressed in terms of characteristic classes. Let $w_i(\xi)$ for $i=1,2$ be the first two Stiefel-Whitney classes of ξ. We refer to Giambalvo [6] for the proof of the following results. It shows that we can stabilize; a bundle ξ admits a suitable structure if and only if $\xi\oplus 1$ admits this structure.

Lemma 1.2 *Let ξ be as before.*

- *The bundle ξ admits a spin structure $\iff$ $w_1(\xi)=0$ and $w_2(\xi)=0$.*
- *The bundle ξ admits a spin^c structure $\iff$ $w_1(\xi)=0$ and if $w_2(\xi)$ lifts from $H^2(M;\mathbb{Z}_2)$ to $H^2(M;\mathbb{Z})$.*
- *The bundle ξ admits a pin$^-$ structure $\iff$ $w_2(\xi)=0$.*
- *The bundle ξ admits a pin^c structure $\iff$ $w_2(\xi)$ lifts from $H^2(M;\mathbb{Z}_2)$ to $H^2(M;\mathbb{Z})$.*

For examples of manifolds with these structures, consider $\mathbb{R}P^l$, the real projective manifold of dimension l, since $T(\mathbb{R}P^l)\oplus 1=(l+1)L$, where L is the Hopf bundle, we have:

- $\mathbb{R}P^{4l}$ and $(4l+1)L$ admit *pin*$^+$ structures.
- $\mathbb{R}P^{4l+1}$ and $(4l+2)L$ admit *spin*^{c} structures.
- $\mathbb{R}P^{4l+2}$ and $(4l+3)L$ admit *pin*$^-$ structures.

- $\mathbb{R}P^{4l+3}$ and $(4l+4)L$ admit *spin* structures.
- Other examples of manifolds that admit *spin* structures are: S^n for $n \geq 2$, $\mathbb{C}P^n$ for n odd.

Now with the notion of spin structure we can define the spin bordism groups $MSpin_n(B\pi)$. Two spin manifolds M and N of dimension n are spin bordant if there exist a spin manifold W of dimension $n+1$ such that its boundary is the disjoint union of M and N and the restriction of the spin structure on W coincides with the spin structures on M and N.

2 Non existence of metrics of positive scalar curvature

We saw that the Euler-Poincaré characteristic is the invariant that tell us when a 2-dimensional manifold admits a metric with positive scalar curvature, so we are looking for a generalization of this invariant, the pioneer of the solution for the non-existence of metrics of positive scalar curvature was Lichnerowicz, see [17], his method is based on the "Bochner-Lichnerowicz-Weitzenböck formula". In order to state Lichnerowicz Theorem we need to explain the following concepts:

2.1 Spinor Bundle

Let M be a Riemannian manifold of dimension $n = 2k$, the spinor bundle is a vector bundle $\mathcal{S} \to M$.

$$\mathcal{S} = Spin(M) \times_{Spin(n)} \Delta$$

where Δ is a certain representation of $Spin(n)$ called the spinor representation, which is constructed as follows: identify $Spin(n)$ with a subgroup of units of the Clifford algebra $Cliff(n)$, and Δ is a certain $Cliff^c(n)$-module considered as a representation of $Spin(n)$ in the units of $Cliff^c(n) = Cliff^c(2k)$ which is the algebra $\mathbb{C}(2^k) = \mathcal{M}_{2^k \times 2^k}(\mathbb{C})$ of $2^k \times 2^k$ matrices over $\mathbb{C}$ (see [16]).
Let Δ be $\mathbb{C}^{2^k}$ with the $\mathbb{C}(2^k)$-module structure given by multiplying a $2^k \times 2^k$-matrix by a 2^k-vector. We consider Δ as a module over $Cliff^c(2k)$ and define a $\mathbb{Z}_2$ grading, $\Delta := \Delta^+ \oplus \Delta^-$ where $\Delta^{\pm}$ are the ± 1-eigenspace of the involution given by the multiplication by the complex volume element $\omega_{\mathbb{C}} = \iota^{2k} e_1 \cdots e_{2k}$ in $Cliff^c(n)$, the vectors $\{e_1, \dots e_{2k}\}$ form an

orthonormal basis of $T_x(M)$ for $x \in M$. The main point here is that we have a Clifford multiplication:

$$T(M) \otimes \mathcal{S} \to \mathcal{S}.$$

It is induced by the module multiplication $\mathbb{R}^m \otimes \Delta \subset \mathit{Cliff}(n) \otimes \Delta \to \Delta$, the $\mathbb{Z}_2$-grading in Δ induces one in $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, in particular Clifford multiplication by tangent vectors maps $\mathcal{S}^+$ to $\mathcal{S}^-$ and viceversa. The Levi–Civita connection on $T(M)$ induces a principal connection on the frame bundle $S(M)$, which lifts to a connection on $Spin(n)$, which induces a connection on the associated bundle $\mathcal{S} \to M$, see [16] for details.

2.2 The Dirac Operator

The Dirac operator (see [16]) is a first order elliptic differential operator defined by the following diagram:

(1)
$$\begin{array}{ccc} C^\infty(\mathcal{S}) & \xrightarrow{D} & C^\infty(\mathcal{S}) \\ \downarrow \nabla & & \uparrow \cdot \\ C^\infty(T^*M \otimes \mathcal{S}) & \xrightarrow{\equiv} & C^\infty(TM \otimes \mathcal{S}) \end{array}$$

or in local coordinates by:

$$D(\varphi)(x) = \sum_{i=1}^{m} e_i \cdot \nabla_{e_i}\varphi$$

Here $\{e_1, \ldots, e_m\}$ is an orthonormal basis of the tangent space $T_x(M)$, $\nabla_{e_i}\varphi$ is the covariant derivative in the direction of e_i ($\nabla_{e_i}\varphi \in \mathcal{S}_x$) and $e_i\cdot$ is Clifford multiplication by e_i. Notice that if $\varphi \in C^\infty(\mathcal{S}^+)$ then $\nabla_{e_i}\varphi \in \mathcal{S}_x^+$ and hence $e_i \cdot \nabla_{e_i}\varphi \in \mathcal{S}_x^-$ so $D\varphi \in C^\infty(\mathcal{S}^-)$, i.e.

$$D^\pm : C^\infty(\mathcal{S}^\pm) \to C^\infty(\mathcal{S}^\mp).$$

The remarkable property of the Dirac operator is that if we look only at $D^+ : C^\infty(\mathcal{S}^+) \to C^\infty(\mathcal{S}^-)$, this is an elliptic operator (therefore is a Fredholm operator, see [16]) so we can define its ”Index” as follows:

$$\mathrm{Index} D^+ = \dim\ker D^+ - \dim\ker D^-.$$

The geometric meaning of the Dirac operator is that its square and the scalar curvature are related via the connection Laplacian (“Bochner-Lichnerowicz-Weitzenböck” formula).

$$D^2 = \nabla^*\nabla + \frac{1}{4}s.$$

Where the connection Laplacian is the operator

$$\nabla^*\nabla : C^\infty(E) \to C^\infty(E) \quad \text{defined by} \quad \nabla^*\nabla(\varphi) = -\sum_{i,j=1}^{m} \nabla_{e_i}\nabla_{e_j}\varphi.$$

One can use this formula to compute

$$|D^2\varphi|^2_{L^2} = |\nabla\varphi|^2_{L^2} + \frac{1}{4}\int_M s(\varphi,\varphi)dvol.$$

Therefore if the metric in question has positive scalar curvature, then there are no elements in $\ker D$ (harmonic spinors).

2.3 $\hat{A}$-genus

We recall that for an oriented vector bundle $E \to M$, $\hat{A}(E) \in H^*(M,\mathbb{Q})$ given by:

$$\hat{A}(E) = 1 - \frac{1}{24}p_1 + \frac{1}{2^7 \cdot 3^2 \cdot 5}(-4p_2 + 7p_1^2) + \dots$$

where $p_j = p_j(E) \in H^{4j}(M,\mathbb{Z})$ are the Pontryagin classes of E, see [16]. The famous Atiyah-Singer Index Theorem (see [16] for details) identifies $\text{Index}D^+$ with the $\hat{A}$-genus.

If M is a manifold of dimension $m = 4k$ the the $\hat{A}$-genus is:

$$\hat{A}(M) = \langle \hat{A}(TM), [M] \rangle \in \mathbb{Q}.$$

Theorem 2.1 (Lichnerowicz) *Let M be a closed spin manifold of dimension $M = 4k$ which admits a metric of positive scalar curvature, then $\hat{A}(M) = 0$.*

Notice that the assumption of *spin* is very important, consider the following example: $\mathbb{C}P^2 = S^5/S^1$ is a manifold with positive scalar curvature, since it is a Riemannian submersion, and

$$\hat{A}(\mathbb{C}P^2) = -(\frac{1}{8})\text{sign}(\mathbb{C}P^2) \neq 0$$

but $\mathbb{C}P^2$ is not a *spin* manifold, see Lemma 1.2.

3 Gromov-Lawson-Rosenberg Conjecture

If g is a Riemannian metric on M, let $D(M,s,g)$ be the associated Dirac operator defined by the *spin* structure s. We define the $\hat{A}$-genus as follows:

1. If $m \equiv 0 \bmod 4$, decompose $D(M,s,g) = D^+(M,s,g) + D^-(M,s,g)$ and let $\hat{A}(M,s,g) := \dim\ker(D^+(M,s,g)) - \dim\ker(D^-(M,s,g)) \in \mathbb{Z}$; the $D^{\pm}$ are the chiral spin operators.
2. If $m \equiv 1 \bmod 8$, let $\hat{A}(M,s,g) = \dim\ker(D(M,s,g)) \in \mathbb{Z}_2$.
3. If $m \equiv 2 \bmod 8$, let $\hat{A}(M,s,g) = \frac{1}{2}\dim\ker(D(M,s,g)) \in \mathbb{Z}_2$.
4. If $m \not\equiv 0,1,2,4 \bmod 8$, let $\hat{A}(M,s,g) = 0$.

One can use the Atiyah-Singer index theorem to show that $\hat{A}(M,s) = \hat{A}(M,s,g)$ is independent of the metric g, also notice that in dimension 2 the invariant (Euler-Poincaré characteristic $\chi(M)$) is independent of the Riemannian metric and certain index invariant introduced by Hitchin are independent of the Riemannian metric, see [10] for details.
If M is simply connected, the spin structure s is unique and we let $\hat{A}(M) = \hat{A}(M,s)$.

If M admits a metric of positive scalar curvature, the formula of Lichnerowicz [17] shows there are no harmonic spinors; consequently $\hat{A}(M,s) = 0$. In other words, if there exists a spin structure s on M so that $\hat{A}(M,s) \neq 0$, then M does not admit a metric of positive scalar curvature. Gromov and Lawson conjectured that the $\hat{A}$-genus might be the only obstruction to the existence of a metric of positive scalar curvature if the dimension n was at least 5 and if M was a simply connected spin manifold. Stolz used deep homotopy theory to identify the kernel of $\hat{A}(M,s)$, see [31] for details, he established this conjecture by proving:

Theorem 3.1 *If M is a simply connected, closed, spin manifold of dimension $n \geq 5$, then M admits a metric of positive scalar curvature if and only if $\hat{A}(M) = 0$.*

The situation in the non-simply connected setting is quite different. Rosenberg has modified the original conjecture of Lawson and Gromov. Fix a group π. Let M be a connected manifold of dimension $n \geq 5$

with fundamental group π and *spin* universal cover. Rosenberg conjectured that M admits a metric of positive scalar curvature if and only if a generalized equivariant index α_π (see [10, 21]) of the Dirac operator vanishes. For the fundamental groups that we shall be considering, α_π can be expressed in terms of the $\hat{A}$-genus defined above. In general Rosenberg's Index α_π lives in the K-theory of a certain C^*-algebra associated to the fundamental groups of the manifold, but it is not in general a number, see [21, 25, 22] for details.

What about the universal cover of M? Consider a manifold of dimension 9 which is homotopy equivalent to a sphere, call it Σ^9 with $\alpha(\Sigma^9) \neq 0$),(see [10]) take the connected sum of $\mathbb{R}P^7 \times S^2$ and Σ^9, notice that $\mathbb{R}P^7 \times S^2$ is *spin* and $\alpha(\mathbb{R}P^7 \times S^2) = 0$, since $\mathbb{R}P^7 \times S^2$ is zero bordant. Since the manifold $M = (\mathbb{R}P^7 \times S^2)\#\Sigma^9$ is spin bordant to the disjoint union of $\mathbb{R}P^7 \times S^2$ and Σ^9, we have that:

$$\alpha(M) = \alpha((\mathbb{R}P^7 \times S^2)\#\Sigma^9) = \alpha(\mathbb{R}P^7 \times S^2) + \alpha(\Sigma^9) = \alpha(\Sigma^9) \neq 0.$$

So M does not admit a metric with positive scalar curvature but its universal cover $\tilde{M} = (S^7 \times S^2)\#\Sigma^9\#\Sigma^9$ which is diffeomerphic to $S^7 \times S^2$ does admit such a metric. So the question whether a *spin* manifold with finite fundamental group π admits a metric with positive scalar curvature cannot be reduced to the universal covering.

Kwasik and Schultz [14] showed that the Gromov-Lawson-Rosenberg conjecture holds for a finite group π if and only if the conjecture holds for all the Sylow subgroups of π. Thus one can work one prime at a time. The Gromov-Lawson-Rosenberg conjecture has been established in the following cases:

- If π is a spherical space form group and if M is *spin* (Botvinnik, Gilkey and Stolz [4]).
- If $\pi = \mathbb{Z}_p \oplus \mathbb{Z}_p$ and if p is an odd prime (Schultz [28]).
- If π belongs to a short list of infinite fundamental groups including free groups, free abelian groups and fundamental groups of orientable surfaces (Rosenberg & Stolz [23]).

For more information about results concerning the Gromov-Lawson-Rosenberg conjecture, see the article of Joachim and Shick [11].
Note that Schick [26, 27] has shown that this conjecture fails in some

instances so it is crucial to investigate the precise conditions under which the $\hat{A}$-genus carries the full set of obstructions.

The spin bordism groups are too big, so it is useful to reformulate the Gromov-Lawson-Rosenberg conjecture in terms of more manageable groups, which are the connective K-theory groups.

3.1 Connective K-theory

Let KO be the periodic real K-theory spectrum and ko the connective cover of KO. The generalized homology theory associated with ko is called the real connective K-theory. We are interested on the connective K-theory of the classifying space of a group π, $ko_n(B\pi)$.

Let $\mathbb{H}P^2$ be the quaternion projective space with the usual homogeneous metric of positive scalar curvature. Let $\mathbb{H}P^2 \to E \to B$ be a fiber bundle where the transition functions are the group of isometries PSp^3 of $\mathbb{H}P^2$. Since $\mathbb{HP}^2$ is simply connected, the projection $p : E \to B$ induces an isomorphism on the fundamental group. Let $T_n(B\pi)$ be the subgroup of $MSpin_n(B\pi)$ generated by the total space of geometric fibrations with fiber $\mathbb{H}P^2$. Using some work of Jung and deep homotopy theory, Stolz [31] has given the following geometrical characterization of the real connective K-theory groups localized at the special prime 2:

$$ko_n(B\pi)_{(2)} = \{MSpin_n(B\pi)/T_n(B\pi)\}_{(2)}.$$

Let $MSpin_n^+(B\pi)$ be the classes in $MSpin_n(B\pi)$ which can be represented by manifolds which admit metrics of positive scalar curvature. The invariant α_π extends to the bordism groups $MSpin_n(B\pi)$; the formula of Lichnerowicz [17] show that it vanishes on $MSpin_n^+(B\pi)$. One therefore has the following equivalent formulation of the Gromov-Lawson-Rosenberg conjecture, see [31] for details:

Theorem 3.2 *Let π be a finite group, if $n \geq 5$, then the following assertions are equivalent:*

- *Let M be any closed connected spin manifold of dimension n with fundamental group π. Then M admits a metric of positive scalar curvature if and only if $\alpha_\pi(M) = 0$.*
- $MSpin_n^+(B\pi) = \ker(\alpha_\pi) \cap MSpin_n(B\pi)$.

Let $ko_n^+(B\pi)$ be the image of $MSpin_n^+(B\pi)$ in $ko_n(B\pi)$. The Gromov-Lawson-Rosenberg conjecture has the following reformulation in terms of connective K theory:

Theorem 3.3 *Let π be an Abelian 2 group, if $n \geq 5$, then the following assertions are equivalent:*

- *Let M be any closed connected spin manifold of dimension n with fundamental group π. Then M admits a metric of positive scalar curvature if and only if $\alpha_\pi(M) = 0$.*

- $ko_n^+(B\pi) = \ker(\alpha_\pi) \cap ko_n(B\pi)$.

Algebraic topology (spectral sequences) give upper bounds of connective K-theory and using Spectral invariants of the Dirac operator (eta invariant) give geometric generators and lower bounds of connective K-theory, using this approach the conjecture is valid for certain non-orientable manifolds with fundamental group an Abelian 2 group, see [1, 7] for details.

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