Morfismos, Vol. 12, No. 1, 2008, pp. 45–61

A proof of the multiplicative property of the Berezinian *

Manuel Vladimir Vega Blanco

Abstract

The Berezinian is the analogue of the determinant in super-linear algebra. In this paper we give an elementary and explicative proof that the Berezinian is a multiplicative homomorphism. This paper is self-conatined, so we begin with a short introduction to super-algebra, where we study the category of supermodules. We write the matrix representation of super-linear transformations. Then we can define a concept analogous to the determinant, this is the superdeterminant or Berezinian. We finish the paper proving the multiplicative property.

2000 Mathematics Subject Classification: 81R99.
Keywords and phrases: Berezinian, superalgebra, supergeometry, super-calculus.

1 Introduction

Linear superalgebra arises in the study of elementary particles and its fields, specially with the introduction of super-symmetry. In nature we can find two classes of elementary particles, in accordance with the statistics of Einstein-Bose and the statistics of Fermi, the bosons and the fermions, respectively. The fields that represent it have a parity, there are bosonic fields (even) and fermionic fields (odd): the former commute with themselves and with the fermionic fields, while the latter anti-commute with themselves. This fields form an algebra, and with the above non-commutativity property they form an algebraic structure

*Research supported by a CONACyT scholarship. This paper is part of the autor's M. Sc. thesis presented at the Department of Mathematics of CINVESTAV-IPN.

called a which we call a superalgebra. Super-linear algebra is the first stage in the development of super-geometry [1, 2, 3, 4].

The purpose of this paper is to develop super-linear algebra as far as proving in an elementary fashion that the super-determinant or Berezinian[1] satisfies:

$$\mathrm{Ber}(TR) = \mathrm{Ber}(T)\mathrm{Ber}(R).$$

In order to do this we follow an elegant idea sketched in a paragraph of Manin's book [2] filling up all the details. Let us start by introducing some elementary definitions.

2 Rudiments of superlinear algebra

Definition 2.0.1. *A supervector space is a $\mathbb{Z}_2$-graded vector space V over a field $\mathbb{K}$, this means that there are two subspaces V_0 and V_1 of V such that $V = V_0 \oplus V_1$. The elements of $V_0 \cup V_1$ are called* homogeneous*. In particular the elements of V_0 (V_1) are called even (odd). The parity function $p : V_0 \cup V_1 \setminus \{0\} \longrightarrow \mathbb{Z}_2$ over the homogeneous elements is defined by the rule $v \mapsto \alpha$ for each $v \in V_\alpha$. There is a problem with the parity of 0, our convention is that 0 is of any parity. Let V and W be supervector spaces. A linear map $f : V \longrightarrow W$ that is $\mathbb{Z}_2$-degree preserving (namely $p(f(v)) = p(v)$ for each v in V homogeneous) is called a* graded morphism or a superlinear morphism*.*

Thus supervector spaces and the graded morphisms between them naturally form a category, denoted by SVect.

Example 2.0.2. *Consider the vector space $\mathbb{R}^{p+q}$. We have that $\mathbf{R}^p$ is naturally embedded in $\mathbb{R}^{p+q}$ and with its orthogonal complement $\mathbf{R}^q$ in $\mathbb{R}^{p+q}$ forms a supervector space $\mathbb{R}(p|q)$, naturally isomorphic to $\mathbb{R}^{p+q}$, where $\mathbb{R}(p|q)_0 = \mathbb{R}^p$ and $\mathbb{R}^{p|q}_1 = \mathbb{R}^q$.*

Example 2.0.3. *Let $\mathcal{M}(2n;\mathbb{R})$ the vector space of $2n \times 2n$ matrices with entries in $\mathbb{R}$. Note that each matrix in $\mathcal{M}(2n;\mathbb{R})$ can be written as*

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

[1] The Berezinian is named after the russian physicist Felix A. Berezin (1931-1980), and it is fundamental in the theory of integration over super-manifolds also known as Berezin integration.

where each X_{ij} is a $n \times n$ matrix for $i = 1, 2$.

Define $\mathcal{M}(2n;\mathbb{R})_0$ as the matrices of the form

$$\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}$$

and $\mathcal{M}(2n;\mathbb{R})_1$ as the matrices of the form

$$\begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix}.$$

Given this structure $\mathcal{M}(2n;\mathbb{R})$ becomes a supervector space of even (odd) dimension $2n^2$.

As in the case of usual vector spaces it is possible to define a tensor product for supervector spaces. Making this category into a symmetric monoidal category.

Definition 2.1.4. *Let V and W supervector spaces, the* tensor product *of V and W is a supervector space $V \otimes W$ together with a bilinear map $u : V \times W \longrightarrow V \otimes W$ satisfying the following universal property: for each supervector space U and for each bilinear map $f : V \times W \longrightarrow U$ there exists a unique graded morphism $k : V \otimes W \longrightarrow U$ such that $k \circ u = f$. The bilinear map u is called a universal bilinear map. This is equivalent to the existence of a unique k filling up the following commutative diagram:*

(1)
$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[dr, "u"'] & U \\ & V \otimes W \arrow[u, dashed, "k"'] \end{tikzcd}$$

Proposition 2.1.5. *The tensor product is unique up to unique isomorphism. That is, if T is a supervector space and $u' : V \times W \longrightarrow T$ is a graded morphism satisfying the universal property of the previous definition, then $T \cong V \otimes W$ by an unique isomorphism. We say that the unique isomorphism k is canonical or natural.*

Proof. The hypothesis give unique graded morphisms $k : V \otimes W \longrightarrow T$ and $k' : T \longrightarrow V \otimes W$ such that $u' = k \circ u$ and $u = k' \circ u'$. Hence $k' \circ k \circ u = u$ and $k \circ k' \circ u' = u'$. Since the identities $id_{V\otimes W}$ and id_T are unique, then $k' \circ k = id_{V\otimes W}$ and $k \circ k' = id_T$. Whence $T \cong V \otimes W$. □

Proposition 2.1.6. *Let V and W supervector spaces. The tensor product $V \otimes W$ satisfies*

$$(V \otimes W)_k = \bigoplus_{i+j=k} V_i \otimes W_j$$

for each k in $\mathbb{Z}_2$, where $V_i \otimes W_j$ is the usual tensor product of vector spaces.

Proof. Define the map

$$\phi : V \times W \longrightarrow T := \bigoplus_{k \in \mathbb{Z}_2} \bigoplus_{i+j=k} V_i \otimes W_j$$

by $(v,\ w) \mapsto v \otimes w$ on homogeneous elements, and extend it on all $V \times W$ by linearity. Whence ϕ is bilinear. Let U a supervector space and let $f : V \times W \longrightarrow U$ a bilinear map. Define $k : T \longrightarrow U$ by $k(v \otimes w) = f(v,\ w)$ on homogeneous elements, extending by linearity is clear that k is graded morphism and $f = k \circ \phi$, moreover is unique: if $k' : V \otimes W \longrightarrow U$ is other graded morphism such that $f = k' \circ \phi$, must be happen that $k = k'$ on homogeneous elements. Therefore $k = k'$ on T. As a consequence, by the uniqueness up to unique isomorphism of the tensor product we have $T \cong V \otimes W$.$\square$

Proposition 2.1.7. *Let V and W supervector spaces, then $V \otimes W$ exists*

Proof. Consider the set $G = (V_0 \cup V_1) \setminus \{0\} \times (W_0 \cup W_1) \setminus \{0\}$. Let S a supervector space. Define a parity map $p : G \longrightarrow \mathbb{Z}_2$ by $p(v,\ w) = p(v) + p(w)$. Now consider the set $F(G, S, p)$ of functions from G to S that vanishes at each g in G except at finitely many elements. We show that this is a supervector space. In fact, we define the sum by $(f + f')(g) = f(g) + f'(g)$ and the left product by elements in K as $(af)(g) = af(g)$. Thus $F(G, S, p)$ is a vector space. Now for each i in $\mathbb{Z}_2$ set $F(G, S, p)_i$ as the elements in $F(G, p)$ with image into $S_{i-p(g)}$. It is easy to verify that $F(G, S, p)$ satisfies the definition of supervector space.

Define the following sets: H_{sum} as the set that contains the elements of the form $f(v + v', w) - f(v + w) - f(v', w)$ and H_{prod} as the set that contains the elements $f(av, w) - af(v, w)$, $f(v, aw) - f(va, w)$ for all v, v' in V_i, w in W_{1-i} and a in K for each i in $\mathbb{Z}_2$. Let I the space generated by $H_{\text{sum}} \cup H_{\text{prod}}$, write $V \otimes W := F(G, S, p)/I$. The *canonical projection* is the map $\pi : F(G, S, p) \longrightarrow V \otimes W$ defined by $f(v, w) \mapsto [f(v, w)]$.

Where $[f(v,w)]$ is the equivalence class with representant $f(v,w)$. Set $u : F(G,S,p) \longrightarrow V \otimes W$ as

$$f(v,w) \longrightarrow \sum_{i,j \in \mathbb{Z}_2} \pi(f(v_i, w_j)).$$

We have that u is a bilinear map satisfying the universal property of the tensor product. $\square$

The tensor product of supervector spaces satisfies the following properties whose proofs can be found in [4].

Proposition 2.1.8. *Let U, V and W supervector spaces. Then*

$$U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$$

canonically.

Definition 2.1.9. *Let V and W supervector spaces. The* commutativity isomorphism $c_{V,W} : V \otimes W \longrightarrow W \otimes V$ *is defined by $v \otimes w \mapsto (-1)^{p(v)p(w)} w \otimes v$ on homogeneous elements, hence by linear extension can be defined over all $V \otimes W$.*

Proposition 2.1.10. *Let $V_1, \ldots, V_n$ a finite collection of supervector spaces, let k and l two integers such that $1 \leq k < l \leq n$ and let τ the transposition with $\tau(k) = l$,$\tau(l) = k$, $\tau(j) = j$ if $j \neq k$ and $j \neq l$. Then, there exists a canonical isomorphism*

$$\eta : \bigotimes_{1 \leq i \leq n} V_i \longrightarrow \bigotimes_{1 \leq i \leq n} V_{\sigma(i)}$$

such that, on homogeneous elements, η is given by

$$\bigotimes_{1 \leq i \leq n} v_i \longmapsto (-1)^N \bigotimes_{1 \leq i \leq n} v_{\tau(i)}$$

where N is the number of pairs of odd elements such that $i < j$ and $\tau(i) > \tau(j)$.

3 Superalgebras

Definition 3.0.11. *Let $\mathcal{A}$ a supervector space and $\mathbb{K}$ an algebraically closed field (v.g $\mathbb{R}$ or $\mathbb{C}$). A graded bilinear morphism $\mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$; $a \otimes b \mapsto ab$, is called product and $\mathcal{A}$ with this graded morphism is called* *a superalgebra over the field $\mathbb{K}$. The superalgebra $\mathcal{A}$ is associative if $x(yz) = (xy)z$ for all x,y and z in $\mathcal{A}$. A unit is an even element 1 in $\mathcal{A}$ such that $1x = x1 = x$ for each x in $\mathcal{A}$. It is common to reserve the name superalgebra only for an associative superalgebra $\mathcal{A}$ with unity.*

Remark 3.0.12. *Note that a superalgebra $\mathcal{A}$ is not necessary commutative as an usual algebra. Also it is not difficult to show that if the unit exists, then it is unique.*

Definition 3.0.13. *Let $\mathcal{A}$ a superalgebra we say that a supervector space M is called a* left (resp. right) $\mathcal{A}$-module *if there exist a graded morphism that is bilinear and posses a unity element $1 \neq 0$, also called product, $\mathcal{A} \otimes M \longrightarrow M$ (resp. $M \otimes \mathcal{A} \longrightarrow M$). The superalgebra $\mathcal{A}$ is called* supercommutative *if $xy = (-1)^{p(x)p(y)}yx$.*

Remark 3.0.14. *If $\mathcal{A}$ is supercommutative, clearly a left $\mathcal{A}$-module M is also a right $\mathcal{A}$-module. This can be done writing*

$$m \cdot a := (-1)^{p(m)p(a)} a \cdot m$$

for each a in $\mathcal{A}$ and m in M.

Example 3.0.15. *(The exterior algebra of a real vector space with the wedge product). Let V a real n-dimensional vector space. Define the* exterior algebra $\Lambda^*(V)$ *of V as the direct sum of the exterior powers $\Lambda^1(V)$, ..., $\Lambda^n(V)$. Remember that the k-th exterior power $\Lambda^k V$ is defined as the quotient of the k-fold tensor product $V^{\otimes k}$ modulo the ideal $\mathfrak{I}$ generated by the elements of the form $v \otimes v$ with v in V. The wedge product*

$$\wedge : \Lambda^n(V) \times \Lambda^m(V) \longrightarrow \Lambda^{n+m}(V)$$

induces in a natural way a product

$$\wedge : \Lambda^*(V) \otimes \Lambda^*(V) \longrightarrow \Lambda^*(V).$$

A well-know fact is that $\Lambda^*(V) \cong \mathbb{R}[\theta_1, \dots, \theta_n]$ where $\mathbb{R}[\theta_1, \dots, \theta_n]$ is the ring of polynomials in n variables with coefficients in $\mathbb{R}$ and the variables θ_j satisfies $\theta_i\theta_j = -\theta_j\theta_i$ for each $i, j = 1, \dots, n$. Thus a typical element can be written as $\sum_J a_J \theta_{i_1} \cdot \ldots \cdot \theta_{i_k}$ where J is the ordered set $\{1 \leq i_1 < \ldots < i_k \leq n\}$ and k varying between 1 up to n. Such element will be even (odd) iff k is even (odd).

Definition 3.0.16. *Let M and N $\mathcal{A}$-modules. The* tensor product *of M and N respect to A is defined by* $M \otimes_{\mathcal{A}} N := (M \text{ as right module}) \otimes (N \text{ as left module})$. *With this definition we get a new $\mathcal{A}$-module. Also we have the commutativity isomorphism and its definition is analogous to the tensor product of supervector spaces. Namely we can write:*

$$(M \otimes_{\mathcal{A}} N)_\gamma = \bigotimes_{\gamma=\alpha+\beta} M_\alpha \otimes_{\mathcal{A}} N_\beta$$

Definition 3.0.17. *On* SVect *we define the* parity reversing functor $\prod$ *by*

$$(\prod V)_0 := V_1, \; (\prod V)_1 := V_0.$$

4 Supermodules and linear transformations

In this section we study the linear maps between supervector spaces. These can be represented by a matrix with entries in a superalgebra. We shall define the analogous concepts of trace, transpose and bases. In what follows $\mathcal{A}$ denotes a superalgebra over $\mathbb{R}$.

Definition 4.0.18. *Let $\mathcal{A}$ be a superalgebra and let $\{e_1, \ldots, e_p\}$ and $\{o_1, \ldots, o_q\}$ be finite sets. Define*

$$\mathcal{A}^{p|q} := \left\{ \sum_{i=1}^{p} a_i e_i + \sum_{j=1}^{q} b_j o_j | \forall i, j \; a_i, b_j \in \mathcal{A} \right\},$$

we call to $\mathcal{A}^{p|q}$ the standard free supermodule of rank $p|q$.

Remark 4.0.19. *The ranks of the free supermodules are pairs of integers rather that just integers*

Naturally in the case $\mathcal{A} = \mathbb{R}$ we have a real supervector space of finite dimension $p+q$, with even dimension p and odd dimension q. We write $\mathbb{R}^{p|q}$ for the free supermodule of rank $p|q$ on $\mathbb{R}$. A standard free supermodule is $\mathbb{Z}_2$-graded: let $x = \sum_{i=1}^{p} a_i e_i + \sum_{j=1}^{q} b_j o_j$ where the a_i, b_j are in A, we say that the element x is even (odd) when $p(a_i) = 0 (p(a_i) = 1)$ and $p(b_j) = 1$ $(p(b_j) = 0)$ for each i and j. The standard free supermodules over the super algebra $\mathcal{A}$ form a category.

Definition 4.0.20. *Let* $\mathcal{A}^{p|q}$ *and* $\mathcal{A}^{r|s}$ $\mathcal{A}$*-free supermodules. A* morphism of supermodules $T : \mathcal{A}^{p|q} \longrightarrow \mathcal{A}^{r|s}$ *is a usual linear transformation such that exclusively preserve (or exclusively anti-preserve) the parity of the homogeneous elements, that is* $p(T(v)) = p(v)$ (or $p(T(v)) = -p(v)$), *for all* v *in* $\mathcal{A}^{p|q}$. *We denote the morphisms from* $\mathcal{A}^{p|q}$ *to* $\mathcal{A}^{r|s}$ *as* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})$, *if* $p = s$ *and* $q = s$ *we write* $\mathrm{End}(\mathcal{A}^{p|q})$ *for that morphisms and we use the suggestive notation* $\mathrm{GL}(\mathcal{A}^{p|q})$ *for the invertible morphisms in* $\mathrm{End}(\mathcal{A}^{p|q})$. *If* T *is in* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})$ *and preserves (anti-preserves) the parity of the homogeneous elements, then we say that* T *is an* even morphism *(*odd morphism*). Clearly the even morphisms form a real vector space (similarly the odd), we denote the even morphisms by* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})_0$ *and* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})_1$ *for the odd. It is also clear that*

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})_0 \cap \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})_1 = \{0\}$$

Proposition 4.0.21. $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})$ *is a graded vector space.*

Proof Let T in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})$ and choose ordered basis for $\mathcal{A}^{p|q}$ and $\mathcal{A}^{r|s}$ $\mathcal{B} = \{e_1, \ldots, e_p, o_1, \ldots, o_q\}$ and $\mathcal{B}' = \{e'_1, \ldots, e'_r, o'_1, \ldots, o'_s\}$ respectively. For each element X in $\mathcal{B}$ we have that

$$T(x) = \sum_{k=1}^{r} a_k e'_k + \sum_{l=1}^{s} b_k o'_k.$$

Since $a_l = a_{l,0} + a_{l,1}$ and $b_l = b_{l,0} + b_{l,1}$, where $a_{k,0}, b_{l,0}$ are in $\mathcal{A}_0^{r|s}$ and $a_{k,1}, b_{k,1}$ are in $\mathcal{A}_1^{r|s}$ for each $1 \leq k \leq r$ and for each $1 \leq l \leq s$, we can define T_0 and T_1 on each element in $\mathcal{B}$ by

$$T_0(x) := \sum_{k=1}^{r} a_{k,0} e'_k + \sum_{l=1}^{s} b_{l,0} o'_l$$

and

$$T_1(x) := \sum_{k=1}^{r} a_{k,1} e'_l + \sum_{l=1}^{s} b_{l,1} o'_l.$$

It is clear that for each x in $\mathcal{B}$, $T(x) = T_0(x) + T_1(x)$, $p(T_0(x)) = p(x)$ and $p(T_1(x)) = p(x) + 1$, hence T_0 is even and T_1 is odd, and extending by linearity we have that $T(v) = T_0(v) + T_1(v)$, then

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})_0 \oplus \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})_1.\square$$

Let $\mathcal{A}$ a superalgebra, $\mathcal{A}^{p|q}$ and $\mathcal{A}^{r|s}$ standard free supermodules an even morphism $T : A^{p|q} \longrightarrow A^{r|s}$ have a matrix representation

$$X_T = \begin{pmatrix} X_{11} & X_{12} \\ \\ X_{21} & X_{22} \end{pmatrix} \tag{2}$$

where the sizes of X, X_{11}, X_{12}, X_{21} and X_{22} are $(p+q) \times (r+s)$, $r \times p$, $r \times q$, $s \times p$ and $s \times q$ respectively. The matrices X_{ii} are composed by even elements and the other two by odd elements. When the morphism is odd, the matrices in the diagonal X_{ii} are composed by odd elements and the off-diagonal matrices by even elements. An element x in $\mathcal{A}^{p|q}$ can be represented by a column vector

$$u_x = \begin{pmatrix} a_1 \\ \vdots \\ a_p \\ b_1 \\ \vdots \\ b_q \end{pmatrix}$$

then $T(x)$ is represented by the matrix product $X_T u_x$. And by abuse of notation we simply write T for X_T, and x for u_x, hence Tx represents to $X_T u_x$.

The appropriate generalization for the concept of trace of a square matrix in the case when T is in $\mathrm{End}(\mathcal{A}^{p|q})$ is as follows:

Definition 4.0.22. *Let T in $\mathrm{End}(\mathcal{A}^{p|q})$ with matrix representation as in (2), the* supertrace *of T, denoted by $\mathrm{Str}(T)$ is defined by*

$$\mathrm{Str}(T) := \mathrm{tr}(X_{11}) - (-1)^{p(T)}\mathrm{tr}(X_{22})$$

for each homogeneous morphism T in $\mathrm{End}(A^{p|q})$

In the case $q = 0$, we have $\mathrm{Str}(T) = \mathrm{tr}(T)$, the usual trace. It is straightforward that $\mathrm{Str}(T + T') = \mathrm{Str}(T) + \mathrm{Str}(T')$ for each T and T' in $\mathrm{End}(A^{p|q})$.

In ordinary linear algebra when we have a matrix X we can construct its transpose denoted by X^t defining $X^t_{ij} := X_{ji}$. In the super case we need to consider whether the matrix is even or odd. The following definition is useful

Definition 4.0.23. *Let X in* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})$ *with matrix representation, again, as in 2. Then the* graded transpose o supertranspose *of X, denoted by X^t is defined by*

$$T^t = \begin{cases} \begin{pmatrix} X_{11}^t & X_{21}^t \\ -X_{12}^t & X_{21}^t \end{pmatrix}, & T \text{ even;} \\ \begin{pmatrix} X_{11}^t & -X_{12}^t \\ X_{12}^t & X_{22}^t \end{pmatrix}, & T \text{ odd.} \end{cases}$$

An immediate consequence of this definition is that for each X and Y matrices of appropriated dimensions we have that $(XY)^t = (-1)^{p(X)p(Y)} Y^t X^t$. Also if X is in $\mathrm{End}(\mathcal{A}^{p|q})$ we have that $\mathrm{Str}(X) = \mathrm{Str}(X^t)$.

Now we motivate the concept of superdeterminant. As the superdeterminant will be a generalization of the usual determinant it must be satisfies, at least, the following requisites:

1. For each X and Y matrices, $\mathrm{Ber}(XY) = \mathrm{Ber}(X)\mathrm{Ber}(Y)$ that is: the Berezinian is multiplicative.

2. If X is an endomorphism of $\mathbb{R}^{p|0}$, then $\mathrm{Ber}(X) = \det(X)$.

An important difference between the classical determinant and the Berezinian is that while the determinant is defined on any endomorphism, the Berezinian is defined only on invertible endomorphisms. As a first step, to understand the concept, we can define the Berezinian in the simplest case $\mathcal{A} = \mathbb{R}$, each T in $\mathrm{End}(\mathbb{R}^{p|q})$ has a matrix representation

$$\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}.$$

We provide a provisional definition of Ber as

$$\mathrm{Ber}(T) := \det(X_{11}) / \det(X_{22}).$$

In this case if

$$X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \text{ and } Y = \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix}$$

$$\begin{aligned}\mathrm{Ber}(XY) &= \mathrm{Ber}(\begin{pmatrix} X_{11}X'_{11} & 0 \\ 0 & X_{22}X'_{22}\end{pmatrix}) \\ &= \det(X_{11}X'_{11})\det(X_{22}X'_{22})^{-1} \\ &= \frac{\det(X_{11})}{\det(X_{22})}\cdot\frac{\det(X'_{11})}{\det(X'_{22})} \\ &= \mathrm{Ber}(X)\mathrm{Ber}(Y).\end{aligned}$$

The invertible matrices can be nicely characterized as the following result shows:

Lemma 4.0.24. *If T is in* $\mathrm{End}(\mathcal{A}^{p|q})$ *and has a matrix representation as in (2) then X is invertible if and only if X_{11} and X_{22} are invertible matrices over the commutative ring $\mathcal{A}_0$, equivalently,* $\det(X_{11})$ *and* $\det(X_{22})$ *are units of $\mathcal{A}_0$.*

Proof. First we need proof the case when all the odd variables are zero. For this define the ideal J of $\mathcal{A}$ as $J = \mathcal{A}_1 + \mathcal{A}_1^2$, this is the ideal generated by $\mathcal{A}_1$. Then consider the quotient $[\mathcal{A}] = \mathcal{A}/J$. We have a natural quotient map $q : \mathcal{A} \longrightarrow [\mathcal{A}]$ defined by $a \mapsto a + I$, so to each matrix X with entries in $\mathcal{A}$ correspond a matrix $[X]$ with entries the images of entries by q of the entries of X, that is $[X_{ij}] = q(X_{ij})$, then q can be extended to $\mathrm{End}(\mathcal{A}^{p|q})$.

The first step is to prove that X is invertible if and only if $[X]$ is invertible. Is clear that if X is invertible, then $[X]$ is invertible, for $XX^{-1} = I$ implies $[X][X^{-1}] = [I]$. Reciprocally suppose that $[X]$ is invertible, then we can find a matrix Z such that $XZ = I + Y$ where Y is a matrix with Y_{ij} is in J for all i, j. To see that $I + Y$ is invertible note that $Y^r = 0$ for some r integer, that is Y is nilpotent. For, if Y is nilpotent

$$(I + Y)\sum_{i=0}^{r-1}(-1)^i Y^i = I - Y^r = I,$$

where $r = min\{k \in \mathbb{N} | Y^k = 0\}$, hence

$$X^{-1} = Z\sum_{i=0}^{r-1}(-1)^i Y^i.$$

To see that Y is nilpotent note that there are odd elements $o_1, \ldots, o_k$ in $\mathcal{A}$ such that any entry of Y take the form $\sum_i a_i o_i$ for some suitable elements a_i in $\mathcal{A}$, thus when we calculate X^r, we have that each entry of Y^r have the form

$$\sum_{i_1 \dots i_r} a'_{i_1 \dots i_r} o_{i_1} \cdots o_{i_r}$$

then choose r such that the product $o_{i_1} \cdots o_{i_r}$ has at least two equal factors, so it vanishes. As consequence $[X]$ invertible if and only if X is invertible. Observe that if X is an even matrix we have that

$$[X] = \begin{pmatrix} [X_{11}] & 0 \\ 0 & [X_{22}] \end{pmatrix},$$

thus X is invertible if and only if $[X_{11}]$ and $[X_{22}]$ are invertible, and this will be true if and only if X_{11} and X_{22} are invertible. This proves the lemma. □

We are in position to define the Berezinian in the general case, that is when $\mathcal{A}_1 \neq 0$.

Definition 4.0.25. *Let T even in* $\mathrm{GL}(\mathcal{A}^{p|q})$ *that have the matrix representation as in 2. The* Berezinian *of T,denoted by* $\mathrm{Ber}(T)$ *is defined by*

$$\mathrm{Ber}(X) := \det(X_{11} - X_{12}X_{22}^{-1}X_{21}) \det(X_{22})^{-1}.$$

Remark 4.0.26. *Note that* $\mathrm{Ber}(X) = 1$ *for the identity matrix $X_{11} = I_p$, $X_{22} = I_q$, $X_{12} = 0$ and $X_{21} = 0$, the two latter matrices has odd entries.*

In the way to prove the multiplicative property of the Berezinian we require the following lemma, whose easy proof we leave to the reader as an exercise.

Lemma 4.0.27. *For each matrix*

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

we have

$$X = \begin{pmatrix} 1 & X_{12}X_{22}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_{11} - X_{12}X_{22}^{-1}X_{21} & 0 \\ 0 & X_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X_{22}^{-1}X_{21} & 1 \end{pmatrix},$$

when X_{22} is invertible, and

$$X = \begin{pmatrix} 1 & 0 \\ X_{21}X_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} - X_{21}X_{11}^{-1}X_{12} \end{pmatrix} \begin{pmatrix} 1 & X_{11}^{-1}X_{21} \\ 0 & 1 \end{pmatrix},$$

when X_{11} is invertible.

The proof of the multiplicative property of the Berezinian can be divided into two steps. The first one is to prove the property for certain families of matrices, this is the content of lemmas 1 y 2 and, then, show that any matrix can be decomposed in terms of matrices of those families. Let us to begin defining the following subsets of $\mathrm{GL}(\mathcal{A}^{p|q})$:

$$\Delta^1 = \{X \in \mathrm{GL}(\mathcal{A}^{p|q}) | X = \begin{pmatrix} I & 0 \\ X_{21} & I \end{pmatrix}\}$$

$$\Delta^{-1} = \{X \in \mathrm{GL}(\mathcal{A}^{p|q}) | X = \begin{pmatrix} I & X_{12} \\ 0 & I \end{pmatrix}\}$$

$$\Delta^0 = \{X \in \mathrm{GL}(\mathcal{A}^{p|q}) | X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}\},$$

each one of this is closed by the usual multiplication of matrices, as easily one can prove. Then we have:

Lemma 4.0.28. *For each X in Δ^k and Y in Δ^l, where $k, l \in \{-1, 0, 1\}$, we have*

$$\mathrm{Ber}(XY) = \mathrm{Ber}(X)\mathrm{Ber}(Y). \tag{3}$$

Proof. The statement is clear for X and Y in Δ^k with $k \in \{-1, 0, 1\}$ since in the case $k = -1$ if we let

$$X = \begin{pmatrix} I & 0 \\ X_{21} & I \end{pmatrix} \quad and \quad Y = \begin{pmatrix} I & 0 \\ X'_{21} & I \end{pmatrix}$$

we have that

$$\begin{pmatrix} I & 0 \\ X_{21} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ X'_{21} & I' \end{pmatrix} = \begin{pmatrix} I & 0 \\ X_{21} + X'_{21} & I \end{pmatrix},$$

and

$$\begin{aligned} \mathrm{Ber}(\begin{pmatrix} I & 0 \\ X_{21} + X'_{21} & I \end{pmatrix}) &= \det(I)\det(I)^{-1} \\ &= \det(I)\det(I)\det(I)^{-1}\det(I)^{-1} \\ &= [\det(I)\det(I)^{-1}][\det(I)\det(I)^{-1}] \\ &= \mathrm{Ber}(X)\mathrm{Ber}(Y), \end{aligned}$$

the other cases are similar since for each X in Δ^0 $X\Delta^k \subset \Delta^k$. The important case is X in $\Delta^{-1} \cup \Delta^1$, then we have that

$$X = \begin{pmatrix} I & 0 \\ X_{21} & I \end{pmatrix} \text{ and } \quad Y = \begin{pmatrix} I & X'_{12} \\ 0 & I \end{pmatrix},$$

and

$$XY = \begin{pmatrix} I & X'_{12} \\ X_{21} & X_{21}X'_{12} + I \end{pmatrix}$$

hence

$$\begin{aligned} \mathrm{Ber}(XY) &= \mathrm{Ber}\left(\begin{pmatrix} I & X'_{12} \\ X_{21} & X_{21}X'_{12} + I \end{pmatrix} \right) \\ &= \det(I - X'_{12}(X_{21}X'_{12} + I)^{-1}X_{21}) \det((X_{21}X'_{12} + I))^{-1}, \end{aligned}$$

then we need to prove that

$$\det(I - X'_{12}(X_{21}X'_{12} + I)^{-1}X_{21}) = \det((X_{21}X'_{12} + I)).$$

For this suppose that X'_{12} is elementary, this means that there exists a single nonzero entry, namely b, and the others are zero. This b is odd, hence $b^2 = 0$. Therefore $(X_{21}X'_{12})^2 = 0$ and $(X'_{12}X_{21}) = 0$ since each entry is multiple of b. So is clear that $(I + X_{21}X'_{12})^{-1} = I - X_{21}X'_{12}$, as consequence

$$\begin{aligned} I - X'_{12}(I + X_{21}X'_{12})^{-1}X_{21} &= I - X'_{12}(I - X_{21}X'_{12})X_{21} \\ &= I - X'_{12}X_{21} - (X'_{12}X_{21})^2 \\ &= I - X'_{12}X_{21}. \end{aligned}$$

We have that $X'_{12}X_{21}$ is a square matrix, then

$$\det(I - X'_{12}X_{21}) = 1 + \det(-X'_{12}X_{21}) + \mathrm{tr}(-X'_{12}X_{21})$$

but $\det(X'_{12}X_{21}) = 0$ for $b^2 = 0$, hence

$$\det(I - X'_{12}X_{21}) = 1 + \mathrm{tr}(-X'_{12}X_{21}).$$

In the other hand $\det(I + X_{21}X'_{12}) = 1 + \mathrm{tr}(X_{21}X'_{12})$, and $\mathrm{tr}(X_{21}X'_{12}) = -\mathrm{tr}(X'_{12}X_{21}) = \mathrm{tr}(-X'_{12}X_{21})$ for X'_{12} and X_{21} are odd. Thus $\det(I - X'_{12}X_{21}) = \det(I + X_{21}X'_{12})$, hence

$$\det(I - X'_{12}(X_{21}X'_{12} + I)^{-1}X_{21}) \det((X_{21}X'_{12} + I))^{-1} = 1$$

as we want, and this proves the lemma.□

Theorem 4.0.29. *Let T and R in $\mathrm{GL}(\mathcal{A}^{p|q})$ even morphisms such that its matrix representations are X and Y respectively, then*

$$\mathrm{Ber}(TR) = \mathrm{Ber}(T)\mathrm{Ber}(R).$$

Equivalently we can say that the Berezinian is an homomorphism.

Proof. Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \text{ and } Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

that represents to T and R in $\mathrm{GL}(\mathcal{A}^{p|q})$ respectively. Now, by the lemma we have that $X = U_x V_x W_x$ and $Y = U_y V_y W_y$, where U_x, U_y is in Δ^{-1},V_x, V_y is in Δ^0 and W_x, W_y is in Δ^1. Precisely we have

$$\begin{array}{ll} U_x = \begin{pmatrix} I & X_{12}X_{22} \\ 0 & I \end{pmatrix} & U_y = \begin{pmatrix} I & 0 \\ Y_{21}Y_{11}^{-1} & I \end{pmatrix} \\ V_x = \begin{pmatrix} X_{11} - X_{12}X_{22}^{-1}X_{21} & 0 \\ 0 & X_{22} \end{pmatrix} & V_y = \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} - Y_{21}Y_{11}^{-1}Y_{12} \end{pmatrix} \\ W_x = \begin{pmatrix} I & 0 \\ X_{22}^{-1}X_{21} & I \end{pmatrix} & W_y = \begin{pmatrix} I & Y_{11}^{-1}Y_{12} \\ 0 & I \end{pmatrix}. \end{array}$$

By direct calculation we have that

$$\mathrm{Ber}(U_x) = \mathrm{Ber}(W_x) = \mathrm{Ber}(U_y) = \mathrm{Ber}(W_y) = 1,$$

and $\mathrm{Ber}(V_x) = \mathrm{Ber}(T)$, $\mathrm{Ber}(V_y) = \mathrm{Ber}(R)$, hence by the lemma 4.0.28 is clear that

$$\begin{aligned} \mathrm{Ber}(TR) &= \mathrm{Ber}(U_x)\mathrm{Ber}(V_x)\mathrm{Ber}(W_x)\mathrm{Ber}(U_y)\mathrm{Ber}(V_y)\mathrm{Ber}(W_y) \\ &= \mathrm{Ber}(U_xV_xW_x)\mathrm{Ber}(U_yV_yW_y) \\ &= \mathrm{Ber}(T)\mathrm{Ber}(R).\square \end{aligned}$$

Corollary 4.0.30. *Let $\mathbb{R}^{p|q}$, $\mathrm{Ber} \in \mathrm{Hom}_{\mathcal{A}}(\mathrm{GL}(\mathcal{A}^{p|q})_0, \mathcal{A}_0)$ is well defined, that is let*

$$\mathcal{B} = \{e_1, \ldots, e_q, o_1, \ldots, o_q\} \quad \textit{and} \quad \mathcal{B}' = \{e_1', \ldots, e_p', o_1', \ldots, o_q'\}$$

be two ordered basis of $\mathbb{R}^{p|q}$, T in $\mathrm{GL}(\mathcal{A}^{p|q})$; X the matrix representation of T for $\mathcal{B}$ and X' the matrix representation of T for $\mathcal{B}'$, we have that

$$\mathrm{Ber}(X) = \mathrm{Ber}(X').$$

Proof. Each element in $\mathcal{B}'$ can be expressed in terms of the elements of $\mathcal{B}$ as a linear combination in such way that for each $1 \leq i \leq p$ and $1 \leq j \leq q$ we have that

$$e'_i = \sum_{k=1}^{p} a_{i,k}e_k + \sum_{l=1}^{q} c_{i,l}o_l \text{ and } \quad o'_j = \sum_{k=1}^{p} b_{j,k}e_k + \sum_{l=1}^{q} d_{j,l}o_l.$$

Define the matrices A, B, C and D by $A_{ki} := a_{i,k}$, $B_{kj} := b_{j,k}$, $C_{li} := c_{i,l}$ and $D_{lj} := d_{j,l}$ respectively, now define the matrix Y as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

The matrix Y is even and invertible for $\mathcal{B}$ is linearly independent. Hence $X' = PXP^{-1}$, then by theorem 4.0.29

$$\begin{aligned} \mathrm{Ber}(X') &= \mathrm{Ber}(PXP^{-1}) \\ &= \mathrm{Ber}(P)\mathrm{Ber}(X)\mathrm{Ber}(P^{-1}) \\ &= \mathrm{Ber}(PP^{-1})\mathrm{Ber}(X) \\ &= \mathrm{Ber}(I)\mathrm{Ber}(X) \\ &= \mathrm{Ber}(X). \square \end{aligned}$$

When X is an odd matrix its Berezinian can be defined as follows. First suppose that $p = q$, and consider the following matrix

$$\Upsilon = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}$$

where I_p denote the identity of size $p \times p$. Then ΥX is an invertible matrix. Define

$$\mathrm{Ber}(X) := \mathrm{Ber}(\Upsilon X).$$

It is immediate that again $\mathrm{Ber}(X)$ satisfies the multiplicative property.

Acknowledgement

The author express its gratitude to Dr. Ernesto Lupercio Lara for his appreciable guide and valuable suggestions. Also the author wishes to thank to his wife Griselda Ortigoza Alcalá for her patience and unconditional support.

Manuel Vladimir Vega Blanco
Departamento de Matemáticas,
CINVESTAV,
A. Postal 14-740,
07000, México D.F., MÉXICO,
mvega@math.cinvestav.mx

References

[1] Claudio Bartocci, Ugo Bruzzo and Daniel Hernndez-Ruiprez. *The geometry of supermanifolds*, Kluwer Academic Publishers, (1991).

[2] Yuri I. Manin, *Gauge Field Theory and Complex Geometry*, Translated from the Russian by N. Koblitz and J. R. King, Springer-Verlag, Berlin, Heidelberg, (1988).

[3] Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison, and Edward Witten, *Quantum Field Theory and Strings: A Course for Mathematicians*, American Mathematical Society, (1999).

[4] Manuel Vladimir Vega Blanco, *Supermanifolds as superalgebras*, M. Sc. Thesis, Departamento de Matemáticas del CINVESTAV-IPN, México, (2007).