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Motion planning in tori revisited*

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Abstract

The topological complexity (TC) of the complement of a complex hyperplane arrangement, which is either linear generic or else affine in general position, has been computed by Yuzvinsky. This is accomplished by noticing that efficient homotopy models for such spaces are given by skeletons of Cartesian powers of circles. Soon after, Cohen and Pruidze noticed that the topological complexity of the complement of the corresponding redundant subspace arrangement, as well as of right-angled Artin groups, can be obtained by considering general subcomplexes of cartesian powers of higher dimensional spheres. Unfortunately Cohen-Pruidze's TC-calculations are flawed, and our work describes and mends the problems in order to validate the extended applications. In addition, we generalize Farber-Cohen's computation of the topological complexity of oriented surfaces, now to the realm of Rudyak's higher topological complexity.

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1 Introducción

Michael Farber proposed in [5, 6] a topological model to study the continuity instabilities of the motion planning problem in robotics. Following Farber, a *motion planning algorithm* (or *motion planner*) $\mathcal{P} =$

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$\{F_i, s_i\}_{i=1,\dots,k}$ for a space X consists of a collection of k pairwise disjoint subsets F_i of $X \times X$, each admitting a continuous section $s_i : F_i \to X^{[0,1]}$ for the end-points evaluation map $\pi : X^{[0,1]} \to X \times X$, $\pi(\gamma) = (\gamma(0), \gamma(1))$, such that $\{F_i\}_i$ is a covering of $X \times X$ by ENR's. The sets F_i and the maps s_i are respectively called the *local domains* and the *local rules* of $\mathcal{P}$. The motion planner is said to be *optimal* when the number of local domains is minimal possible. The *topological complexity* of X, $\mathrm{TC}(X)$, is one less than the number of local domains in any optimal motion planner for X.

A lower bound for $\mathrm{TC}(X)$ is described in Proposition 1.1 below through the concept of the *zero-divisor cup-length* of X with respect to a cohomology theory with products h^*. An h^*-zero-divisor of X is an element in the kernel of the induced map

$$\pi^*\colon h^*(X \times X) \to h^*(X^{[0,1]}). \tag{1}$$

The h^*-zero-divisor cup-length of X, denoted by $\mathrm{zcl}_{h^*}(X)$, is the maximal number of h^*-zero-divisors whose product in $h^*(X \times X)$ is non-zero. The "zero-divisor" adjective comes from the fact that $\pi : X^{[0,1]} \to X \times X$ is a fibrational substitute for the diagonal map $X \to X \times X$. Thus, if the strong form of the Künneth formula holds for h^*, the kernel of (1) can be identified with the kernel of the cup-product map $h^*(X) \otimes h^*(X) \to h^*(X)$.

Proposition 1.1 ([5, Theorem 7])**.** *The topological complexity of X is bounded from below by the h^*-zero-divisor cup-length of X, i.e.*

$$\mathrm{zcl}_{h^*}(X)) \leq \mathrm{TC}(X).$$

Instead of the usual upper bound for $\mathrm{TC}(X)$ given by homotopy theory (see [7, 8, 11]), the novel ingredient in Yuzvinsky's [16] and Cohen-Pruidze's [4] works relies on the explicit construction of motion planners whose optimality is then guaranteed by Proposition 1.1. The relevant spaces arise as follows. Fix a positive integer k and consider the standard (minimal) cellular structure in the k-dimensional sphere $\mathbb{S} = S^k = e^0 \cup e^k$. Here e^0 is the base point, which we denote by e. Then take the product cell decomposition in

$$\mathbb{S}^n = \underbrace{\mathbb{S} \times \cdots \times \mathbb{S}}_{n \text{ times}} = \bigsqcup_J e_J, \tag{2}$$

where the cells e_J, indexed by subsets $J \subseteq [n] = \{1, \dots, n\}$, are defined by $e_J = \prod_{i=1}^n e^{d_i}$ with $d_i = 0$ if and only if $i \notin J$. Explicitly, $e_J =$

$\{(x_1, \ldots, x_n) \in \mathbb{S}^n \mid x_i = e^0$ if and only if $i \notin J\}$. Cohen and Pruidze's main result is stated next.

Theorem 1.2. *For a subcomplex X of the cell decomposition* (2)*,*

1. $\mathrm{TC}(X) = \frac{2\dim(X)}{k}$ *for even k;*
2. $\mathrm{TC}(X) = \mathrm{zcl}_{H^*}(X)$

$$= \max\{|J| + |K| : J \cap K = \varnothing,\ e_J \text{ and } e_K \text{ cells of } X\}$$

for odd k.

It is illustrative to compare Theorem 1.2 to its Lusternik-Schnirelmann category counterpart (in terms of the polyhedral power notation). Félix and Tanré prove in [10] the equality

$$\mathrm{cat}((S^k, \star)^L) = \mathrm{cat}(S^k)(\dim(L) + 1).$$

Here L is an abstract simplicial complex with vertices in $[n]$, and cat denotes the (reduced) Lusternik-Schnirelmann category of X. For even k, this corresponds to the equality $\mathrm{TC}((S^k, \star)^L) = \mathrm{TC}(S^k)(1 + \dim(L))$ in item 1 of Theorem 1.2. However, for an odd k, the answer

$$\mathrm{TC}((S^k, \star)^L) = \mathrm{zcl}_{H^*}((S^k, \star)^L)$$

in item 2 of Theorem 1.2 has a value which is arbitrarily lower than that in item 1, as we explain next.

Item 2 in Theorem 1.2 yields the calculations in [4, 16] of the topological complexity of complements of complex hyperplane arrangements (either linear generic, or affine in general position), and of Eilenberg-MacLane spaces $K(\pi, 1)$ for π a right-angled Artin group. It is also interesting to notice that, while the value of $\mathrm{TC}(X)$ in item 1 of Theorem 1.2 is maximal possible (see [6, Theorem 5.2]), item 2 in Theorem 1.2 gives instances where the actual value of $\mathrm{TC}(X)$ can be arbitrarily lower than the dimension-vs-connectivity bound. In fact item 2 in Theorem 1.2 implies that the general estimate "cat $\leq$ TC $\leq$ 2 cat" in [5, Theorem 5] can reach any possible combination[1]—besides the standard facts that $\mathrm{TC}(X) = \mathrm{cat}(X)$ for H-spaces ([12, Theorem 1]), and

[1] The authors learned of this fact at Dan Cohen's lecture during the student workshop *Applied Topology at ABACUS: Motion Planning in Robotics*, that took place a week after the 2013 Mathematical Congress of the Americas, in México.

$\mathrm{TC}(X) = 2\,\mathrm{cat}(X)$ for closed simply connected symplectic manifolds ([9, Corollary 3.2]). Indeed, for any positive integers c and t with $c \leq t \leq 2c$, there is a space X with $\mathrm{cat}(X) = c$ and $\mathrm{TC}(X) = t$. In detail, for a fixed non-negative odd integer k, let $X = \mathbb{S}^c \vee \mathbb{S}^{t-c}$. Since $\mathrm{cat}(\mathbb{S}^c) = c \geq t - c = \mathrm{cat}(\mathbb{S}^{t-c})$, we see $\mathrm{cat}(X) = \max\left\{\mathrm{cat}(\mathbb{S}^c), \mathrm{cat}(\mathbb{S}^{t-c})\right\} = c$. On the other hand, item 2 in Theorem 1.2 yields $\mathrm{TC}(X) = t$. (The case $k = 1$ is treated in [14] with different techniques.)

We noticed that the inequality $\mathrm{TC}(X) \leq \frac{2\dim(X)}{k}$ in item 1 of Theorem 1.2 is standard. Cohen and Pruidze assert to have constructed an explicit motion planning algorithm realizing this upper bound, but as described in the next section, their construction is flawed on several fronts. Similar problems hold in item 2 of Theorem 1.2, but in this case the situation is critical because the needed upper bound is not available by other means.

The first main goal of this paper, addressed in the next section, is to fix the problems in [4]. Then, in Sections 3, we compute the higher topological complexity of oriented surfaces.

2 Correction of gaps in Cohen-Pruidze's work

The simplest motion planner on $\mathbb{S}^n$ holds for k odd, assumption which will be in force in this section until further notice. When $n = 1$, the motion planner has two local domains described as follows: Let $F_1 \subset \mathbb{S} \times \mathbb{S}$ and $s_1 : F_1 \to \mathbb{S}^{[0,1]}$ be given by

$$F_1 = \{(x, -x) | x \in \mathbb{S}\}$$

and, for a fixed nowhere zero tangent vector field ν on $\mathbb{S}$, $s_1(x, -x)$ is the path from x to $-x$ at constant speed along the semicircle determined by the tangent vector $\nu(x)$. The second local domain is given by the complement of F_1,

$$F_0 = \mathbb{S} \times \mathbb{S} - F_1,$$

with local rule $s_0 : F_0 \to \mathbb{S}^{[0,1]}$ where $s_0(x, y)$ is the path from x to y at constant speed along the shortest geodesic arc. It is elementary to see that $\mathrm{zcl}_{H^*}(\mathbb{S}) = 1$, so $\mathrm{TC}(\mathbb{S}) = 1$ and the above motion planner is optimal.

The corresponding product motion planner in $\mathbb{S}^n$ (described in [5, Theorems 11 and 13], and simplified in [6, p. 24]) is recalled in Proposition 2.1 below. The needed preliminaries go as follows: For a subset

$I \subset [n]$, let

$$F_I = \{(x,y) \in \mathbb{S}^n \times \mathbb{S}^n |\ x_i = -y_i \text{ iff } i \in I\}$$

and define $s_I : F_I \to (\mathbb{S}^n)^{[0,1]}$ using the maps s_1 and s_0 defined above, namely

$$s_I(x,y) = (t_1(x_1,y_1), \ldots, t_n(x_n,y_n)) \tag{3}$$

where $t_i = s_1$ if $i \in I$, and $t_i = s_0$ if $i \notin I$. (Here and below we use the shorthand $x = (x_1, \ldots x_n)$, $y = (y_1, \ldots, y_n)$, etc.) The sets F_I's are conveniently separated as $\overline{F_I} \cap F_J = \varnothing$ for $I \not\subseteq J$. In particular, $\overline{F_I} \cap F_J = \varnothing = F_I \cap \overline{F_J}$ when $|I| = |J|$ with $I \neq J$. This allows us to set

$$W_j = \bigcup_{|I|=n-j} F_I \cong \bigsqcup_{|I|=n-j} F_I \tag{4}$$

for $j = 0, 1, \ldots, n$, and define a local rule $\sigma_j : W_j \to (\mathbb{S}^n)^{[0,1]}$ by $\sigma_j|_{F_I} = s_I$.

Proposition 2.1 ([6, p. 24])**.** *For k odd, the subsets $W_j \subset \mathbb{S}^n \times \mathbb{S}^n$ and maps $\sigma_j : W_j \to (\mathbb{S}^n)^{[0,1]}$, $j = 0, 1, \ldots, n$, determine an optimal motion planner for $\mathbb{S}^n$, thus* $\mathrm{TC}(\mathbb{S}^n) = n$.

Still assuming k is odd, let X be a subcomplex of the cell decomposition (2) and, for $J \subset [n]$, let $\mathbb{T}_J$ denote the subcomplex of $\mathbb{S}^n$ generated by e_J, i.e. $\mathbb{T}_J = \overline{e_J} = \{x \in \mathbb{S}^n \mid x_i = e^0 \text{ if } i \notin J\}$. If $J \cap K = \varnothing$, $\mathbb{T}_J \cup \mathbb{T}_K$ sits inside $\mathbb{S}^n$ as the wedge union $\mathbb{T}_J \vee \mathbb{T}_K$. Therefore the term on the right of the second item in Theorem 1.2 takes the form

$$z(X) := \max \{\, |J| + |K| \mid J \cap K = \varnothing \text{ and } \mathbb{T}_J \vee \mathbb{T}_K \subseteq X \,\}.$$

Cohen-Pruidze's critical assertion $\mathrm{TC}(X) = z(X)$ in [4, Theorem 3.4] is argued by (i) constructing a motion planner for X with $z(X) + 1$ local domains, and then (ii) showing that the H^*-zero-divisor cup-length of X is at least $z(X)$. Their proof of (ii) is correct and straightforward, but their construction in (i) is flawed. Explicitely, the authors assert that the local rules in the product motion planner for $\mathbb{S}^n$ constructed in Proposition 2.1 restrict to give local rules for any subcomplex X of $\mathbb{S}^n$. But such an assertion is false in most of the cases. We exhibit an explicit (but typical) counterexample (Example 2.2), and then show how the combinatorics of the cell decomposition of X need to be taken into consideration to fix the construction—and, therefore, Cohen-Pruidze's proof of Theorem 1.2.

Example 2.2. Take $n = 2$ and $k = 1$, so $\mathbb{S}^n = T^2$, the 2-torus. Let $X = S^1 \vee S^1$ be the 1-dimensional skeleton in the minimal cell structure of T^2, so X has the cell decomposition $X = e_\varnothing \cup e_{\{1\}} \cup e_{\{2\}}$, and $z(X) = 2$. The local domain decomposition proposed in [4] is

$$X \times X = \bigcup_{j=0}^{2} (X \times X) \cap W_j$$

with local rules given by the restricted sections $\sigma_j(X) = \sigma_{j|W_j \cap (X \times X)}$. Now let us focus attention on the local domain

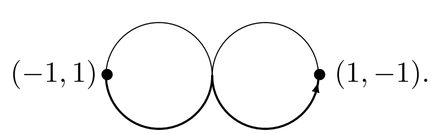
$$(X \times X) \cap W_0 = \{(-1, 1, 1, -1)\} \cup \{(1, -1, -1, 1)\} \tag{5}$$

with corresponding local rule given by

$$\sigma_0(X)(x, -x, -x, x) = (s_1(x, -x), s_1(-x, x)).$$

The authors of [4] claim that the local rule $\sigma_0(X)$ lands in $X^{[0,1]}$— rather than in $(T^2)^{[0,1]}$. But such an assertion is clearly false. In fact, for any $p \in (X \times X) \cap W_0$, the path $\sigma_0(X)(p)$ takes values in X only at $t = 0, 1$.

The assertion in [4] that $(X \times X) \cap W_j = \varnothing$ for $j < n - z(X)$ is true (and easy to verify). As illustrated above, the problem comes with the claim that the restrictions $\sigma_j(X)$ of σ_j to $(X \times X) \cap W_j$, where $(n - z(X) \leq j \leq n)$, give local motion planners in X. The gap noted in Example 2.2 is typical and can be corrected by taking into account the combinatorial properties of the cell structure in X. For instance, in the explicit situation considered in Example 2.2, rather than insisting on performing the motion in both coordinates in a "parallel" way, one should move in two halves; the first part of the motion should be on the coordinate different to 1, keeping the other coordinate fixed. Only when this part of the motion is complete, and we have arrived to the base point $(1, 1)$, it will be safe to move the missing coordinate. For instance, the motion planning algorithm from $(-1, 1)$ to $(1, -1)$ —the first of the two "tasks" represented in (5)— is depicted by the thick curve in



The correct general motion planner is described next.

Fixed motion planner for item 2 *in Theorem* 1.2 *(k odd).* The description simplifies by normalizing $\mathbb{S}$ so to have great semicircles of length $1/2$. For $x, y \in \mathbb{S}$, we let $d(x, y)$ stand for the length of the shortest geodesic between x and y over $\mathbb{S}$, so $d(x, -x) = 1/2$. Likewise, the local rules s_0 and s_1 for $\mathbb{S}$ defined at the beginning of the section need to be adjusted. For $i = 0, 1$ and $(x, y) \in F_i$ with $x \neq y$, set

$$S_i(x,y)(t) = \begin{cases} s_i(x,y)\left(\frac{1}{d(x,y)}t\right), & 0 \leq t \leq d(x,y); \\ y, & d(x,y) \leq t \leq 1 \end{cases}$$

(if $x = y$, $S_i(x, y)(t) = x = y$ for all $t \in [0, 1]$). Thus, S_i reparametrizes s_i so to perform the motion at speed 1, keeping still at the final position once it is reached—which happens at most at time $1/2$. In view of the homeomorphism in (4), it suffices to define a local rule on each $(X \times X) \cap F_I$ taking values in $X^{[0,1]}$. Replace (3) by the map $S_I\colon F_I \cap (X \times X) \to (\mathbb{S}^n)^{[0,1]}$ defined by $S_I(x, y) = (T_1(x_1, y_1), \ldots, T_n(x_n, y_n))$ where $T_i(x_i, y_i) : [0, 1] \to \mathbb{S}$ is the path

$$T_i(x_i,y_i)(t) = \begin{cases} x_i, & 0 \leq t \leq t_{x_i}, \\ \Sigma_i(x_i,y_i)(t - t_{x_i}), & t_{x_i} \leq t \leq 1. \end{cases}$$

Here $t_{x_i} = \frac{1}{2} - d(x_i, 1)$, and $\Sigma_i = S_1$ if $i \in I$ while $\Sigma_i = S_0$ if $i \notin I$.

It is clear that S_I is a continuous section on $F_I \cap (X \times X)$ for the end-points evaluation map $\pi\colon (\mathbb{S}^n)^{[0,1]} \to \mathbb{S}^n \times \mathbb{S}^n$. We only need to check that S_I takes values in $X^{[0,1]}$. With that in mind, note that the motion described by the local rule S_I, from an "initial coordinate" x_i to the corresponding "final coordinate" y_i, is executed according to the relevant instruction S_j ($j \in \{0, 1\}$), except that the movement is delayed a time $t_{x_i} \leq 1/2$. The closer x_i gets to 1, the closer the delaying time t_{x_i} gets to $1/2$. It is then convenient to think of the path $S_I(x, y)$ as happening in two stages. In the first stage ($t \leq 1/2$) all initial coordinates $x_i = 1$ keep still, while the rest of the coordinates (eventually) start traveling to their corresponding final position y_i. Further, at the time the second stage starts ($t = 1/2$), any final coordinate $y_i = 1$ will already have been reached. As a result, S_I *never leaves* X. In more detail: Let $e_J, e_K \subset X$ be cells of X. For $(x, y) = ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \in F_I \cap (e_J \times e_K)$, coordinates corresponding to indexes i not in J, keep their initial position $x_i = 1$ through time $t \leq 1/2$. Therefore $S_I(x, y)[0, 1/2]$ stays within $\mathbb{T}_J \subseteq X$. On the other hand, by construction, $T_i(x_i, y_i)(t) = y_i = 1$ whenever $t \geq 1/2$ and $i \notin K$. Thus, $S_I(x, y)[1/2, 1]$ stays within $\mathbb{T}_K \subseteq X$. □

The motion planning algorithm constructed above is a variation of the one carefully described in [16] for skeletons of the minimal cell decomposition in the n-th Cartesian power of the circle. In the current case, we are considering all possible subcomplexes of the minimal CW-complex $(S^k)^n$ for any odd k.

Proof of Theorem 1.2 *(an additional fixing).* It remains to check that $\mathrm{zcl}_{H^*}(X)$ is bounded from below by $2\dim(X)/k$ when k is even, and by $z(X)$ when k is odd. The case for odd k is addressed correctly in [4, Proposition 3.7], however the case for even k requires some tune-up. We thus assume in this proof that k is even.

Say $\dim(X) = k\ell$. Cohen and Pruidze's argument starts by noticing that X must contain a copy of $\mathbb{S}^\ell$ as a subcomplex, and that the inclusion $\iota\colon \mathbb{S}^\ell \to X$ induces an epimorphism $\iota^*\colon H^*(X) \to H^*(\mathbb{S}^\ell)$. From this, they *infer*

$$\mathrm{zcl}_{H^*}(X) \geq \mathrm{zcl}_{H^*}(\mathbb{S}^\ell), \tag{6}$$

and obtain the desired conclusion from the well-known equality

$$\mathrm{zcl}(H^*(\mathbb{S}^\ell)) = 2\ell.$$

The subtlety here is that the surjectivity of the ring morphism ι^* is not enough to deduce (6). One actually needs to know that each factor in a non-zero product of zero-divisors realizing $\mathrm{zcl}(H^*(\mathbb{S}^\ell))$—as the one described in the proof of Proposition 6.2 in [4]—is the image of a zero-divisor in $\mathrm{zcl}(H^*(X))$. But such a property does hold in the current situation, as Proposition 3.6 in [4] holds true also for k even (cf. [1, Theorem 2.35]). Alternatively, the composition of the inclusion $X \hookrightarrow \mathbb{S}^n$ with the obvious projection $\mathbb{S}^n \to \mathbb{S}^\ell$ gives a retraction ρ for the inclusion $\iota\colon \mathbb{S}^\ell \to X$ and, evidently, both ι and ρ are compatible with diagonal inclusions. □

Remark 2.3. The problem noted in Example 2.2 (for k odd) also holds in [4] for k even. The new issue is more subtle, and this is reflected in part by noticing an additional gap in the proof of [4, Theorem 6.3]. Here we illustrate the new error (and some of the subtleties needed to sort it out), so we assume in this remark that the reader is familiar with the notation set in the final section of [4] (where k is even). For $X =$

$S^2 \vee S^2 \subset S^2 \times S^2$, the first paragraph in the proof of [4, Theorem 6.3] asserts that $(X \times X) \cap W_j = \varnothing$ for $j = 0, 1$. In particular $(A_J \times A_K) \cap F_\alpha$ would have to be empty for $J = \{1\}$, $K = \{2\}$, and $\alpha = (1, 0)$. However $((-e, e), (e, -e))$ clearly lies in the latter intersection. Of course, Cohen and Pruidze's gap in the argument of their Theorem 6.3 comes from their assertion (in the second paragraph of their proof) that there should be some index in $\{1, 2\}$ missing $J \cup K$.

As part of her Ph.D. studies, the second author of this paper has managed to construct an optimal motion planner for any subcomplex of $\mathbb{S}^n$ when k is even. The construction, carried over in more general terms (for Rudyak's higher TC and any parity of k), will be discussed elsewhere.

3 Higher TC of oriented surfaces

In [13] Yuli B. Rudyak introduced the concept of the higher topological complexity of a path connected space X, denoted by $\mathrm{TC}_s(X)$. In this section we extend Farber and Cohen's calculation of $\mathrm{TC}_2(\Sigma_g)$ in [3] to the realm of higher topological complexity. Here Σ_g stands for an oriented surface of genus g.

Let J_s, $s \in \mathbb{N}$, denote the wedge of n closed intervals $[0,1]_i$, $i = 1, \ldots, s$, where the zero points $0_i \in [0,1]_i$ are identified. Consider X^s (the s-th cartesian product of X) and X^{J_s}, where X is a path connected space. There is a fibration

$$(7) \qquad e_s : X^{J_s} \longrightarrow X^s, \qquad e_s(f) = (f(1_1), \ldots, f(1_s))$$

where $1_i \in [0,1]_i$. Recall that the s-th topological complexity of X, denoted by $\mathrm{TC}_s(X)$, is defined as the reduced Schwarz genus of e_s. Note that (7) is a fibrational substitute of the iterated diagonal map $d_s^X : X \longrightarrow X^s$. Hence $\mathrm{TC}_s(X)$ coincides with the Schwarz genus of $d_s^X : X \longrightarrow X^s$. Using the iterated diagonal map d_s^X and allowing cohomology with local coefficients we have the following standard definition:

Definition 3.1. *Given a space X and a positive integer n, we denote by $\mathrm{zcl}_s(H^*(X))$ the maximal length of non-zero products of elements in the kernel of the map induced in cohomology by d_s^X. Thus, $\mathrm{zcl}_s(H^*(X))$ is the largest integer m for which there exist cohomology classes $u_i \in H^*(X^s, A_i)$ with $d_s^X(u_i) = 0$, $i = 1, \ldots, m$, and*

$$0 \neq u_1 \otimes \cdots \otimes u_m \in H^*(X^s,\, A_1 \otimes \cdots \otimes A_m).$$

Note that $\mathrm{zcl}_2\,(H^*(X))$ recovers $\mathrm{zcl}_{H^*}(X)$. The following result bounds $\mathrm{TC}_s(X)$ from below by $\mathrm{zcl}_s(H^*(X))$ and from above by a number which involves the homotopy dimension of X, $\mathrm{hdim}(X)$ (the smallest dimension of CW complexes having the homotopy type of X), and the connectivity of X, $\mathrm{conn}(X)$.

Theorem 3.1. *For any path-connected space X we have:*

$$\mathrm{zcl}_s\,(H^*(X)) \le \mathrm{TC}_s(X) \le \frac{s\ \mathrm{hdim}(X)}{\mathrm{conn}(X)+1}$$

For a proof of Theorem 3.1 see [15, Theorems 4 and 5].

Proposition 3.2. *For $g, s \ge 2$, the s-th higher topological complexity of Σ_g is $\mathrm{TC}_s(\Sigma_g) = 2s$.*

This should be compared to the facts that $\mathrm{TC}_s(\Sigma_0) = s$ and

$$\mathrm{TC}_s(\Sigma_1) = 2(s-1)$$

proved in [2, Corollary 3.12] (see also [13, Section 4]). In addition, it should be noted that Proposition 3.2 was also mentioned by Ibai Basabe during his talk at the conference "Applied Algebraic Topology" held at the *Centro Internacional de Encuentros Matemáticos* on July 2014.

Proof of Proposition 3.2. We use cohomology with rational coefficients. Let a_i, b_i, $i = 1, \ldots, g$, be the generators of $H^1(\Sigma_g)$ which satisfy $a_i b_j = a_i a_j = b_i b_j = 0$ for $i \ne j$, $a_i^2 = b_i^2 = 0$ and $a_i b_i = \omega$ for any i, where ω generates $H^2(\Sigma_g)$. Let $H_{\Sigma_g} = H^*(\Sigma_g^{\times s}) = [H^*(\Sigma_g)]^{\otimes s}$. For each $i = 2, \ldots, s$, consider the elements

$$\begin{aligned} \alpha_i &= a_1 \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes a_1 \otimes \cdots \otimes 1, \\ \beta_i &= b_1 \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes b_1 \otimes \cdots \otimes 1, \end{aligned}$$

where the factor a_1 (resp. b_1) in $1 \otimes \cdots \otimes a_1 \otimes \cdots \otimes 1$ (resp. $1 \otimes \cdots \otimes b_1 \otimes \cdots \otimes 1$) appears in the i-th tensor coordinate, and

$$\begin{aligned} \gamma_1 &= a_2 \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes a_2 \otimes 1 \otimes \cdots \otimes 1, \\ \gamma_2 &= b_2 \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes b_2 \otimes 1 \otimes \cdots \otimes 1 \end{aligned}$$

of H_{Σ_g}. These elements lie in the kernel of the cup-product map

$$[H^*(\Sigma_g)]^{\otimes s} \to H^*(\Sigma_g),$$

and satisfy $\gamma_1 \cdot \gamma_2 \cdot \alpha_2 \cdot \beta_2 \cdots \alpha_s \cdot \beta_s = 2\omega \otimes \cdots \otimes \omega \ne 0$. Therefore, $2s \le \mathrm{zcl}_s(H^*(\Sigma_g)) \le \mathrm{TC}_s(\Sigma_g)$. On the other hand, using the upper bound in Theorem 3.1, we get $\mathrm{TC}_s(\Sigma_g) \le 2s$. Thus, $\mathrm{TC}_s(\Sigma_g) = 2s$. $\square$

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