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Toeplitz operators with piecewise quasicontinuous symbols *

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Abstract

For a fixed subset of the unit circle $\partial\mathbb{D}$, $\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we define the algebra PC of piecewise continuous functions in $\partial\mathbb{D} \setminus \Lambda$ with one sided limits at each point $\lambda_k \in \Lambda$. Besides, we let QC stands for the C^*-algebra of quasicontinuous functions on $\partial\mathbb{D}$ defined by D. Sarason in [5]. We define then PQC as the C^*-algebra generated by PC and QC.

$\mathcal{A}^2(\mathbb{D})$ stands for the Bergman space of the unit disk $\mathbb{D}$, that is, the space of square integrable and analytic functions defined on $\mathbb{D}$. Our goal is to describe $\mathcal{T}_{PQC}$, the algebra generated by Toeplitz operators whose symbols are certain extensions of functions in PQC acting on $\mathcal{A}^2(\mathbb{D})$. Of course, a function defined on $\partial\mathbb{D}$ can be extended to the disk in many ways. The more natural extensions are the harmonic and the radial ones. In the paper we describe the algebra $\mathcal{T}_{PQC}$ and we prove that this description does not depend on the extension chosen.

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1 Introduction

We consider the C^*-algebra of quasicontinuous functions QC, which consists of all functions $f : \partial\mathbb{D} \to \mathbb{C}$ such that both f and its complex

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conjugate $\bar{f}$ belong to $H^\infty + C$. Here H^∞ denotes the set (algebra) of boundary functions for bounded analytic functions on the unit disk $\mathbb{D}$, and C stands for the algebra of continuous functions on $\partial\mathbb{D}$. The space QC has two natural extensions to the disk, namely, the radial and the harmonic extension, we denote these extensions by QC_R and QC_H, respectively.

We use $\mathcal{A}^2(\mathbb{D})$ to denote the Bergman space of $L^2(\mathbb{D})$ which consists in all analytic functions. For $\mathcal{A}^2(\mathbb{D}) \subset L^2(\mathbb{D})$, we denote by $B_{\mathbb{D}}$ the Bergman projection $B_{\mathbb{D}} : L^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$. Let $\mathcal{K}$ denote the ideal of compact operators acting on $\mathcal{A}^2(\mathbb{D})$.

Recall that, for a bounded function f on $\mathbb{D}$, the Toeplitz operator T_f acting on $\mathcal{A}^2(\mathbb{D})$ is defined by the formula $T_f(g) = B_{\mathbb{D}}(fg)$. For a linear subespace $\mathcal{A} \subset L^\infty(\mathbb{D})$ we denote by $\mathcal{T}_{\mathcal{A}}$ the (closed) operator algebra generated by Toeplitz operators with defining simbols in $\mathcal{A}$.

In this paper we describe the Calkin algebras $\mathcal{T}_{QC_R}/\mathcal{K}$ and $\mathcal{T}_{QC_H}/\mathcal{K}$. We use the characterization of QC as the set of bounded functions with vanishing mean oscillation to prove that the Calkin algebras $\mathcal{T}_{QC_R}/\mathcal{K}$ and $\mathcal{T}_{QC_H}/\mathcal{K}$ are commutative, moreover $\mathcal{T}_{QC_R} \cong \mathcal{T}_{QC_H}$.

For a finite set of points $\Lambda := \{\lambda_1, \ldots, \lambda_n\}$ of $\partial\mathbb{D}$, we define the space of piecewise continuous functions $PC := PC_\Lambda$ as the algebra of continuous functions on $\partial\mathbb{D} \setminus \Lambda$ with one sided limits at each point $\lambda_k \in \Lambda$. We denote by PQC the C^*-algebra generated by both PC and QC. We use an extension of PQC to the disk and thus define the Toeplitz operator algebra $\mathcal{T}_{PQC} \subset \mathcal{B}(\mathcal{A}^2(\mathbb{D}))$. There are several ways to extend the functions in PQC to $\mathbb{D}$; two of them are: the radial extension, PQC_R, and the harmonic extension, PQC_H. The main goal of this paper is the description of the Calkin algebra $\mathcal{T}_{PQC}/\mathcal{K}$, which is stated in Theorem 3.15. Finally, in Section 4, we prove that the result does not depend on the extension chosen for PQC, that is, $\mathcal{T}_{PQC_R} = \mathcal{T}_{PQC_H}$.

2 Preliminaries

First of all, we set some notation that will be used throughout the paper. Any mathematical symbol not described here will be used in its more common sense, $\|\cdot\|_{\mathcal{A}}$ stands for the norm in the space $\mathcal{A}$. We denote by $\mathbb{D}$ the unit disk and by $\partial\mathbb{D}$ its boundary, the unit circle. The sets $\mathbb{D}$ and $\partial\mathbb{D}$ are endowed with the standard topology and with the Lebesgue measures $dz = dxdy$ and $d\theta$, where the point $z = x + iy$ belongs to $\mathbb{D}$ and $e^{i\theta}$ belongs to $\partial\mathbb{D}$. All the functions in the paper are considered as complex-valued.

This section includes some basic facts about the space of Vanishing Mean Oscillation functions on $\partial\mathbb{D}$, denoted here by VMO. The importance of this space lies in the fact that $QC = VMO \cap L^\infty$ (see [4]). For the convenience of the reader we recall the relevant material from [5] omitting proofs, thus making the exposition self contained.

We define the following spaces of functions on $\partial\mathbb{D}$:

- $L^\infty := L^\infty(\partial\mathbb{D})$ = the algebra of bounded measurable functions $f : \partial\mathbb{D} \to \mathbb{C}$,
- $H^\infty := H^\infty(\partial\mathbb{D})$ = the algebra of radial limits of bounded analytic functions defined on $\mathbb{D}$,
- $C := C(\partial\mathbb{D})$ = the algebra continuous functions on $\partial\mathbb{D}$.

Definition 2.1. [5, page 818] We define the C^*-algebra of quasicontinuous functions QC as the algebra of all bounded functions f on $\partial\mathbb{D}$, such that, both f and its complex conjugate $\bar{f}$ belong to $H^\infty + C$, that is;

$$QC := (H^\infty + C) \cap \left(\overline{H}^\infty + C\right).$$

Some of the statements below are formulated for segments in the real line, but they can also be formulated for arcs in $\partial\mathbb{D}$.

By an interval on $\mathbb{R}$ we always mean a finite interval. The length of the interval I will be denoted by $|I|$.

For $f \in L^1(I)$, the average of f over I is given by

$$I(f) := |I|^{-1} \int_I f(t)dt. \tag{1}$$

For $a > 0$, let

$$M_a(f, I) := \sup_{J \subset I, |J| < a} \frac{1}{|J|} \int_J |f(t) - J(f)|dt.$$

Note that $0 \leq M_a(f, I) \leq M_b(f, I)$ if $a \leq b$, then let $M_0(f, I) := \lim\limits_{a \to 0} M_a(f, I)$.

Definition 2.2. [5, page 81] A function $f \in L^1(I)$ is of *vanishing mean oscillation* in the interval I (or the arc I), if $M_0(f, I) = 0$. The set of all vanishing mean oscillation functions on I is denoted by $VMO(I)$.

In particular, if we replace I by $\partial\mathbb{D}$ in definitions above we get $VMO := VMO(\partial\mathbb{D})$.

A useful characterization of the space VMO is as follows: a function f belongs to VMO if and only if for any $\epsilon > 0$ there exists $\delta > 0$, depending on ϵ, such that

$$|J|^{-2} \int_J \int_J |f(t) - f(s)| ds dt < \epsilon,$$

for every interval $J \subset I$ with $|J| < \delta$.

Definition 2.3. [5, Page 818] Let f be an integrable function defined in an open interval containing the point λ. We define the integral gap of f at λ by

$$\gamma_\lambda(f) := \limsup_{\delta \to 0} \left| \delta^{-1} \int_{\lambda}^{\lambda+\delta} f(t) dt - \delta^{-1} \int_{\lambda-\delta}^{\lambda} f(t) dt \right|.$$

Obviously, if f belongs to $VMO(I)$, then $\gamma_\lambda(f) = 0$ for each interior point λ of I. The most important use of Definition 2.3 is stablished in the following lemma:

Lemma 2.4. *[5, Lemma 2] Let $I = (a, b)$ be an open interval, λ a point of I, and f a function on I which belongs to both $VMO((a, \lambda))$ and $VMO((\lambda, b))$. If $\gamma_\lambda(f) = 0$, then f belongs to $VMO(I)$.*

We denote by $M(QC)$ the space of all non-trivial multiplicative linear functionals on QC, endowed with the Gelfand topology. In the same way define $M(C)$ and identify it with $\partial\mathbb{D}$ via the evaluation functionals. Since C is a subset of QC, every functional in $M(QC)$ induces, by restriction, a functional in C.

Here and subsequently, f_0 denotes the function $f_0(\lambda) = \lambda$. The Stone-Weierstrass theorem implies that f_0 and the function $f(\lambda) = 1$ generate the C^*-algebra of all continuous functions on $\partial\mathbb{D}$.

Definition 2.5. [5, Page 822] For every $\lambda \in \partial\mathbb{D}$, we denote by $M_\lambda(QC)$ the set of all functionals x in $M(QC)$ such that $x(f_0) = \lambda$, that is

$$M_\lambda(QC) := \{x \in M(QC) : x(f_0) = f_0(\lambda) = \lambda\}.$$

In other words, x belongs to $M_\lambda(QC)$ if the restriction of x to the continuous functions is the evaluation functional at the point λ.

Definition 2.6. [5, Page 822] We let $M_\lambda^+(QC)$ denote the set of $x \in M_\lambda(QC)$ with the property that $f(x) = 0$ whenever f in QC is a function such that $\lim_{t \to \lambda^+} f(t) = 0$. $M_\lambda^-(QC)$ is defined in an analogous way.

Let f be bounded function on $\partial\mathbb{D}$. The harmonic extension of f to the unit disk is denoted by f_H and is given by the formula

$$f_H(z) := f_H(r,\theta) := \frac{1}{2\pi}\int_{\partial\mathbb{D}} P_r(\theta-\lambda)f(\lambda)d\lambda, \tag{2}$$

where

$$P_r(\theta) := \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2rcos(\theta)+r^2}$$

is the Poisson kernel for the unit disk.

For every point z in $\mathbb{D}$ we define a functional in QC by the following rule: $z(f) = f_H(z)$, so, we consider $\mathbb{D}$ as a subset of the dual space of QC. Under this identification we have that the weak-star closure of $\mathbb{D}$ contains $M(QC)$ [5, Lemma 7].

Lemma 2.7. *Let f be a function in QC which is continuous at the point λ_0. Then $x(f) = f(\lambda_0)$ for every functional x in $M_{\lambda_0}(QC)$.*

Proof. Consider the case where the function f is continuous at λ_0 and such that $f(\lambda_0) = 0$. Let x be a point in $M_{\lambda_0}(QC)$. For $\epsilon > 0$ there is $\delta_0 > 0$ such that $|f(\lambda)| < \epsilon$ for all λ in the arc $V_{\lambda_0} = (\lambda_0 - \delta_0, \lambda_0 + \delta_0)$. The values taken by the Poisson extension of f should be small if we evaluate points in $\mathbb{D}$ of an open disk with center at λ_0, i.e, there is a δ_1 such that $|f_H(z)| < \epsilon/2$ if $\mathrm{dist}(z, \lambda_0) < \delta_1$ and $z \in \mathbb{D}$.

Using $\epsilon_1 = \min\{\delta_0, \delta_1, \epsilon\}$ we construct a neighbourhood V_x in QC^* with parameters f, f_0, ϵ_1.

By Lemma 7 in [5], there is a z in $\mathbb{D}$ such that $z \in V_x$, that is

$$|f_H(z) - x(f)| < \epsilon_1 < \epsilon \quad \text{and} \quad |f_0(z) - f_0(\lambda_0)| = |z - \lambda_0| < \epsilon_1 < \delta_1.$$

This implies that $\mathrm{dist}(z, \lambda_0) \leq \delta_1$ and then $|f_H(z)| < \epsilon/2$.

Now we estimate $x(f)$,

$$|x(f)| \leq |x(f) - f_H(z)| + |f_H(z)| < \epsilon,$$

consequently $x(f) = 0$.

In the general case, when $f(\lambda_0) \neq 0$, we apply the previous argument to the function $g = f - f(\lambda_0)$. For g we obtain $0 = x(g) = x(f) - f(\lambda_0)$ and then $x(f) = f(\lambda_0)$ for all $x \in M_{\lambda_0}(QC)$. □

For $z \neq 0$ in $\mathbb{D}$, we let I_z denote the closed arc of $\partial\mathbb{D}$ whose center is $z/|z|$ and whose length is $2\pi(1 - |z|)$. For completeness, $I_0 = \partial\mathbb{D}$.

Lemma 2.8. *[5, Lemma 5] For f in QC and any positive number ϵ, there is a positive number δ such that $|f_H(z) - I_z(f)| < \epsilon$ whenever $1 - |z| < \delta$.*

The average of a function f over an arc I defines a linear functional on QC. Let us identify each arc I with the "averaging" functional in QC. The set of all these functionals is denoted by $\mathcal{G}$. By Lemma 7 in [5] and Lemma 2.8 we come to the following lemma.

Lemma 2.9. *[5, Page 822] $M(QC)$ is the set of points in the weak-star closure of $\mathcal{G}$ (denoted here by $\overline{\mathcal{G}}^*$) which does not belong to $\mathcal{G}$.*

For $\lambda \in \partial\mathbb{D}$ we denote by $\mathcal{G}_\lambda^0$ the set of all arcs I in $\mathcal{G}$ with center at λ. Let $M_\lambda^0(QC)$ be the set of functionals in $M_\lambda(QC)$ that lie in the weak-star closure of $\mathcal{G}_\lambda^0$. By Lemma 2.8, the set $M_\lambda^0(QC)$ coincides with the set of functionals in $M_\lambda(QC)$ that lie in the weak-star closure of the radius of $\mathbb{D}$ terminating at λ.

In [5], D. Sarason splits the space $M_\lambda(QC)$ into three sets:

$$M_\lambda^+(QC) \setminus M_\lambda^0(QC), \quad M_\lambda^-(QC) \setminus M_\lambda^0(QC) \quad \text{and} \quad M_\lambda^0(QC).$$

These three sets are mutually disjoint due to the next lemma:

Lemma 2.10. *[5, Lemma 8] $M_\lambda^+(QC) \cup M_\lambda^-(QC) = M_\lambda(QC)$ and $M_\lambda^+(QC) \cap M_\lambda^-(QC) = M_\lambda^0(QC)$.*

The result in Lemma 2.10 allows us to draw the maximal ideal space $M(QC)$. We consider the unit circle as the interval $[0, 2\pi]$, where the points 0 and 2π represent the same point. At each point λ in $[0, 2\pi]$ we draw a segment representing the fiber $M_\lambda(QC)$. The segment $M_\lambda(QC)$ is splitted into two parts, the upper part $M_\lambda^+(QC)$ and the lower part $M_\lambda^-(QC)$. Their intersection is $M_\lambda^0(QC)$, the central part of the fiber.

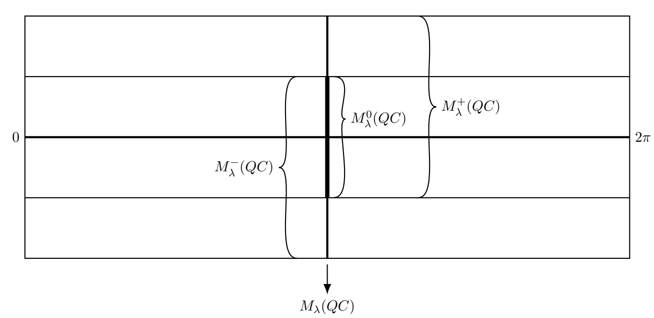

Figure 1: The maximal ideal space of QC.

3 Toeplitz operators with piecewise quasicontinuous symbols on the Bergman space

This section deals with Toeplitz operators with symbols in certain extension of PQC acting on $\mathcal{A}^2(\mathbb{D})$. The C^*-algebra PQC is generated by both, the space PC of piecewise continuous functions, and QC, the space of quasicontinuous functions, both extended from $\partial\mathbb{D}$ to the unit disk $\mathbb{D}$. The main result of this section (Theorem 3.15) describes the Calkin algebra $\hat{\mathcal{T}}_{PQC} := \mathcal{T}_{PQC}/\mathcal{K}$ as the C^*-algebra of continuous sections over a bundle ξ constructed from the operator algebra $\mathcal{T}_{PQC}$.

Definition 3.1. Let $\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a fixed set of n different points on $\partial\mathbb{D}$. Define $PC := PC_\Lambda$ as the set of continuous functions on $\partial\mathbb{D} \setminus \Lambda$ with one sided limits at every point λ_k in Λ. For a function a in PC we set

$$a_k^+ := \lim_{\lambda \to \lambda_k^+} a(\lambda) \quad \text{and} \quad a_k^- := \lim_{\lambda \to \lambda_k^-} a(\lambda),$$

following the standard positive orientation of $\partial\mathbb{D}$.

Definition 3.2. PQC is defined as the C^*-algebra generated by PC and QC.

Our interest is to describe a certain Toeplitz operator algebra acting on the Bergman space $\mathcal{A}^2(\mathbb{D})$. For this we need to extend the functions in PQC to the whole disk. There are two most natural ways of such extensions

- the harmonic extension g_H, given by the Poisson formula 2,
- the radial extension g_R, defined by $g_R(r, \theta) = g(\theta)$.

In this section we use the radial extension, however, we emphasize that the main result of this paper does not depend on the extensions mentioned above (Theorem 4.11).

Recall that the Bergman space $\mathcal{A}^2(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D})$ which consists of all functions analytic in $\mathbb{D}$. Being closed, the space $\mathcal{A}^2(\mathbb{D})$ has the orthogonal projection $B_\mathbb{D} : L^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$, called the Bergman projection. Let $\mathcal{K}$ denote the ideal of compact operators acting on $\mathcal{A}^2(\mathbb{D})$.

Given a function g in $L^\infty(\mathbb{D})$, the Toeplitz operator $T_g : \mathcal{A}^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$ with generating symbol g is defined by $T_g(f) = B_\mathbb{D}(gf)$.

In [7], K. Zhu describes the largest C^*-algebra $Q \subset L^\infty(\mathbb{D})$ such that the map

$$\begin{array}{rcccl} \psi & : & Q & \to & \mathcal{B}(\mathcal{A}^2(\mathbb{D}))/\mathcal{K} \\ & & f & \mapsto & T_f + K, \end{array}$$

is a C^*-algebra homomorphism. This algebra is closely related to QC because both can be described using spaces of vanishing mean oscillation functions.

Definition 3.3. [7, Page 633] Consider

$$\Gamma := \{f \in L^\infty(\mathbb{D}) : T_f T_g - T_{fg} \in \mathcal{K} \text{ for all } g \in L^\infty(\mathbb{D})\}.$$

Let $Q := \bar{\Gamma} \cap \Gamma$.

For z in $\mathbb{D}$, we define

$$S_z := \{w \in \mathbb{D} : \ |w| \geq |z| \ \text{ and } \ |\arg(z) - \arg(w)| \leq 1 - |z|\},$$

The area of S_z, denoted by $|S_z|$, is $\pi(1-|z|)^2(1+|z|)$.

Definition 3.4. [7, Page 621] A function f in $L^1(\mathbb{D})$ belongs to $VMO_\partial(\mathbb{D})$, the space of functions with vanishing mean oscillation near the boundary of $\mathbb{D}$, if

$$\lim_{|z|\to 1^-} \frac{1}{|S_z|} \int_{S_z} \left| f(w) - \frac{1}{|S_z|} \int_{S_z} f(u) dA(u) \right| dA(w) = 0.$$

Theorem 3.5. *[7, Theorem 13] The algebra Q is the set of bounded functions with vanishing mean oscillation near the boundary,* i.e.,

$$Q = VMO_\partial(\mathbb{D}) \cap L^\infty(\mathbb{D}).$$

For the proof we refer the reader to [7].

Lemma 3.6. *Let f be a function in QC. Then, the function f_R belongs to Q.*

Proof. According to Theorem 3.5 and Definition 3.4, we need to estimate

$$\frac{1}{|S_z|} \int_{S_z} \left| f(w) - \frac{1}{|S_z|} \int_{S_z} f(u) dA(u) \right| dA(w). \tag{3}$$

Using polar coordinates we get that this quantity is equal to

$$(4) \qquad \frac{2}{|I_z|^2}\int_{I_z}\int_{I_z}|f(\theta)-f(\phi)|dA(\theta)dA(\phi).$$

If z is close to the boundary, then the measure of $|I_z|$ is small. Hence, the expresion in (4) goes to zero because f is in QC. This implies that the expresion in (3) goes to zero if $|z|$ goes to 1, thus f_R is in Q as required. □

In Lemma 4.5 we prove that the harmonic extension f_H also belongs to Q, but the tools needed for the proof of this fact are not stablished yet.

From now on, and until further notice, we use only the radial extension of a function in PQC. To simplify the notation, we use PQC to denote functions defined on $\partial\mathbb{D}$ as well as radial extensions of such functions. Moreover g will denote both, the function on $\partial\mathbb{D}$ and its radial extension to $\mathbb{D}$.

By $\mathcal{T}_{PQC}$ we denote the C^*-algebra generated by Toeplitz operators with symbols in PQC. We use $\hat{\mathcal{T}}_{PQC}$ to denote the Calkin algebra $\mathcal{T}_{PQC}/\mathcal{K}$. The main goal of this paper is to describe the C^*-algebra $\hat{\mathcal{T}}_{PQC}$.

We use the Douglas-Varella Local Principle (DVLP for short) to describe the C^*-algebra $\hat{\mathcal{T}}_{PQC}$. A complete description of this principle can be found, for example, in [6, Chapter 1].

Let $\mathcal{A}$ be a C^*-algebra with identity, $\mathcal{Z}$ be some of its central C^*-subalgebras with the same identity, T be the compact of maximal ideals of $\mathcal{Z}$. Furthermore, let J_t be the maximal ideal of $\mathcal{Z}$ corresponding to the point $t \in T$, and $J(t) := J_t \cdot \mathcal{A}$ be the two sided closed ideal in the algebra $\mathcal{A}$ generated by J_t .

We define $E_t := \mathcal{A}/J(t)$ as the local algebra at the point t. $[a]_t$ stands for the class of the element a in the quotient algebra E_t. Two elements a, b of $\mathcal{A}$ are say locally equivalents at the point $t \in T$ if $[a]_t = [b]_t$ in E_t.

Using the spaces

$$E := \bigcup_{t\in T} E_t$$

and T, there is a standard procedure to construct the C^*-bundle $\xi = (p, E, T)$, where $p : E \to T$ is a projection such that $p|_{E_t} = t$. This procedure gives to E a compatible topology such that the function $\hat{a} : T \to E$ with $\hat{a}(t) = [a]_t \in E_t$ is continuous for each a in $\mathcal{A}$.

A function $\gamma : T \to E$ is called a section of the C^*-bundle ξ, if $p(\gamma(t)) = t$. Let $\Gamma(\xi)$ denote the C^*-algebra of all continuous sections defined on ξ.

Theorem 3.7 (Douglas-Varela Local Principle)**.** *The C^*-algebra $\mathcal{A}$ is isomorphic and isometric to the C^*-algebra $\Gamma(\xi)$, where ξ is the C^*-bundle constructed from $\mathcal{A}$ and its central algebra $\mathcal{Z}$.*

Lemma 3.6 and the results in [7] imply that the quotient $\hat{\mathcal{T}}_{QC} = \mathcal{T}_{QC}/\mathcal{K}$ is a commutative C^*-subalgebra of $\hat{\mathcal{T}}_{PQC}$. Thus we use $\hat{\mathcal{T}}_{QC}$ as the central algebra needed to apply the DVLP in the description of $\hat{\mathcal{T}}_{PQC}$. The algebra $\hat{\mathcal{T}}_{QC}$ can be identified with QC

$$\hat{\mathcal{T}}_{QC} = \{T_f + \mathcal{K} | f \in QC\},$$

hence we localize by points in $M(QC)$. We first construct the system of ideals parametrized by points x in $M(QC)$.

Definition 3.8. For every point $x \in M(QC)$, we define the maximal ideal of $\hat{\mathcal{T}}_{QC}$, $J_x := \{f \in QC : f(x) = 0\} = \{T_f + \mathcal{K} | f(x) = 0\}$. The ideal $J(x)$ is defined as the set $J_x \cdot \mathcal{T}_{PQC}/\mathcal{K}$.

We set the notation $\hat{\mathcal{T}}_{PQC}(x) := \hat{\mathcal{T}}_{PQC}/J(x)$ for the local algebra at the point x. The class of the element $T_f + \mathcal{K} \in \hat{\mathcal{T}}$ in the quotient algebra $\hat{\mathcal{T}}(x)$ shall be denoted by $[\hat{T}_f]_x$, in order to simplify the notation we say "T_f is locally equivalent..." instead of "the class of $[\hat{T}_f]_x$ is loccaly equivalent..."

Lemma 3.9. *Let f be a function in QC and x a point of $M(QC)$. The Toeplitz operator T_f is locally equivalent, at the point x, to the complex number $f(x)$ (realized as the operator $f(x)I$).*

Proof. Let x be a point in $M(QC)$ and f be a function in QC. The function $f - f(x)$ belongs to $J(x)$, thus, the operator $T_f - T_{f(x)} = T_{f-f(x)}$ is zero in $\hat{\mathcal{T}}_{PQC}(x)$. This means that the operator T_f is locally equivalent to the operator $T_{f(x)} = f(x)I$ and then, the operator T_f is locally equivalent to the complex number $f(x)$. □

Lemma 3.10. *Let x be a point of $M_\lambda(QC)$ with $\lambda \notin \Lambda$ and a be a function in PC. Then, the Toeplitz operator T_a, in the local algebra $\hat{\mathcal{T}}_{PQC}(x)$, is equivalent to the complex number $a(\lambda)$ (realized as the operator $a(\lambda)I$).*

The proof is very similar to the proof of Lemma 3.9 and is omitted. For the case when $x \in M_{\lambda_k}(QC)$, we use Lemma 2.10 to split the fiber $M_{\lambda_k}(QC)$ into three disjoints sets: $M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$, $M^-_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$ and $M^0_{\lambda_k}(QC)$.

Lemma 3.11. *Let x be a point of $M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$ and a be a function in PC. Then, the Toeplitz operator T_a, in the local algebra $\hat{\mathcal{T}}_{PQC}(x)$, is equivalent to the complex number a_k^+ (realized as the operator $a_k^+ I$).*

Proof. Let a be a function for which $a_k^+ = 0$. If x belongs to

$$M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC),$$

then x belongs to $M^+_{\lambda_k}(QC)$ and does not belong to $M^-_{\lambda_k}(QC)$. This implies the existence of a function g in QC such that

$$\lim_{\lambda \to \lambda_k^-} g(\lambda) = 0$$

and $g(x) = 1$.

The product ag is continuous at λ_k and $ag(\lambda_k) = 0$. The difference $T_a - T_{ag}$ can be rewritten as $T_{(1-g)a} = T_{1-g}T_a + K$ where K is a compact operator. Since the function $1 - g$ vanishes at x, T_{1-g} belongs to J_x, and then $T_a - T_{ag}$ belongs to $J(x)$.

From this we conclude that the Toeplitz operator with symbol a is locally equivalent to the Toeplitz operator with symbol ag. At the same time, the Toeplitz operator T_{ag} is locally equivalent to the complex number $0 = ag(\lambda_k)$, hence, the operator T_a is locally equivalent to the complex number $a_k^+ = 0$.

For the general case, if the function a in PC has limit $a_k^+ \neq 0$, we construct the function $b(\lambda) = a(\lambda) - a_k^+$. The function b has lateral limit $b_k^+ = 0$, fulfilling the initial assumption of the proof. By the first part of the proof, the Toeplitz operator $T_b = T_a - a_k^+ I$ is locally equivalent to the complex number 0, thus the Toeplitz operator T_a is locally equivalent to the complex number a_k^+. □

Similarly the following lemma holds:

Lemma 3.12. *Let x be a point of $M^-_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$ and a be a function in PC. Then, the Toeplitz operator T_a, in the local algebra $\hat{\mathcal{T}}_{PQC}(x)$, is equivalent to the complex number a_k^- (realized as the operator $a_k^- I$).*

Now we analize the case when x belongs to central part of the fiber $M_{\lambda_k}(QC)$, *i.e,* $x \in M^0_{\lambda_k}(QC)$. For this case we use some results regarding Toeplitz operators with zero-order homogeneous symbols defined in the upper half plane Π.

We consider $\mathcal{A}^2(\Pi)$ as the Bergman space of Π, that is, the (closed) space of square integrable and analytic functions on Π. Let B_Π stands for the Bergman projection $B_\Pi : L^2(\Pi) \to \mathcal{A}^2(\Pi)$.

Denote by $\mathcal{A}_\infty$ the C*-algebra of bounded mesureable homogeneous functions on Π of zero-order, or functions depending only in the polar coordinate θ. We introduce the Toeplitz operator algebra $\mathcal{T}(\mathcal{A}_\infty)$ generated by all Toeplitz operators

$$T_a : \phi \in \mathcal{A}^2(\Pi) \mapsto B_\Pi(a\phi) \in \mathcal{A}^2(\Pi)$$

with defining symbols $a(r,\theta) = a(\theta) \in \mathcal{A}_\infty$.

Theorem 3.13. *[6, Theorem 7.2.1] For $a = a(\theta) \in \mathcal{A}_\infty$, the Toeplitz operator T_a acting in $\mathcal{A}^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_a I$ acting on $L^2(\mathbb{R})$. The function $\gamma_a(s)$ is given by*

$$\gamma_a(s) = \frac{2s}{1 - e^{-2s\pi}} \int_0^\pi a(\theta) e^{-2s\theta} d\theta.$$

Let $\partial\mathbb{D}^+_k$ denote the upper half of the circunference and $\mathbb{D}^+_k$ the upper half of the disk $\mathbb{D}$ both determined by the diameter passing through λ_k and $-\lambda_k$. Denote by $\partial\mathbb{D}^-_k$ and $\mathbb{D}^-_k$ the complement of $\partial\mathbb{D}^+_k$ and $\mathbb{D}^+_k$, respectively.

Let H be a function in PC with lateral limits H^+_k and H^-_k, we construct the function h in PC such that $h = H^+_k$ in $\partial\mathbb{D}^+$ and $h = H^+_k$ in $\partial\mathbb{D}^-$. The function $H - h$ is continuous at λ_k and $(H - h)(\lambda_k) = 0$. For any point $x \in M^0_{\lambda_k}$, the Toeplitz operator T_{H-h} belongs to $J(x)$ and thus T_H and T_h are locally equivalent at the point x.

The previous paragraph implies that the C*-algebra generated by T_H in $\hat{\mathcal{T}}_{PQC}(x)$ depends only on the values H^+_k and H^-_k. To describe the local algebra at x, we need to analize the algebra generated by the Toeplitz operator which symbol h is constant on both $\partial\mathbb{D}^+$ and $\partial\mathbb{D}^-$. The radial extension of such function h is the function which is constant in $\mathbb{D}^+$ with value A and constant in $\mathbb{D}^-$ with value B for some complex constants A and B.

Let ϕ be a Möbius transformation which sends the upper half plane to the unit disk and such that: $\phi(0) = \lambda_k$, $\phi(i) = 0$ and $\phi(\infty) = -\lambda_k$.

Using the function ϕ we construct a unitary transformation W which sends $L^2(\mathbb{D})$ onto $L^2(\Pi)$. Under the unitary transformation W, the Toeplitz operator with symbol h, acting on $\mathcal{A}^2(\mathbb{D})$, is unitary equivalent to the Toeplitz operator $T_{h(\phi(w))}$ acting on $\mathcal{A}^2(\Pi)$. The corresponding symbol $h(\phi(w))$ is a homogeneous function of zero-order.

By Theorem 3.13, the Toeplitz operator with symbol $h(\phi(w))$, acting on $\mathcal{A}^2(\Pi)$, is unitary equivalent to the multiplication operator by the function $\gamma_{h(\phi(w))}$, acting on $L^2(\mathbb{R})$. Following the unitary equivalences we deduce that T_h is unitary equivalent to $\gamma_{h(\phi(w))}$.

By Corollary 7.2.2 in [6], the function $\gamma_{h(\phi)}(s)$ is continuous in $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, the two point compactification of $\mathbb{R}$; furthermore, for the function $h = \chi_{\mathbb{D}_k^+}$ we have $\gamma_{h(\phi)}(s) = \frac{1-e^{-s\pi}}{1-e^{-2s\pi}} = \frac{1}{1+e^{-2s\pi}}$. The function $\gamma_{h(\phi)}(s)$ and the identity function 1 generate the algebra of continuous functions on $\bar{\mathbb{R}}$ [6, Corollary 7.2.6].

Recall that all piecewise constant functions are generated by linear combinations of the identity and the function $\chi_{\mathbb{D}_k^+}$. Thus, using the change of variables

$$t = \frac{1}{1+e^{-2s\pi}},$$

which is a homeomorphism between $[0,1]$ and $\bar{\mathbb{R}}$, we conclude that the local algebra $\hat{\mathcal{T}}_{PQC}(x)$ is isomorphic to $C[0,1]$ for every $x \in M^0_{\lambda_k}(QC)$; further, such isomorphism, denoted here by ψ, acts on the generator $T_{\chi_{\mathbb{D}_k^+}}$ as follows:

$$T_{\chi_{\mathbb{D}_k^+}} \mapsto t.$$

This implies that the Toeplitz operator with symbol a in PC is sent to $C([0,1])$, via ψ, to the function $a_k^-(1-t) + a_k^+ t$. Thus we come to the following lemma.

Lemma 3.14. *If x belongs to $M^0_{\lambda_k}(QC)$, then the local algebra generated by the Toeplitz operators with symbols in PQC is isometric and isomorphic to the algebra of all continuous functions in* $[0,1]$.

With the set $M(QC)$, we construct the C*-bundle

$$\xi_{\mathrm{PQC}} := (p, E, M(QC)).$$

We use the description of the local algebras given by Lemmas 3.10, 3.11, 3.12 and 3.14 to construct the bundle

$$E := \bigcup_{x \in M(QC)} E_x$$

where

- $E_x = \mathbb{C}$, if $x \in M_\lambda(QC)$ with $\lambda \notin \Lambda$,
- $E_x = \mathbb{C}$, if $x \in M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$, $\lambda_k \in \Lambda$,
- $E_x = \mathbb{C}$, if $x \in M^-_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$, $\lambda_k \in \Lambda$,
- $E_x = C([0,1])$, if $x \in M^0_{\lambda_k}(QC)$, $\lambda_k \in \Lambda$.

The function p is the natural projection from E to $M(QC)$.

Let $\Gamma(\xi_{PQC})$ denote the algebra of all continuous sections of the bundle ξ_{PQC}. Applying the DVLP (Theorem 3.7) we get the following theorem:

Theorem 3.15. *The C^*algebra $\hat{\mathcal{T}}_{PQC}$ is isometric and isomorphic to the C^*-algebra of continuous sections over the C^*-bundle ξ_{PQC}.*

As a corollary of Theorem 3.15, the algebra $\hat{\mathcal{T}}_{PQC}$ is commutative, thus there exists a compact space $X = M(\hat{\mathcal{T}}_{PQC})$, such that $\hat{\mathcal{T}}_{PQC} \cong C(X) = C(M(\hat{\mathcal{T}}_{PQC}))$. The compact space $M(\hat{\mathcal{T}}_{PQC})$ can be constructed using the irreducible representations of $\hat{\mathcal{T}}_{PQC}$.

Let $\hat{\partial\mathbb{D}}$ be the set $\partial\mathbb{D}$ cut by the points λ_k of Λ. The pair of points of $\hat{\partial\mathbb{D}}$ which correspond to the point λ_k will be denoted by λ_k^+ and λ_k^-, following the positive orientation of $\partial\mathbb{D}$. Let $I^n := \sqcup_{i=1}^n [0,1]_k$ be the disjoint union of n copies of the interval $[0,1]$.

Denote by Σ the union of $\hat{\partial\mathbb{D}}$ and I_n with the point identification

$$\lambda_k^- \equiv 0_k \qquad \lambda_k^+ \equiv 1_k,$$

where 0_k and 1_k are the boundary points of $[0,1]_k$, $k = 1, \ldots, n$.

Let $M(\hat{\mathcal{T}}_{PQC}) := \bigcup_{\lambda\in\Sigma} M_\lambda(\mathcal{T}_{PQC})$ where each fiber corresponds to

$$\begin{aligned}
M_\lambda(\hat{\mathcal{T}}_{PQC}) :=& M_\lambda(QC) \text{ if } \lambda \in \partial\hat{\mathbb{D}}, && \lambda_k \notin \Lambda\\
M_{\lambda_k^+}(\hat{\mathcal{T}}_{PQC}) :=& \Big(M^+_{\lambda_k}(QC) \setminus M^-_{\lambda_k}(QC)\Big) \cup M^0_{\lambda_k}(QC), && \lambda_k \in \Lambda,\\
M_{\lambda_k^-}(\hat{\mathcal{T}}_{PQC}) :=& \Big(M^-_{\lambda_k}(QC) \setminus M^+_{\lambda_k}(QC)\Big) \cup M^0_{\lambda_k}(QC), && \lambda_k \in \Lambda,\\
M_t(\hat{\mathcal{T}}_{PQC}) :=& M^0_{\lambda_k}(QC) \text{ if } t \in (0,1)_k, && k = 1, \ldots, n.
\end{aligned}$$

With the help of Figure 1, we draw the maximal ideal space for $\hat{\mathcal{T}}_{PQC}$. The idea is to duplicate the set $M^0_\lambda(QC)$ and then connect this two copies by the interval $[0,1]$.

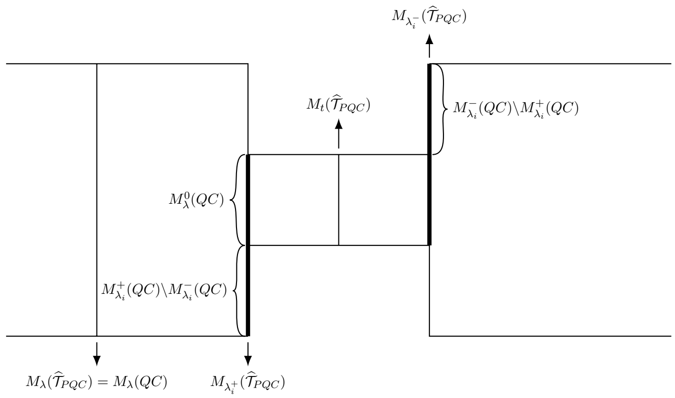

Figure 2: The maximal ideal space of $\widehat{\mathcal{T}}_{PQC}$.

We use the topology of $M(QC)$ in order to describe the topology of $M(\hat{\mathcal{T}}_{PQC})$. We only describe the topology of the fibers $M_{\lambda_k^\pm}(\hat{\mathcal{T}}_{PQC})$ and $M_t(\hat{\mathcal{T}}_{PQC})$, since the topology on the other fibers corresponds to the topology of $M_\lambda(QC)$. For x in $M(QC)$, let $\Omega(x)$ denote the family of open neighbourhoods of x. For $x \in M_\lambda(QC)$ and N in $\Omega(x)$, let $N_\lambda = N \cap M_\lambda(QC)$, and let N_{λ^+} and N_{λ^-} denote the sets of points in N that lie above the semicircles $\partial\mathbb{D}_k^+$ and $\partial\mathbb{D}_k^-$, respectively.

Consider the fiber $M_{\lambda_k^+}(\hat{\mathcal{T}}_{PQC})$. The sets N in $\Omega(x)$ satisfying $N = N_{\lambda_k} \cup N_{\lambda_k^+}$ form neighbourhoods of $x \in M^+_{\lambda_k}(QC)\backslash M^-_{\lambda_k}(QC)$. Let $\Omega_+(x)$ be the set of neighbourhoods N in $\Omega(x)$ satisfying $N = N_\lambda \cup N_{\lambda^+}$. The sets

$$(N_{\lambda_k} \times (1-\epsilon, 1]) \cup N_{\lambda_k^+} \qquad N \in \Omega_+(x), \quad \text{and} \quad 0 < \epsilon < 1,$$

form open neighbourhoods of points x in $M^0_{\lambda_k}(QC)$.

The open neighbourhoods for points in the fiber $M_{\lambda_k^-}(\hat{\mathcal{T}}_{PQC})$ are constructed analogously.

The sets N in $\Omega(x)$ satisfying $N = N_{\lambda_k} \cup N_{\lambda_k^-}$ form neighbourhoods of $x \in M^-_{\lambda_k}(QC) \setminus M^+_{\lambda_k}(QC)$.

The sets

$$(N_{\lambda_k} \times [0, \epsilon)) \cup N_{\lambda_k^-} \qquad N \in \Omega_-(x), \quad \text{and} \quad 0 < \epsilon < 1,$$

form open neighbourhoods of points x in $M^0_{\lambda_k}(QC)$.

Each set $M^0_{\lambda_k}(QC) \times (0,1)$ is open in $M(\hat{\mathcal{T}}_{PQC})$ and carries the product topology.

Theorem 3.16. *Let $X := M(\hat{\mathcal{T}}_{PQC})$ as described above. The algebra $\hat{\mathcal{T}}_{PQC}$ is isomorphic to the algebra of continuous functions over X, the isomorphism acts on the generators in the following way:*

- *For generators which symbols is a function a in PC*

$$\Phi(\hat{T}_a)(x) = \begin{cases} a(\lambda), & \text{if } x \in M_\lambda(\hat{\mathcal{T}}_{PQC}) \text{ with } \lambda \neq \lambda_k; \\ a_k^+, & \text{if } x \in M_{\lambda_k^+}(\hat{\mathcal{T}}_{PQC}); \\ a_k^-, & \text{if } x \in M_{\lambda_k^-}(\hat{\mathcal{T}}_{PQC}); \\ a_k^-(1-t) + a_k^+ t, & \text{if } x \in M_t(\hat{\mathcal{T}}_{PQC}). \end{cases}$$

- *For generators which symbols are functions f in QC, $\Phi(\hat{T}_f)(x) = f(x)$.*

4 Independence of the result on the extension chosen

In this section we prove that the description of the algebra $\hat{\mathcal{T}}_{PQC}$ does not depend of the extension chosen for functions in PQC. Recall that PQC is the algebra generated by PC and QC. This algebra is defined on $\partial\mathbb{D}$ and then extended to the whole disk by two different ways:

- the harmonic extension g_H given by the Poisson formula 2,
- the radial extension g_R, defined by $g_R(r, \theta) = g(\theta)$.

Let a be a function in PC. At the point $x \in M_\lambda(QC)$, for $\lambda \notin \Lambda$; the Toeplitz operator T_{a_R} is locally equivalent to the complex number $a(\lambda)$. The same still true if we use the harmonic extension a_H. For points x in $M_{\lambda_k}^+(QC) \setminus M_{\lambda_k}^0(QC)$ (respectively $M_{\lambda_k}^-(QC) \setminus M_{\lambda_k}^0(QC)$), the Toeplitz operators T_{a_R} and T_{a_H} are equivalent to the number a_k^+ (Respectively a_k^-), and then, the local algebras are the same.

Now, we analize the case when x belongs to $M_{\lambda_k}^0(QC)$. Let $\hat{a}$ be a function in PC, we construct a function a such that $a = \hat{a}_k^+$ in $\partial\mathbb{D}_k^+$ and $a = \hat{a}_k^-$ in $\partial\mathbb{D}_k^-$. The Toeplitz operator with symbol $\hat{a}_H$ is locally equivalent to T_{a_H}.

As in Section 2, we use a Möbius transformation ϕ to generate a unitary operator between $L^2(\mathbb{D})$ and $L^2(\Pi)$. For the function a in PC described earlier, the function $a_H(\phi(z))$ is harmonic in Π and corresponds to the harmonic extension of $a(\phi(t))$.

The harmonic extension of $a(\phi(t))$ is $a_H^\Pi := \frac{\theta}{\pi}(a_k^- - a_k^+) - a_k^+$, which is a zero-order homogeneous function on Π. By Theorem 3.13, the Toeplitz operator $T_{a_H^\Pi}$ is unitary equivalent to the multiplication operator $\gamma_{a_H^\Pi}$. The function $\gamma_{a_H^\Pi}$ is given by

$$\gamma_{a_H^\Pi} = A\left(\frac{1}{2s\pi} - \frac{1}{e^{-2s\pi} - 1}\right) + B,$$

for suitable complex constants A and B. Corollary 7.2.7 of [6] shows that the algebra generated by $\gamma_{a_H^\Pi}$ and the identity is the algebra of continuous functions on $\bar{\mathbb{R}}$.

Following the unitary equivalences from T_{a_H} to $\gamma_{a_H^\Pi}$ and making a change of variables, we have that the algebra generated by T_{a_H} is isomorphic to the algebra of continuous functions over the segment $[0, 1]$.

We already know, from Theorem 3.14, that the Toeplitz operator with symbol a_R generates the algebra of continuous functions over $[0, 1]$ as well, so, locally, the algebras genereated by T_{a_H} and T_{a_R} are the same. We have thus proved

Theorem 4.1. *Consider the algebra PC defined on $\partial\mathbb{D}$ and its extensions PC_R and PC_H. The local algebras $\hat{\mathcal{T}}_{PC_R}(x)$ and $\hat{\mathcal{T}}_{PC_H}(x)$ are the same.*

To show the same theorem for functions f in QC we need to stablish some definitions related to the space Q in Definition 3.3. Further information on the theorems and definitions below can be found in [7].

Definition 4.2. For a function $g \in L^\infty(\mathbb{D})$ we define its Berezin transform $\tilde{g}$ by the formula

$$\tilde{g}(z) := \int_{\mathbb{D}} g(w) \frac{1 - |w|^2}{(1 - z\bar{w})^2} dA(w).$$

Note that $\tilde{g}$ belongs to $L^\infty(\mathbb{D})$ and $\|\tilde{g}\|_\infty \leq \|g\|_\infty$.

Definition 4.3. Define B as the set of bounded functions on $\mathbb{D}$ such that its Berezin transform goes to zero as z approaches to the boundary of $\mathbb{D}$, that is,

$$B := \{f \in L^\infty(\mathbb{D}) \ : \lim_{|z|\to 1} \tilde{f}(z) = 0\}.$$

In [1], S. Axler and D. Zheng proved that a Toeplitz operator T_g, with bounded symbol g, is compact if and only if g is in B. The next lemma is due to D. Sarason and is a combination of some results in [5].

Theorem 4.4. *The set Q in Definition 3.3 is described as*

$$Q = \{f \in L^\infty(\mathbb{D}) : \lim_{|z|\to 1} \widetilde{|f|}^2(z) - |\tilde{f}(z)|^2 = 0\}.$$

The set $B \cap Q$ is an ideal of Q and, for $f \in Q$, the Toeplitz operator T_f is compact if and only if f belongs to $B \cap Q$.

Lemma 4.5. *For a function f in QC, the function f_H belongs to Q.*

Proof. For this proof we use two facts:

1. The Berezin transform of a harmonic function is the function itself, in our case, $\tilde{f}_H = f_H$.

2. By [3], the harmonic extension is asymptotically multiplicative in QC, that is
$$\lim_{|z|\to 1^-} |f|^2_H(z) - |f_H(z)|^2 = 0.$$

Now we proceed with the proof:

$$\begin{aligned} \widetilde{|f_H|^2}(z) - |\widetilde{f_H}(z)|^2 &\le \left| \widetilde{|f_H|^2}(z) - |f|^2_H(z) \right| + \left| |f|^2_H(z) - |f_H(z)|^2 \right| \\ &\le \left| |f_H|^2 \widetilde{(z) - |f|}^2_H(z) \right| + \left| |f|^2_H(z) - |f_H(z)|^2 \right|, \end{aligned}$$

the last two sumands goes to zero as z approaches to the boundary $\partial\mathbb{D}$; the later because of item 2, and the former is due to items 2 and 1. Finally, using Theorem 4.4, we have that f_H is in Q. □

Definition 4.6 (page 626, [7])**.** For each point z in $\mathbb{D}$ we define

$$S'_z := \left\{ w \in \mathbb{D} \ : \ |w| \ge |z| \quad \text{and} \quad |\arg(z) - \arg(w)| \le \frac{1-|z|}{2} \right\}.$$

Definition 4.7 (page 627, [7])**.** For a function f in $L^\infty(\mathbb{D})$ define

$$\hat{f}(z) := \frac{1}{|S'_z|} \int_{S'_z} f(w) dA(w).$$

Definition 4.8 (page 626, [7]). Let f be in $L_\infty(\mathbb{D}, dA)$. We say f is in $ESV(\mathbb{D})$ if and only if for any $\epsilon > 0$, and $\sigma \in (0,1)$, there exists $\delta_0 > 0$ such that $|f(z) - f(w)| < \epsilon$ whenever $w \in S'_z$ and $|z|, |w| \in [1-\delta, 1-\delta\sigma]$, with $\delta < \delta_0$.

The notation $ESV(\mathbb{D})$ means eventually slowly varying and was introduced by C. Berger and L. Coburn in [2].

Theorem 4.9. *[7, Theorem 5] $Q = ESV + Q \cap B$. A decomposition is given by $f = \hat{f} + (f - \hat{f})$. Moreover*

$$ESV(D) \cap B = \{f \in L_\infty(D) \mid \ f(z) \to 0 \ as \ \ |z| \to 1^-\}.$$

We calculate $\hat{f}_R$ and get $\hat{f}_R(z) = I_z(f)$. Then, Theorem 4.9 gives us the decomposition $f_R(z) = I_z(f) + (f_R(z) - I_z(f))$, where $I_z(f)$ belongs to $ESV(\mathbb{D})$ and $f_R(z) - I_z(f)$ belongs to $Q \cap B$.

Lemma 4.10. *Consider the function f in QC. The Toeplitz operator with symbol $f_R - f_H$ is compact.*

Proof. We write $f_R(z) - f_H(z) = (I_z(f) - f_H(z)) + (f_R(z) - I_z(f))$. The first summand goes to zero as $|z|$ goes to 1 by Theorem 2.8. Then by Theorem 4.9, the function $I_z(f) - f_H(z)$ belongs to $ESV(\mathbb{D}) \cap B$. By the decomposition of Q as $ESV(\mathbb{D}) + Q \cap B$ we have that $(f_R(z) - I_z(f))$ belongs to $Q \cap B$.

In summary, the function $f_R(z) - f_H(z)$ belongs to $Q \cap B$ and then the Toeplitz operator with symbol $f_R - f_H$ is compact. □

Now we stablish the main result of this section: the algebra described in Theorem 3.15 does not depend on the extension chosen for the symbols in PQC.

Theorem 4.11. *Let PQC_R and PQC_H denote, respectively, the radial and the harmonic extension to the disk of functions in PQC. Then, the Calkin algebras $\mathcal{T}_{PQC_R}/\mathcal{K}$ and $\mathcal{T}_{PQC_H}/\mathcal{K}$ are the same.*

Proof. The proof follows from Theorem 4.1 and Lemma 4.10. □

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