Morfismos, Vol. 22, No. 1, 2018, pp. 27–39

Non-contractible configuration spaces

Cesar A. Ipanaque Zapata [1]

Abstract

Let $F(M,k)$ be the configuration space of ordered $k-$tuples of distinct points in the manifold M. Using the Fadell-Neuwirth fibration, we prove that the configuration spaces $F(M,k)$ are never contractible, for $k \geq 2$. As applications of our results, we will calculate the LS category and topological complexity for its loop space and suspension.

2010 Mathematics Subject Classification: 55R80, 55S40, 55P35 (primary) 55M30 (secondary).
Keywords and phrases: Ordered configuration spaces, Fadell-Neuwirth fibration, pointed loop spaces, suspension, Lusternik-Schnirelmann category, Topological complexity.

1 Introduction

Let X be the space of all possible configurations or states of a mechanical system. A motion planning algorithm on X is a function which assigns to any pair of configurations $(A,B) \in X \times X$, an initial state A and a desired state B, a continuous motion of the system starting at the initial state A and ending at the desired state B. The elementary problem of robotics, *the motion planning problem*, consists of finding a motion planning algorithm for a given mechanical system. The motion planning algorithm should be continuous, that is, it depends continuously on the pair of points (A,B). Absence of continuity will result in the instability of behavior of the motion planning. Unfortunately, a continuous motion

[1]This work is a part of my PhD's thesis under the supervision of professor Denise de Mattos at the Universidade de São Paulo and it is supported by FAPESP 2016/18714-8.

planning algorithm on space X exists if and only if X is contractible, see [10]. The design of effective motion planning algorithms is one of the challenges of modern robotics, see, for example Latombe [18] and LaValle [19].

Investigation of the problem of simultaneous motion planning without collisions for k robots in a topological manifold M leads one to study the (ordered) configuration space $F(M, k)$. We want to know if exists a continuous motion planning algorithm on the space $F(M, k)$. Thus, an interesting question is whether $F(M, k)$ is contractible.

It seems likely that the configuration space $F(M, k)$ is not contractible for certain topological manifolds M. Evidence for this statement is given in the work of F. Cohen and S. Gitler, in [4], they described the homology of loop spaces of the configuration space $F(M, k)$ whose results showed that this homology is non trivial. In a robotics setting, the (collision-free) motion planning problem is challenging since it is not known an effective motion planning algorithm, see [20].

In this paper, using the Fadell-Neuwirth fibration, we will prove that the configuration spaces $F(M, k)$ of topological manifolds M, are never contractible (see Theorem 2.1). Note that the configuration space $F(X, k)$ can be contractible, for any $k \geq 1$ (e.g. if X is an infinite indiscrete space or if $X = \mathbb{R}^\infty$). As applications of our results, we will calculate the LS category and topological complexity for the (pointed) loop space $\Omega F(M, k)$ (see Theorem 4.7) and the suspension $\Sigma F(M, k)$ (see Theorem 4.11 and Proposition 4.17).

Conjecture 1.1. *If X is a path-connected and paracompact topological space with covering dimension $1 \leq dim(X) < \infty$. Then the configuration spaces $F(X, k)$ are never contractible, for $k \geq 2$.*

Computation of LS category and topological complexity of the configuration space $F(M, k)$ is a great challenge. The LS category of the configuration space $F(\mathbb{R}^m, k)$ has been computed by Roth in [21]. In Farber and Grant's work [11], the authors computed the TC of the configuration space $F(\mathbb{R}^m, k)$. Farber, Grant and Yuzvinsky determined the topological complexity of $F(\mathbb{R}^m - Q_r, k)$ for $m = 2, 3$ in [12]. Later González and Grant extended the results to all dimensions m in [15]. Cohen and Farber in [2] computed the topological complexity of the configuration space $F(\Sigma_g - Q_r, k)$ of orientable surfaces Σ_g. Recently in [24], the author computed the LS category and TC of the configuration space $F(\mathbb{CP}^m, 2)$. The LS category and TC of the configuration space of ordered $2-$tuples of distinct points in $G \times \mathbb{R}^n$ has been computed by the author in [25]. Many more related results can be found in the recent survey papers [1] and [9].

2 Main Results

Let M denote a connected $m-$dimensional topological manifold (without boundary), $m \geq 1$. The *configuration space* $F(M,k)$ of ordered $k-$tuples of distinct points in M (see [8]) is the subspace of M^k given by

$$F(M,k) = \{(m_1, \ldots, m_k) \in M^k | \ \ m_i \neq m_j, \ \forall i \neq j\}.$$

Let $Q_r = \{q_1, \ldots, q_r\}$ denote a set of r distinct points of M.

Let M be a connected finite dimensional topological manifold (without boundary) with dimension at least 2 and $k > r \geq 1$. It is well known that the projection map

$$(1) \qquad \pi_{k,r} : F(M,k) \longrightarrow F(M,r), \ (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_r)$$

is a fibration with fibre $F(M-Q_r, k-r)$. It is called the Fadell-Neuwirth fibration [6]. In contrast, when the manifold M has nonempty boundary, $\pi_{k,r}$ is not a fibration. The fact that the map $\pi_{k,r}$ is not a fibration may be seen by considering, for example, the manifold $M = \mathbb{D}^2$ that is with boundary but the fibre $\mathbb{D}^2 - \{(0,0)\}$ is not homotopy equivalent to the fibre $\mathbb{D}^2 - \{(1,0)\}$.

Let X be a space, with base-point x_0. The pointed loop space is denoted by ΩX, as its base-point, if it needs one, we take the function w_0 constant at x_0. We recall that a topological space X is weak-contractible if all homotopy groups of X are trivial, that is, $\pi_n(X, x_0) = 0$ for all $n \geq 0$ and all choices of base point x_0.

In this paper, using the Fadell-Neuwirth fibration, we prove the following theorem

Theorem 2.1. *[Main Theorem] If M is a connected finite dimensional topological manifold, then the configuration space $F(M,k)$ is not contractible (indeed, it is never weak-contractible), for any $k \geq 2$.*

Remark 2.2. Theorem 2.1 can be proved using classifying spaces. I am very grateful to Prof. Nick Kuhn for his suggestion about the following proof. Let M be a connected finite dimensional topological manifold. If the configuration space $F(M,k)$ was contractible, then the quotient $F(M,k)/S_k$ would be a finite dimensional model for the classifying space of the k^{th} symmetric group S_k. But if G is a nontrivial finite group or even just contains any nontrivial elements of finite order, then there is no finite dimensional model for BG because $H^*(G)$ is periodic. Thus $F(M,k)$ is never contractible for $k \geq 2$.

3 PROOF of Theorem 2.1

The proof of Theorem 2.1 is greatly simplified by actually working on two main steps:

S1. We first get the Theorem 2.1 when $\pi_1(M) = 0$ (Proposition 3.5).

S2. Then we prove the Theorem 2.1 when $\pi_1(M) \neq 0$ (It follows from Lemma 3.6).

Here we note that the manifolds being considered are without boundary.

Step $S1$ above is accomplished proving the next four results.

Lemma 3.1. *Let M denote a connected $m-$dimensional topological manifold, $m \geq 2$. If $r \geq 1$, then the configuration space $F(M - Q_r, k)$ is not contractible (indeed, it is not weak-contractible), $\forall k \geq 2$.*

Proof. Recall that if $p : E \longrightarrow B$ is the projection map in a fibration with inclusion of the fibre $i : F \longrightarrow E$ such that p supports a cross-section σ, then (1) $\pi_q(E) \cong \pi_q(F) \oplus \pi_q(B)$, $\forall q \geq 2$ and (2) $\pi_1(E) \cong \pi_1(F) \rtimes \pi_1(B)$.

If $r \geq 1$, then the first coordinate projection map

$$\pi : F(M - Q_r, k) \longrightarrow M - Q_r$$

is a fibration with fibre $F(M - Q_{r+1}, k-1)$ and π admits a section ([8], Theorem 1). Thus (1) $\pi_q(F(M - Q_r, k)) \cong \bigoplus_{i=0}^{k-1} \pi_q(M - Q_{r+i})$, $\forall q \geq 2$ ([8], Theorem 2) and (2) $\pi_1(F(M - Q_r, k))$ is isomorphic to

$$\Bigg(\bigg(\cdots\Big(\pi_1(M - Q_{r+k-1}) \rtimes \pi_1(M - Q_{r+k-2})\Big)\cdots\bigg)$$
$$(\rtimes\, \pi_1(M - Q_{r+1})\Bigg) \rtimes \pi_1(M - Q_r)$$

Finally, notice that $M - Q_{r+k-1}$ is homotopy equivalent to

$$\bigvee_{i=1}^{r+k-2} \mathbb{S}^{m-1} \vee (M - V),$$

where V is an open $m-$ball in M such that $Q_{r+k-1} \subset V$ ([7], Proposition 3.1). Thus $M - Q_{r+k-1}$ is not weak contractible, therefore $F(M - Q_r, k)$ is not weak-contractible. □

Lemma 3.2. *If M is a simply-connected finite dimensional topological manifold which is not weak-contractible, then the singular homology (with coefficients in a field $\mathbb{K}$) of ΩM does not vanish in sufficiently large degrees.*

Proof. By contradiction, we will suppose the singular homology of ΩM vanishes in sufficiently large degrees, that is, there exists an integer $q_0 \geq 1$ such that, $H_q(\Omega M; \mathbb{K}) = 0, \forall q \geq q_0$, where $\mathbb{K}$ is a field. Let f denote a nonzero homology class of maximal degree in $H_*(\Omega M; \mathbb{K})$. As M is finite dimensional and not weak-contractible, let b denote a nonzero homology class in $\widetilde{H}_*(M; \mathbb{K})$ of maximal degree (here $\widetilde{H}_*(-; \mathbb{K})$ denote reduced singular homology, with coefficients in a field $\mathbb{K}$). Notice that $b \otimes f$ survives to give a non-trivial class in the Serre spectral sequence abutting to $H_*(P(M, x_0); \mathbb{K})$, since M is simply-connected, the local coefficient system $H_*(\Omega M; \mathbb{K})$ is trivial, where

$$P(M, x_0) = \{\gamma \in PM \mid \gamma(0) = x_0\},$$

it is contractible. This is a contradiction and so the singular homology of ΩM does not vanish in sufficiently large degrees. □

Proposition 3.3. *Let M be a simply-connected topological manifold which is not weak-contractible with dimension at least 2. Then the configuration space $F(M, k)$ is not contractible (indeed, it is never weak-contractible), $\forall k \geq 2$.*

Proof. By hypothesis, M is a connected finite dimensional topological manifold of dimension at least 2. Consequently, there is a fibration

$$F(M, k) \longrightarrow M$$

with fibre $F(M - Q_1, k - 1)$ $(k \geq 2)$. We just have to note that in sufficiently large degrees, the singular homology, with coefficients in a field $\mathbb{K}$, of $F(M - Q_1, k - 1)$ vanishes, since $F(M - Q_1, k - 1)$ is a connected finite dimensional topological manifold.

On the other hand, if $F(M, k)$ were weak-contractible, then the pointed loop space of M is weakly homotopy equivalent to $F(M - Q_1, k - 1)$ which it cannot be by Lemma 3.2. Thus, the configuration space $F(M, k)$ is not weak-contractible. □

Proposition 3.4. *Let M be a weak-contractible topological manifold with dimension at least 2. Then the configuration space $F(M,k)$ is not contractible (indeed, it is never weak-contractible), $\forall k \geq 2$.*

Proof. By the homotopy long exact sequence of the fibration

$$F(M,k) \longrightarrow M$$

with fibre $F(M-Q_1,k-1)$, we can conclude the inclusion

$$i : F(M-Q_1,k-1) \hookrightarrow F(M,k)$$

is a weak homotopy equivalence. If $k \geq 3$, then Lemma 3.1 implies that $F(M-Q_1,k-1)$ is not weak-contractible and so $F(M,k)$ is not weak-contractible. If $k=2$, we consider the cover

$$M = A \cup B,$$

where $A = M - \{q\}, B = M - \{q'\}, \quad q, q'$ distinct. Here we note that $A = M - \{q\}$ and $B = M - \{q'\}$ are homeomorphic to $M - Q_1$ and $A \cap B = M - \{q, q'\}$ is not weak-contractible, because $M - \{q, q'\}$ is homotopy equivalent to the wedge $\mathbb{S}^{m-1} \vee (M - V)$, where V is an open $m-$ball in M such that $\{q, q'\} \subset V$ ([7], Proposition 3.1). Thus, the Mayer-Vietoris sequence, for the given cover, implies $M - Q_1$ is not weak-contractible and so $F(M,2)$ is not weak-contractible. Therefore, $F(M,k)$ is not weak-contractible. □

By Propositions 3.3 and 3.4 we have the following statement.

Proposition 3.5. *If M is a simply-connected topological manifold with dimension at least 2, then the configuration space $F(M,k)$ is not contractible (indeed, it is never weak-contractible), $\forall k \geq 2$.*

A key ingredient for step $S2$ is given by the next result.

Lemma 3.6. *If M is a connected finite dimensional topological manifold with dimension at least 2, then the inclusion map $i : F(M,k) \longrightarrow M^k$ induces a homomorphism $i_* : \pi_1 F(M,k) \longrightarrow \pi_1 M^k$ which is surjective.*

Proof. We will prove it by induction on k. We just have to note that the inclusion map $j : M - Q_k \longrightarrow M$ induces an epimorphism

$$j_* : \pi_1(M - Q_k) \longrightarrow \pi_1 M,$$

for any $k \geq 1$. The following diagram of fibrations (see Figure 1) is commutative.

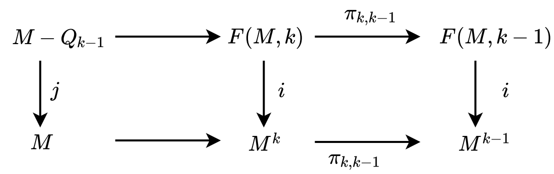

Figure 1: Commutative diagram.

Thus by induction, we can conclude the inclusion map

$$i : F(M,k) \longrightarrow M^k$$

induces a homomorphism $i_* : \pi_1 F(M,k) \longrightarrow \pi_1 M^k$ which is surjective and so we are done. □

Remark 3.7. Lemma 3.6 is actually a very special case of a general theorem of Golasiński, Gonçalves and Guaschi in ([13], Theorem 3.2). Also, it can be proved using braids ([14], Lemma 1).

Proof of Theorem 2.1. The case $dim\ M = 1$ is straightforward, so we assume that $dim\ M \geq 2$. If $\pi_1(M) = 0$ then the result follows easily from the Proposition 3.5. If $\pi_1(M) \neq 0$ then $\pi_1(M^k) \neq 0$ and by Lemma 3.6

$$i_* : \pi_1(F(M,k)) \longrightarrow \pi_1(M^k)$$

is an epimorphism. Thus $\pi_1(F(M,k)) \neq 0$ and $F(M,k)$ is not weak-contractible. Therefore, $F(M,k)$ is not contractible. □

4 Lusternik-Schnirelmann category and topological complexity

As applications of our results, in this section, we will calculate the LS category and topological complexity for the (pointed) loop space $\Omega F(M,k)$ and the suspension $\Sigma F(M,k)$.

Here we follow a definition of category, one greater than category given in [5].

Definition 4.1. We say that the Lusternik-Schnirelmann category or category of a topological space X, denoted $cat(X)$, is the least integer m such that X can be covered with m open sets, which are all contractible within X. If no such m exists we will set $cat(X) = \infty$.

Let PX denote the space of all continuous paths $\gamma : [0,1] \longrightarrow X$ in X and $\pi : PX \longrightarrow X \times X$ denotes the map associating to any path $\gamma \in PX$ the pair of its initial and end points $\pi(\gamma) = (\gamma(0), \gamma(1))$. Equip the path space PX with the compact-open topology.

Definition 4.2. [10] The *topological complexity* of a path-connected space X, denoted by $TC(X)$, is the least integer m such that the Cartesian product $X \times X$ can be covered with m open subsets U_i,

$$X \times X = U_1 \cup U_2 \cup \cdots \cup U_m$$

such that for any $i = 1, 2, \ldots, m$ there exists a continuous function $s_i : U_i \longrightarrow PX$, $\pi \circ s_i = id$ over U_i. If no such m exists we will set $TC(X) = \infty$.

Remark 4.3. For all path connected spaces X, the basic inequality that relate cat and TC is

$$cat(X) \leq TC(X).$$

On the other hand, by ([10], Theorem 5), for all path connected paracompact spaces X,

$$TC(X) \leq 2cat(X) - 1.$$

It follows from the Definition 4.1 that we have $cat(X) = 1$ if and only if X is contractible. It is also easy to show that $TC(X) = 1$ if and only if X is contractible.

By Remark 4.3 and Theorem 2.1, we obtain the following statement.

Proposition 4.4. *If M is a connected finite dimensional topological manifold, then the Lusternik-Schnirelmann category and the topological complexity of $F(M,k)$ are at least 2, $\forall k \geq 2$.*

Proposition 4.5 and Lemma 4.6 we state in this section are known, they can be found in the paper by Frederick R. Cohen [3]. Here $\Omega_0^j X$ denotes the component of the constant map in the j^{th} pointed loop space of X.

Proposition 4.5. *([3], Theorem 1) If X is a simply-connected finite complex which is not contractible, then the Lusternik-Schnirelmann category of $\Omega_0^j X$ is infinite for $j \geq 1$.*

Lemma 4.6. *Let M be a simply-connected finite dimensional topological manifold with dimension at least 3. If M has the homotopy type of a finite CW complex, then the configuration space $F(M,k)$ has the homotopy type of a finite CW complex, $\forall k \geq 1$.*

As a consequence of Theorem 2.1 we can obtain Proposition 4.5 for configuration spaces.

Theorem 4.7. *Let M be a space which has the homotopy type of a finite CW complex. If M is a simply-connected finite dimensional topological manifold with dimension at least 3, then the Lusternik-Schnirelmann category and the topological complexity of $\Omega_0^j F(M,k)$ are infinite, for any $k \geq 2$ and $j \geq 1$.*

Proof. The assumptions that M is a simply-connected finite dimensional topological manifold with dimension at least 3, imply the configuration space $F(M,k)$ is simply-connected. Furthermore, as M has the homotopy type of a finite CW complex, the configuration space $F(M,k)$ also has the homotopy type of a finite CW complex by Lemma 4.6. Finally the configuration space $F(M,k)$ is not contractible by Theorem 2.1. Therefore we can apply Proposition 4.5 and conclude that the Lusternik-Schnirelmann category of $\Omega_0^j F(M,k)$ is infinite, $\forall k \geq 2$. Moreover, by Remark 4.3, the topological complexity of $\Omega_0^j F(M,k)$ is also infinite, $\forall k \geq 2$. □

Remark 4.8.

1. In Theorem 4.7, the assumption M *has the homotopy type of a finite CW complex* can be reduce to the assumption M *is a CW complex of finite type* (see [22]).
2. By Theorem 4.7, if G is a simply-connected finite dimensional Lie group of finite type with dimension at least 3. Then the topological complexity $TC(\Omega F(G,k)) = \infty$, for any $k \geq 2$. In contrast, we will see that the topological complexity $TC(\Sigma F(G,k)) = 3 < \infty$, for any $k \geq 3$.

Remark 4.9. If X is any topological space and

$$\Sigma X := \frac{X \times [0,1]}{X \times \{0\} \cup X \times \{1\}}$$

is the non-reduced suspension of the space X, it is well-known that $cat(\Sigma X) \leq 2$. We can cover ΣX by two overlapping open sets (e.g, $q(X \times [0,3/4)$ and $q(X \times (1/4,1])$, where $q : X \times [0,1] \longrightarrow \Sigma X$ is the projection map), such that each open set is homeomorphic to the cone $CX := \frac{X\times[0,1]}{X\times\{0\}}$, so they are contractible in itself and thus they are contractible in the suspension ΣX.

Lemma 4.10. *Let X be a simply-connected topological space. If X is not weak-contractible, then*

$$cat(\Sigma X) = 2.$$

Proof. It is sufficient to prove that ΣX is not weak-contractible and thus $cat(\Sigma X) \geq 2$. Since contractible implies weak-contractible. If ΣX was weak-contractible then by the Mayer-Vietoris sequence for the open covering $\Sigma X = q(X \times [0,3/4) \cup q(X \times (1/4,1])$ we can conclude $H_q(X;\mathbb{Z}) = 0, \forall q \geq 1$. Thus by ([17], Corollary 4.33) X is weak-contractible (here we have used that X is simply-connected[2]). It is a contradiction with the hypothesis. Therefore ΣX is not weak-contractible. □

Theorem 4.11. *If M is a simply-connected finite dimensional topological manifold with dimension at least 3, then*

$$cat(\Sigma F(M,k)) = 2, \forall k \geq 2.$$

Proof. The arguments M is a simply-connected finite dimensional topological manifold with dimension at least 3, imply the configuration space $F(M,k)$ is simply-connected. The configuration space $F(M,k)$ is not weak-contractible by Theorem 2.1. Therefore we can apply Lemma 4.10 and the Lusternik-Schnirelmann category of $\Sigma F(M,k))$ is two, $\forall k \geq 2$. □

We note that $\Sigma F(M,k)$ is paracompact because $F(M,k)$ is paracompact.

Corollary 4.12. *If M is a simply-connected finite dimensional topological manifold with dimension at least 3, then*

$$2 \leq TC(\Sigma F(M,k)) \leq 3, \forall k \geq 2.$$

Proof. It follows from Remark 4.3 and Theorem 4.11. □

Remark 4.13. By Corollary 4.12 the topological complexity of the suspension of a configuration space is secluded in the range

$$2 \leq TC(\Sigma F(M,k)) \leq 3$$

[2]By Hatcher ([17], Example 2.38) there exists nonsimply-connected acyclic spaces.

and any value in between can be taken (e.g. if $M = \mathbb{S}^m$ or $\mathbb{R}^m$ and $k = 2$).

Now we will recall the definition of the cup-length.

Definition 4.14. [5] Let R be a commutative ring with unit and X be a topological space. The *cup-length* of X, denote $cup_R(X)$, is the least integer n such that all $(n+1)-$fold cup products vanish in the reduced cohomology $\widetilde{H^\star}(X; R)$.

Remark 4.15. ([5], Theorem 1.5) Let R be a commutative ring with unit and X be a topological space. It is well-known that

$$1 + cup_R(X) \leq cat(X).$$

On the other hand, it is easy to verify that the cup-length has the property listed below.

Lemma 4.16. *Let $\mathbb{K}$ be a field and X, Y be topological spaces. Then if $H^k(Y; \mathbb{K})$ is a finite dimensional $\mathbb{K}-$vector space for all $k \geq 0$. We have*

$$cup_{\mathbb{K}}(X \times Y) \geq cup_{\mathbb{K}}(X) + cup_{\mathbb{K}}(Y).$$

Proposition 4.17. *If G is a simply-connected finite dimensional Lie group of finite type with dimension at least 3. Then*

$$TC(\Sigma F(G, k)) = 3, \forall k \geq 3.$$

Proof. We will assume that G is not contractible, the case G is contractible follows easily because $F(G, k)$ is homotopy equivalent to the configuration space $F(\mathbb{R}^d, k)$, where $d = dim(G)$ (see [23], pg. 118). By Corollary 4.12 it is sufficient to prove that $TC(\Sigma F(G, k)) \neq 2$. If $TC(\Sigma F(G, k)) = 2$ then, by ([16], Theorem 1), we have $\Sigma F(G, k)$ is homotopy equivalent to some (odd-dimensional) sphere. Then $F(G, k)$ is homotopy equivalent to some (even-dimensional) sphere and thus $cat(F(G, k)) = 2$. On the other hand, $F(G, k)$ is homeomorphic to the product $G \times F(G - \{e\}, k - 1)$ because G is a topological group. Then $2 = cat(G \times F(G - \{e\}, k - 1)) \geq cup_{\mathbb{K}}(G \times F(G - \{e\}, k - 1)) + 1$ for any field $\mathbb{K}$ (see Remark 4.15). Furthermore, Lemma 4.16 implies that

$$\begin{aligned} cup_{\mathbb{K}}(G \times F(G - \{e\}, k - 1)) &\geq cup_{\mathbb{K}}(G) + cup_{\mathbb{K}}(F(G - \{e\}, k - 1)) \\ &\geq 1 + 1 \\ &= 2 \end{aligned}$$

(here we note that $k-1 \geq 2$ and by Theorem 2.1 we have the cup-length $cup_{\mathbb{K}}(F(G-\{e\},k-1)) \geq 1)$. Thus, $2 = cat(G \times F(G-\{e\},k-1)) \geq 3$ which is a contradiction. □

Acknowledgement

The author is very grateful to Frederick Cohen and Jesús González for their comments and encouraging remarks which were of invaluable mental support.

Cesar A. Ipanaque Zapata
Departamento de Matemática,
Universidade de São Paulo,
Instituto de ciências matemáticas e de computação - USP,
Avenida Trabalhador São-carlense, 400 - Centro CEP: 13566-590 - São Carlos - SP, Brasil,
cesarzapata@usp.br

References

[1] Cohen, Daniel C., *Topological complexity of classical configuration spaces and related objects*, Topological Complexity and Related Topics, American Mathematical Soc., **702** (2018).

[2] Cohen, Daniel C and Farber, Michael., *Topological complexity of collision-free motion planning on surfaces*, Compositio Mathematica, Cambridge Univ Press., **147** (2011), no. 2, 649–660.

[3] Cohen, F. R., *On the Lusternik-Schnirelmann category of an iterated loop space.* Stable and unstable homotopy (Toronto, ON, 1996). (1998), 39–41.

[4] Cohen, F and Gitler, S., *On loop spaces of configuration spaces*, Transactions of the American Mathematical Society., **354** (2002), no. 5, 1705–1748.

[5] Cornea, Octav and Lupton, Gregory and Oprea, John and Tanré, Daniel., *Lusternik-Schnirelmann Category*, American Mathematical Society. (2003).

[6] Fadell, Edward R and Husseini, Sufian Y., *Geometry and topology of configuration spaces.* Springer Science & Business Media. (2001).

[7] Fadell, Edward and Husseini, Sufian., *Configuration spaces on punctured manifolds*, Journal of the Juliusz Schauder Center., **20** (2002), 25–42.

[8] Fadell, Edward and Neuwirth, Lee., *Configuration spaces.* Math. Scand., **10** (1962), no. 4, 111-118.

[9] Farber, Michael., *Configuration spaces and robot motion planning algorithms*, Combinatorial and Toric Homotopy: Introductory Lectures, World Scientific. (2017), 263–303.

[10] Farber, Michael., *Topological complexity of motion planning*, Discrete and Computational Geometry., **29** (2003), no. 2, 211–221.

[11] Farber, Michael and Grant, Mark., *Topological complexity of configuration spaces*, Proceedings of the American Mathematical Society., **137** (2009), no. 5, 1841–1847.

[12] Farber, Michael and Grant, Mark and Yuzvinsky, Sergey., *Topological complexity of collision free motion planning algorithms in the presence of multiple moving obstacles*, Contemporary Mathematics, Providence, RI: American Mathematical Society., **438** (2007), 75–84.

[13] Golasiński, Marek and Gonçalves, Daciberg Lima and Guaschi, John., *On the homotopy fibre of the inclusion map $F_n(X) \hookrightarrow \prod_1^n X$ for some orbit spaces X*, Boletín de la Sociedad Matemática Mexicana, Springer., **23** (2017), no. 1, 457–485.

[14] Goldberg, Charles H., *An exact sequence of braid groups*, Mathematica Scandinavica, JSTOR., **33** (1974), no. 1, 69–82.

[15] González, Jesús and Grant, Mark., *Sequential motion planning of non-colliding particles in Euclidean spaces*, Proceedings of the American Mathematical Society., **143** (2015), no. 10, 4503–4512.

[16] Grant, Mark and Lupton, Gregory and Oprea, John., *Spaces of topological complexity one*, Homology, Homotopy and Applications, International Press of Boston., **15** (2013), no. 2, 73–81.

[17] Hatcher, Allen., *Algebraic topology.* (2002).

[18] Latombe, Jean-Claude., *Robot motion planning*, Springer Science & Business Media., **124** (2012).

[19] LaValle, Steven M., *Planning algorithms*, Cambridge university press. (2006).

[20] Le, Duong and Plaku, Erion., *Multi-Robot Motion Planning with Dynamics Guided by Multi-Agent Search*, IJCAI. (2018), 5314–5318.

[21] Roth, Fridolin., *On the category of Euclidean configuration spaces and associated fibrations*, Groups, homotopy and configuration spaces., **13** (2008), 447–461.

[22] Wilkerson, Clarence W., *Draft-loopspaces and finiteness.* (2006).

[23] Zapata, Cesar Augusto Ipanaque., *Espaços de configurações*, Universidade de São Paulo. Dissertao de mestrado. (2017).

[24] Zapata, Cesar A Ipanaque., *Lusternik-Schnirelmann category of the configuration space of complex projective space*, arXiv preprint arXiv:1708.05830. (2017) (to appear in Topology Proceedings).

[25] Zapata, Cesar A Ipanaque., *Category and Topological Complexity of the configuration space $F(G \times \mathbb{R}^n, k)$*, arXiv preprint arXiv:1711.01718. (2017).