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Derived Mackey functors and profunctors: an overview of results

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Abstract

In this paper we overview the theory of derived Mackey functors and profunctors.

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1 Introduction

"Mackey functors" associated to a finite group G have long been a standard tool both in group theory and in algebraic topology (specifically, in the G-equivariant stable homotopy theory). The notion was originally introduced by Dress [2] and later clarified by several people, in particular by Lindner [7]. The reader can find modern expositions in the topological context e.g. in [6], [8], [1], or a more algebraic treatment in [9].

For any finite group G, G-Mackey functors form an abelian category $\mathcal{M}(G)$. If one wants to develop a homological theory of Mackey functors, it seems natural to consider the derived category $\mathcal{D}(\mathcal{M}(G))$. However, there is an alternative to this suggested in [4]. By modifying the very definition of a Mackey functor, one can construct a triangulated category $\mathcal{DM}(G)$ of "derived Mackey functors" that contains $\mathcal{M}(G)$ but differs from $\mathcal{D}(\mathcal{M}(G))$. The category $\mathcal{DM}(G)$ reflects better the properties of the G-equivariant stable homotopy category, and it also behaves better than $\mathcal{D}(\mathcal{M}(G))$ from the purely formal point of view. For more details, we refer the reader to [4]; here we only mention two things:

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- $\mathcal{DM}(G)$ has a natural semiorthogonal decomposition into pieces that are identified with the derived categories of representations of certain subquotients of G; the gluing functors of this descomposition have a natural interpretation in terms of a certain generalization of Tate cohomology of finite groups.

- $\mathcal{DM}(G)$ has a lot of autoequivalences – in particular, any finite-dimensional real representation V of G gives rise to a "suspension autoequivalence" $\Sigma^V : \mathcal{DM}(G) \to \mathcal{DM}(G)$, a generalization of the homological shift.

Another way to extend to the notion of a Mackey functor is to consider more general groups G. In fact, if one allows groups with non-trivial topology, then G does not have to be finite – one can consider any compact Lie group. However, from the point of view of algebra, a more natural modification is to allow groups that are discrete but infinite. Formally, one can do so without any changes – Lindner's definition works for any group G, albeit the resulting category only depends on its profinite completion $\widehat{G}$. However, once more there is an alternative suggested in [5] under the name of a "G-Mackey profunctor". Mackey profunctors seem to reflect better the structure of the group G. In particular, in good situations, giving a G-Mackey profunctor M is equivalent to giving a system of Mackey functors $M_N \in \mathcal{M}(G/N)$, $N \subset G$ a cofinite normal subgroup, related by some natural compatibility isomorphisms.

One can also combine the two stories and develop the notion of a "derived Mackey profunctor". This has also been done in [5]. It turns out that even if one is interested in non-derived Mackey profunctors, looking at them in the derived context is cleaner and gives stronger results.

Unfortunately, both [4] and [5] are rather long and technical papers. The goal of the present paper is to give a brief overview of them. We do not give any proofs whatsoever, and we keep the technicalities to an absolute minimum. Instead, we try to present the conceptual ideas behind the constructions, and to illustrate the theory by some key examples.

The paper is organized as follows. Section 1 is a recollection of the standard theory of Mackey functors. Section 2 is concerned with derived Mackey functors for a finite group G. Generally, we follow [4], but we use improvements suggested in [5] to clean up some statements and simplify the proofs. In particular, we describe and use a relation between Mackey functors and finite pointed G-sets. We illustrate the theory in the simplest possible example $G = \mathbb{Z}/p\mathbb{Z}$, the finite cyclic group of prime order. In Section 3, we turn to the theory of Mackey profunctors developed in [5]. The main example here is $G = \mathbb{Z}$, the infinite cyclic group.

Notation.

For any category $\mathcal{E}$, we denote by $\mathcal{E}^o$ the opposite category. For every integer $n \geq 0$, we denote by $[n]$ the set of integers i, $0 \leq i \leq n$, and as usual, we denote by Δ the category of the sets $[n]$ and order-preserving maps g between them. Simplicial objects in a category $\mathcal{E}$ are functors from Δ^o to $\mathcal{E}$.

For any small category I and ring R, we denote by $\mathrm{Fun}(I, R)$ the abelian category of functors from I to the category of R-modules, and we denote by $\mathcal{D}(I, R)$ the derived category of the abelian category $\mathrm{Fun}(I, R)$. For any functor $\gamma : I' \to I$ between small categories, we have the natural pullback functor $\gamma^* : \mathrm{Fun}(I, R) \to \mathrm{Fun}(I', R)$, and it has a left and a right adjoint $\gamma_!, \gamma_* : \mathrm{Fun}(I', R) \to \mathrm{Fun}(I, R)$, the left and right Kan extension functors. The derived functors $L^{\bullet}\gamma_!, R^{\bullet}\gamma_* : \mathcal{D}(I', R) \to \mathcal{D}(I, R)$ are left and right-adjoint to the pullback functor $\gamma^* : \mathcal{D}(I, R) \to \mathcal{D}(I', R)$.

2 Recollection on Mackey functors.

We start with a brief recapitulation of the usual theory of Mackey functors (we follow the exposition in [5, Section 2]).

Assume given a category $\mathcal{C}$ that has fibered products. The *category* $Q\mathcal{C}$ is the category wwith same objects as $\mathcal{C}$, and with morphisms from c_1 to c_2 given by isomorphism classes of diagrams

(1) $$c_1 \xleftarrow{\ l\ } c \xrightarrow{\ r\ } c_2$$

in $\mathcal{C}$. The compositions are obtained by taking pullbacks. Note that we have a natural embedding

(2) $$e : \mathcal{C}^o \to Q\mathcal{C}$$

that is identical on objects, and sends a map f to the diagram (1) with $l = f$ and $r = \mathsf{id}$. The category $Q\mathcal{C}$ is obviously self-dual: $Q\mathcal{C}^o$ is identified with $Q\mathcal{C}$ by the functor that flips r and l in (1). By duality, the embedding (2) induces an embedding $e^o : \mathcal{C} \to Q\mathcal{C}$.

Now fix a group G and a ring R, and take as $\mathcal{C}$ the category Γ_G of finite G-sets, that is, finite sets equipped with an action of the group G. The category Γ_G has fibered products, so we can define the category $Q\Gamma_G$ and the embedding (2).

Definition 2.1. An object $E \in \mathrm{Fun}(\Gamma_G^o, R)$ is *additive* if for any $S_1, S_2 \in \Gamma_G^o$, the natural map

$$E(S_1 \sqcup S_2) \to E(S_1) \oplus E(S_2) \tag{3}$$

is an isomorphism. An *R-valued G-Mackey functor* E is an object $E \in \mathrm{Fun}(Q\Gamma_G, R)$ whose restriction $e^*E \in \mathrm{Fun}(\Gamma_G^o, R)$ is additive. The full subcategory in $\mathrm{Fun}(Q\Gamma_G, R)$ formed by Mackey functors is denoted by $\mathcal{M}(G, R)$.

we denote by $\mathrm{Fun}_{add}(\Gamma_G^o, R) \subset \mathrm{Fun}(\Gamma_G^o, R)$ the full subcategory spanned by additive objects, and let $O_G \subset \Gamma_G$ be the subcategory of G-orbits – that is, G-sets of the form $[G/H]$, $H \subset G$ a subgroup. Since every finite G-set decomposes into a disjoint union of G-orbits, (3) insures that restriction to $O_G^o \subset \Gamma_G^o$ provides an equivalence

$$\mathrm{Fun}_{add}(\Gamma_G^o, R) \cong \mathrm{Fun}(O_G^o, R).$$

In other words, an additive object $E \in \mathrm{Fun}(\Gamma_G^o, R)$ is uniquely determined by specifyings its values $E^H = E([G/H])$ at all G-orbits, and maps

$$f^* = E(f) : E^H \to E^{H'},$$

one for each map $f : [G/H'] \to [G/H]$ of G-orbits, compatible with compositions for composable pairs of maps f, f'. To extend E to a Mackey functor, one needs to also specify a map

$$f_* = E(e^o(f)) : E(S') \to E(S) \tag{4}$$

for any map $f : S' \to S$ of finite G-sets. These are sometimes called *transfer maps* (this is how they appear in equivariant stable homotopy theory). By additivity, it suffices to specify the transfer maps for maps G-orbits. They should be compatible with compositions, and for any two maps $f' : S' \to S$, $f'' : S'' \to S$, there is a certain compatibility condition between f''_* and f'^* encoded in the definition of the category $Q\Gamma_G$. Explicitly, if $S = [G/H]$, $S' = [G/H']$, $S'' = [G/H'']$, and f', f''

are induced by embeddings $H', H'' \subset H$, the fibered product $S' \times_S S''$ decomposes into a disjoint union

$$S' \times_S S'' = \coprod_{s \in S} [G/H_s] \tag{5}$$

of G-orbits indexed by the finite set $S = H' \backslash H / H''$, and we must have

$$f'^* \circ f''_* = \sum_{s \in S} \widetilde{f}''_{s*} \circ \widetilde{f}'^*_s, \tag{6}$$

where $f'_s : [G/H_s] \to [G/H'']$, $f''_s : [G/H_s] \to [G/H']$ are projections of the component G/H_s of the decomposition (5). This is known as the *double coset formula.*

For any subgroup $H \subset G$, we have a natural restriction functor $\rho^H : \Gamma_G \to \Gamma_H$ sending a G-set S to the same set S considered as an H-set. The functor ρ^H has a left-adjoint $\gamma^H : \Gamma_H \to \Gamma_G$. Both ρ^H and γ^H preserve fibered products, thus induce functors

$$Q(\gamma^H) : Q\Gamma_H \to Q\Gamma_G, \qquad Q(\rho^H) : Q\Gamma_G \to Q\Gamma_H. \tag{7}$$

One checks that the adjunction between ρ^H and γ^H induces an adjunction between $Q(\gamma^H)$ and $Q(\rho^H)$, and that the functor

$$\Psi^H = Q(\rho^H)_! \cong Q(\gamma^H)^* : \mathrm{Fun}(Q\Gamma_G, R) \to \mathrm{Fun}(Q\Gamma_H, R) \tag{8}$$

preserves additivity, thus sends Mackey functors to Mackey functors. The functor Ψ^H is known as the *categorical fixed points functor*.

On the other hand, for any subgroup $H \subset G$ with normalizer $N_H \subset G$, the quotient $W = N_H/H = \mathrm{Aut}_G([G/H])$ acts on the fixed points subset S^H of any G-set S. Then sending S to S^H gives a functor $\varphi^H : \Gamma_G \to \Gamma_W$ preserving fibered products, and we can consider the adjoint pair of functors

$$\begin{aligned} \Phi^H &= Q(\varphi^H)_! : \mathrm{Fun}(Q\Gamma_G, R) \to \mathrm{Fun}(Q\Gamma_W, R), \\ \mathsf{Infl}^H &= Q(\varphi^H)^* : \mathrm{Fun}(Q\Gamma_W, R) \to \mathrm{Fun}(Q\Gamma_G, R). \end{aligned} \tag{9}$$

Both preserve additivity, thus induce adjoint functors between $\mathcal{M}(G, R)$ and $\mathcal{M}(W, R)$. These are the *geometric fixed points functor* and the *inflation functor*.

The embedding $\mathcal{M}(G, R) \subset \mathrm{Fun}(Q\Gamma_G, R)$ admits a left-adjoint *additivization functor* Add. This allows to define tensor products of Mackey functors. Namely, the cartesian product functor

$$m : \Gamma_G \times \Gamma_G \to \Gamma_G \tag{10}$$

preserves fiber products, thus induces a functor

$$Q(m) : Q\Gamma_G \times Q\Gamma_G \to Q\Gamma_G.$$

Then for any two algebras R_1, R_2 over a commutative ring k and Mackey functors $E_1 \in \mathcal{M}(G, R_1)$, $E_2 \in \mathcal{M}(G, R_2)$, their *tensor product* $E_1 \circ E_2$ is given by

$$E_1 \circ E_1 = \mathsf{Add}(Q(m)_!(E_1 \boxtimes_k E_2)) \in \mathcal{M}(G, R_1 \otimes_k R_2). \tag{11}$$

The tensor product (11) is unital, with the unit given by the so-called *Burnside Mackey functor* $\mathcal{A} \in \mathcal{M}(G, \mathbb{Z})$. Explicitly, for any subgroup $H \subset G$, $\mathcal{A}^H$ is canonically identified with the *Burnside ring* A^H of the group H given by

$$A^H = \mathbb{Z}[\mathrm{Iso}(\Gamma_H)]/\{[S \sqcup S'] - [S] - [S']\}, \tag{12}$$

that is, the free abelian group generated by isomorphism classes $[S]$ of finite H-sets S, modulo the relations $[S \sqcup S'] = [S] + [S']$ (this is a commutative ring, with the product induced by the cartesian product of H-sets). Note that since any G-set uniquely decomposes into a disjoint union of G-orbits, we have

$$A^G \cong \bigoplus_{H \subset G} \mathbb{Z}, \tag{13}$$

where the sum is over all conjugacy classes of subgroups $H \subset G$. Both the categorical fixed points functor Ψ^H and the geometric fixed points functor Φ^H commute with the product (11). If R is a commutative ring, then the product (11) with $R_1 = R_2 = k = R$ turns $\mathcal{M}(G, R)$ into a unital symmetric tensor category. An *R-valued G-Green functor* is an algebra object in the category $\mathcal{M}(G, R)$.

For any subgroup $H \subset G$, the functor γ^H and the categorical fixed points functor Ψ^H can be computed rather explicitly. The geometric fixed points functor Φ^H is harder to describe. However, there is the following result.

Lemma 2.2 ([5, Lemma 2.4])**.** *Assume given a normal subgroup $N \subset G$, with the quotient $W = G/N$. Then the inflation functor Infl^N is fully faithful. Moreover, for any $E \in \mathcal{M}(G, R)$, the adjunction map*

$$M \to \mathsf{Infl}^N \Phi^N M \tag{14}$$

is surjective, and for any $S \in \Gamma_W$, we have a short exact sequence

$$\bigoplus_{f:S' \to S} M(S') \xrightarrow{\sum f_*} M(S) \longrightarrow \Phi^N(M)(S) \longrightarrow 0, \tag{15}$$

where f_* is as in (4), and the sum is over all maps $f : S' \to S$ in Γ_G such that S' has no elements fixed under $N \subset G$.

Note that by additivity of $E \in \mathrm{Fun}(Q\Gamma_G, R)$, the image of the map $\sum f_*$ in (15) is not only the sum of the images of individual maps f_*, it is actually the union of these images.

3 Derived Mackey functors.

We now turn to the derived version of the story of Section 2. To see why the derived category $\mathcal{D}(\mathcal{M}(G,R))$ might be a wrong object to work with, it suffices to recall the definition of the category $Q\mathcal{C}$: we only consider the isomorphism classes of diagrams (1), and completely ignore the fact that those diagrams can have non-trivial automorphism groups with non-trivial homology. On the formal level, the problem appears for example in Lemma 2.2: if one considers the derived categories $\mathcal{D}(\mathcal{M}(G,R))$, $\mathcal{D}(\mathcal{M}(W,R))$, then the functor

$$\mathsf{Infl}^N : \mathcal{D}(\mathcal{M}(W,R)) \to \mathcal{D}(\mathcal{M}(G,R))$$

induced by the inflation functor (9) is not longer fully faithful. To obtain the correct category of derived Mackey functors, one has to take account of the automorphisms of diagrams (1) and treat $Q\Gamma_G$ as what it naturally is – a 2-category rather than simply a category. But then, one has to make sense of the derived category $\mathcal{D}(Q\Gamma_G, R)$.

3.1 Quotient constructions.

In fact, [4] introduces not one but two equivalent ways to do it. Firstly, one can observe that any small category $\mathcal{C}$ defines a small additive category $\mathcal{B}^{\mathcal{C}}$ with the same objects, and morphism groups given by

$$\mathcal{B}^{\mathcal{C}}(c,c') = \mathbb{Z}[\mathcal{C}(c,c')],$$

where $\mathcal{C}(c,c')$ is the set of morphisms from c to c' in the category $\mathcal{C}$. Giving a functor from $\mathcal{C}$ to R-modules is equivalent to giving an additive functor from $\mathcal{B}^{\mathcal{C}}$ to R-modules. If $\mathcal{C}$ is not a category but a 2-category, then $\mathcal{C}(c,c')$ are not sets but small categories. What one can do is consider the geometric realizations $|\mathcal{C}(c,c')|$ of the nerves of the categories $\mathcal{C}(c,c')$, and let

$$\mathcal{B}^{\mathcal{C}}_{\bullet} = C_{\bullet}(|\mathcal{C}(c,c')|,\mathbb{Z})$$

be the singular chain complexes of these geometric realizations. If $\mathcal{C}$ is a strict 2-category – that is, compositions are associative on the nose – then geometric realization is sufficiently functorial so that $\mathcal{B}^{\mathcal{C}}_{\bullet}(-,-)$ define a DG category $\mathcal{B}^{\mathcal{C}}_{\bullet}$ with the same objects as $\mathcal{C}$. In the general case, it is still possible to define $\mathcal{B}^{\mathcal{C}}_{\bullet}$ as an A_∞-category. Then one can define $\mathcal{D}(\mathcal{C},R)$ as the dervied category of A_∞-functors from $\mathcal{B}^{\mathcal{C}}_{\bullet}$ to complexes of R-modules. This is done in [4, Subsection 1.6].

The second approach is that of [4, Section 4]. It uses the nerves more directly. Namely, recall that the nerve $N(\mathcal{C}) : \Delta^o \to \mathrm{Sets}$ of a small category $\mathcal{C}$ is a simplicial set whose value at $[n] \in \Delta^o$ is the set of all diagrams

$$c_0 \xrightarrow{f_0} \dots \xrightarrow{f_{n-1}} c_n \tag{16}$$

in $\mathcal{C}$. For any map $g : [n] \to [n']$ in Δ, a diagram $c_{\bullet} \in N(\mathcal{C})([n'])$ induces a diagram $g^*c_{\bullet} \in N(\mathcal{C})([n])$ such that $g^*c_i = c_{g(i)}$. One can turn $N(\mathcal{C})$ into another small category $\mathcal{N}(\mathcal{C})$ by applying the Grothendieck construction of [3]: the objects of $\mathcal{N}(\mathcal{C})$ are pairs $\langle [n], c_{\bullet}\rangle$ of an object $[n] \in \Delta$ and a diagram (16), and morphisms from $\langle [n], c_{\bullet}\rangle$ to $\langle [n'], c'_{\bullet}\rangle$ are morphisms $g : [n] \to [n']$ in Δ such that $c_{\bullet} = g^*c'_{\bullet}$. Sending a diagram (16) to c_n gives a functor $q : \mathcal{N}(\mathcal{C}) \to \mathcal{C}$, and the corresponding pullback functor

$$q^* : \mathcal{D}(\mathcal{C},R) \to \mathcal{D}(\mathcal{N}(\mathcal{C}),R)$$

is a fully faithful embedding. To characterize its image, say that a morphism in $\mathcal{N}(\mathcal{C})$ defined by $g : [n] \to [n']$ is *special* if $g(n) = n'$, and say that $E \in \mathcal{D}(\mathcal{N}(\mathcal{C}),R)$ is *special* if the map $E(f)$ is invertible for any special f. Then $\mathcal{D}(\mathcal{C},R) \subset \mathcal{D}(\mathcal{N}(\mathcal{C}),R)$ is exactly the full subcategory spanned by special objects.

Now, if one starts with a 2-category $\mathcal{C}$, then diagrams (16) form a category rather than a set. However, one can still define the category $\mathcal{N}(\mathcal{C})$: objects are pairs $\langle [n], c_{\bullet}\rangle$, morphisms from $\langle [n], c_{\bullet}\rangle$ to $\langle [n'], c'_{\bullet}\rangle$ are pairs of a morphism $g : [n] \to [n']$ and a morphism $c_{\bullet} \to g^*c'_{\bullet}$. Then one can keep the same notion of a special map and special object, and define $\mathcal{D}(\mathcal{C},R)$ as the full subcategory in $\mathcal{D}(\mathcal{N}(\mathcal{C}),R)$ spanned by special objects.

As it turns out, the second approach is much better for practical applications: one does not need to keep track of all the higher A_∞-operations, since they are all packed into the structure of the category $\mathcal{N}(\mathcal{C})$.

For applications to Mackey functors, then, one would start with a category $\mathcal{C}$ that has fibered products, define a category $Q\mathcal{C}$ with the same objects and morphisms given by isomorphisms classes of diagrams

$$c \longleftarrow \widetilde{c} \longrightarrow c' \tag{17}$$

in $\mathcal{C}$, and refine it to a 2-category $\mathcal{QC}$ by letting

$$\mathcal{QC}(c, c')$$

be the groupoids of such diagrams and invertible maps between them. Then one would consider the nerve $\mathcal{N}(\mathcal{QC})$. This is more-or-less what is done in [4, Section 4] and [5, Section 4], but with a small modification. It turns out that instead of $\mathcal{N}(\mathcal{QC})$, one can use a smaller category $S\mathcal{C}$. Its objects are pairs $\langle [n], c_\bullet \rangle$ of $[n] \in \Delta$ and a diagram (16) in $\mathcal{C}$, and morphisms from $\langle [n], c_\bullet \rangle$ to $\langle [n'], c'_\bullet \rangle$ are pairs $\langle g, \alpha \rangle$ of a morphism $g : [n] \to [n']$ and a collection of morphisms $\alpha_i : c'_{g(i)} \to c_i$, $0 \leq i \leq n$, such that for any i and j, $1 \leq i \leq j \leq n'$, the diagram

$$\begin{array}{ccc} c'_{g(i)} & \longrightarrow & c'_{g(j)} \\ {\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \alpha_j} \\ c_i & \longrightarrow & c_j \end{array}$$

is a fibered product square in the category $\mathcal{C}$. A map $\langle g, \alpha_\bullet \rangle$ in $S\mathcal{C}$ is special if $g(n) = n'$ and α_n is invertible, and an object $E \in \mathcal{D}(S\mathcal{C}, R)$ is special if $E(f)$ is invertible for any special map f. One denotes by

$$\mathcal{DS}(\mathcal{C}, R) \subset \mathcal{D}(S\mathcal{C}, R) \tag{18}$$

the full subcategory spanned by special objects.

One advantage of the category $S\mathcal{C}$ is that one can define an analog of the embedding (2) without replacing the category $\mathcal{C}^o$ with its nerve: by definition, sending $c \in \mathcal{C}$ to the pair $\langle [0], c \rangle$ gives a natural functor $e : \mathcal{C}^o \to S\mathcal{C}$. Thus every object $E \in \mathcal{D}(S\mathcal{C}, R)$ defines an object

$$\overline{E} = e^* E \in \mathcal{D}(\mathcal{C}^o, R) \tag{19}$$

that we call the *base part* of E. Sending a diagram (16) to c_n gives a natural functor $q : S\mathcal{C} \to Q\mathcal{C}$, and the composition $q \circ e$ is the natural embedding (2). For every special map f in $S\mathcal{C}$, $q(f)$ is invertible in $Q\mathcal{C}$, so that we have a natural functor

$$q^* : \mathcal{D}(Q\mathcal{C}, R) \to \mathcal{DS}(\mathcal{C}, R). \tag{20}$$

This functor is in general not an equivalence, and we take $\mathcal{DS}(\mathcal{C}, R)$ as the correct definition of $\mathcal{D}(\mathcal{QC}, R)$, a refinement of $\mathcal{D}(Q\mathcal{C}, R)$.

Although q^* is not an equivalence, note that the standard t-structure on $\mathcal{D}(S\mathcal{C}, R)$ induces a t-structure on $\mathcal{DS}(\mathcal{C}, R)$, q^* is t-exact, and it identifies the heart of the t-structure on $\mathcal{DS}(\mathcal{C}, R)$ with $\operatorname{Fun}(Q\mathcal{C}, R)$. For any $E \in \operatorname{Fun}(Q\mathcal{C}, R)$, the base part of q^*E is $e^*E \in \operatorname{Fun}(\mathcal{C}^o, R)$; extending this base part to a special object in $\mathcal{D}(S\mathcal{C}, R)$ is equivalent to providing the transfer maps (4).

The category $\mathcal{DS}(\mathcal{C}, R)$ of (18) is functorial in $\mathcal{C}$ in the following sense. Say that a functor $F : \mathcal{C} \to \mathcal{C}'$ between two small categories with fibered product is a *morphism* if it preserves fibered products. Then any morphism F induced a functor $S(F) : S\mathcal{C} \to S\mathcal{C}'$ that commutes with projections to Δ and sends special maps to special maps. Therefore we have a pullback functor

$$S(F)^* : \mathcal{DS}(\mathcal{C}', R) \to \mathcal{DS}(\mathcal{C}, R).$$

It has been shown in [5, Corollary 4.15] that $S(F)^*$ has a left-adjoint functor

$$S(F)_! : \mathcal{DS}(\mathcal{C}, R) \to \mathcal{DS}(\mathcal{C}, R).$$

3.2 Definitions and properties.

Now return to the situation of Section 2: fix a finite group G, and let Γ_G be the category of finite G-sets. This category has fiberd products, so that for any ring R, we can consider the derived category $\mathcal{DS}(\Gamma_G, R)$ of (18).

Definition 3.2.1. An object $E \in \mathcal{D}(\Gamma_G^o, R)$ is *additive* if the natural map (3) is an isomorphism for any $S_1, S_2 \in \Gamma_G$. A *derived R-valued G-Mackey functor* is an object $E \in \mathcal{DS}(\Gamma_G, R)$ with additive base part (19).

We denote the full subcategory of derived R-valued G-Mackey functors by

$$\mathcal{DM}(G, R) \subset \mathcal{DS}(\Gamma_G, R).$$

The standard t-structure on $\mathcal{DS}(\Gamma_G, R)$ induces a t-structure on

$$\mathcal{DM}(G, R)$$

whose heart is identified with $\mathcal{M}(G, R)$ by the functor q^* of (20). It has been proved in [5, Lemma 8.3] that the embedding $\mathcal{DM}(G, R) \subset \mathcal{DS}(\Gamma_G, R)$ admits a left-adjoint *additivization functor*

$$\mathsf{Add} : \mathcal{DS}(\Gamma_G, R) \to \mathcal{DM}(G, R). \tag{21}$$

If $G = \{e\}$ is the trivial group consisting of its unity element e, then $\mathcal{DM}(G, R) \cong \mathcal{D}(R)$ is the derived category of the category of R-modules (note that this is a non-trivial statement that requires a proof, such as the one found in [4, Subsection 3.2]). As in (7), for any subgroup $H \subset G$, we have the adjoint pair of functors ρ^H, γ^H; both are morphisms and induce an adjoint pair of functors

$$S(\gamma^H) : S\Gamma_H \to S\Gamma_G, \qquad S(\rho^H) : S\Gamma_G \to S\Gamma_H.$$

The functor

$$S(\gamma^H)^* \cong S(\rho^H)_! : \mathcal{DS}(\Gamma_G, R) \to \mathcal{DS}(\Gamma_H, R)$$

sends additive objects to additive objects, thus induces a functor

$$\Psi^H : \mathcal{DM}(G, R) \to \mathcal{DM}(H, R). \tag{22}$$

This is the derived counterpart of the categorical fixed points functor (8). On the other hand, for any $H \subset G$ with normalizer N_H and quotient $W = N_H/H$, the fixed points functor $\varphi^H : \Gamma_G \to \Gamma_W$ is also a morphism, and induces an adjoint pair of functors

$$\begin{aligned} \Phi^H &= S(\varphi^H)_! : \mathcal{DS}(\Gamma_G, R) \to \mathcal{DS}(\Gamma_W, R), \\ \mathsf{Infl}^H &= S(\varphi^H)^* : \mathcal{DS}(\Gamma_W, R) \to \mathcal{DS}(\Gamma_G, R), \end{aligned} \tag{23}$$

a derived counterpart the geometric fixed points functor and the inflation functor of (9). As in the non-derived case, both Φ^H and Infl^H preserve additivity, thus induce an adjoint pair of functors between $\mathcal{DM}(G, R)$ and $\mathcal{DM}(H, R)$ (for Φ^H, this is [5, Lemma 6.14]). As in Lemma 2.2, for any normal subgroup $N \subset G$ with quotient $W = G/N$, the inflation functor $\mathsf{Infl}^N : \mathcal{DM}(W, R) \to \mathcal{DM}(G, R)$ is fully faithful (this is [5, Lemma 6.12]).

To define products of derived Mackey functors, one considers the cartesian product functor (10). It is a morphism, but $S(m)_!$ does not automatically preserve additivity. So, for any two algebra R_1, R_2 over a commutative ring k and any $E_1 \in \mathcal{DM}(G,R_1)$, $E_2 \in \mathcal{DM}(G,R_2)$, we let

$$E_1 \circ E_2 = \mathsf{Add}(S(m)_!(E_1 \boxtimes_k E_2)) \in \mathcal{DM}(G, R_1 \otimes_k R_2), \tag{24}$$

where Add is the additivization functor (21). This product has exactly the same properties as the underived product (11). In particular, the fixed points functors Ψ^H, Φ^H are tensor functors. Moreover, we have a derived version $\mathcal{A}_\bullet \in \mathcal{DM}(G,\mathbb{Z})$ of the Burnside Mackey functor that serves as the unit object for the product, and a derived Burnside ring $A^G_\bullet = \mathcal{A}_\bullet([G/G])$. Explicitly, as shown in [5, Subsection 8.3], we have

$$A^G_\bullet = \bigoplus_{H \subset G} C_\bullet(W_H, \mathbb{Z}),$$

where the sum is over all the conjugacy classes of subgroups $H \subset G$, and the corresponding summand is the homology complex of the group $W_H = \operatorname{Aut}_G([G/H])$ with coefficients in $\mathbb{Z}$. In homological degree 0, this recovers the isomorphism (13).

3.3 Pointed G-sets.

To obtain more information about derived Mackey functors, it is convenient to use the following observation. Say that a diagram (1) is *restricted* if the map l is injective, and note that the pullback of an injective map is an injective map. Therefore we have a subcategory $Q_I\Gamma_G \subset Q\Gamma_G$ whose maps are isomorphism classes of restricted diagrams, and a subcategory $S_I\Gamma_G \subset \Gamma_G$ whose maps are pairs $\langle g, \alpha_\bullet \rangle$ with injective α_n. Say that a map in $S_I\Gamma_G$ is special if it is special in $S\Gamma_G$, say that an object $E \in \mathcal{D}(S_I\Gamma_G, R)$ is special if it inverts all special maps, and let $\mathcal{DS}_I(\Gamma_G, R) \subset \mathcal{D}(S_I\Gamma_G, R)$ be the full subcategory spanned by special objects. Then the projection $q : S\Gamma_G \to Q\Gamma_G$ restricts to a projection $q : S_I\Gamma_G \to Q_I\Gamma_G$, and we have a natural functor

$$q^* : \mathcal{D}(Q_I\Gamma_G, R) \to \mathcal{DS}_I(\Gamma_G, R). \tag{25}$$

However, unlike the functor (20), this functor is an equivalence of categories (this is [5, Corollary 4.17]—roughly speaking, the reason this holds is that restricted diagrams (1) have no non-trivial automorphisms).

The category $Q_I\Gamma_G$ is naturally identified with the category Γ_{G+} of finite pointed G-sets (that is, finite G-sets S with distinguished G-invariant element $o \in S$). The equivalence sends S to $S \setminus \{o\}$, and a map $f : S \to S'$ goes to the diagram

$$S \setminus \{o\} \longleftarrow S \setminus f^{-1}(\{o'\}) \xrightarrow{f} S' \setminus \{o'\}.$$

The equivalence (25) together with the restriction with respect to the embedding $S_I\Gamma_G \to S\Gamma_G$ then provide a natural functor

$$r : \mathcal{DS}(\Gamma_G, R) \to \mathcal{D}(\Gamma_{G+}, R), \tag{26}$$

and we introduce the following definition.

Definition 3.3.1 ([5, Definition 6.9])**.** For any derived Mackey functor $E \in \mathcal{DM}(G, R)$ and any simplicial finite pointed G-set $X : \Delta^o \to \Gamma_{G+}$, the *homology complex* of X with coefficients in E is given by

$$C_\bullet(X, E) = C_\bullet(\Delta^o, X^* r(E)),$$

where r is the restriction functor (26), and $C_\bullet(\Delta^o, -)$ is the homology complex of the small category Δ^o.

Moreover, in [5, Subsection 7.4], this definition is extended in the following way: for any $E \in \mathcal{DM}(G, R)$ and $X \in \Delta^o\Gamma_{G+}$, one defines a product $X \wedge E \in \mathcal{DM}(G, R)$. This product is functorial in X and E. For any $S \in \Gamma_G$, one has a natural identification

$$(X \wedge E)(S) \cong C_\bullet(X \wedge S_+, E), \tag{27}$$

where $S_+ \in \Gamma_{G+}$ is obtained by adding a distinguished element o to the set S, and $X \wedge S_+$ stands for pointwise smash-product of pointed sets. With this product, one can prove the following analog of (15).

Definition 3.3.2. A simplicial finite pointed G-set $X \in \Delta^o\Gamma_{G+}$ is *adapted* to a normal subgroup $N \subset G$ if

1. $X^N = [1]_+$ is the constant pointed simplicial set with two elements, one distinguished, one not, and
2. for any subgroup $H \subset G$ not containing N, the reduced chain homology complex $\overline{C}_\bullet(X^H, \mathbb{Z})$ is acyclic.

Lemma 3.3.3 ([5, Lemma 7.14]). *Assume given a finite pointed simplicial G-set $X_N \in \Delta^o\Gamma_{G+}$ adapted to a normal subgroup $N \subset G$. Then for any $E \in \mathcal{DM}(G,R)$, we have a natural isomorphism*

$$X \wedge E \cong \mathsf{Infl}^N(\Phi^N(E)).$$

We note that together with (27), Lemma 3.3.3 provides a canonical identification

$$\Phi^N(E)(S) \cong C_\bullet(X_N \wedge S_+, E) \tag{28}$$

for any $S \in \Gamma_W$, $W = G/N$. It is not difficult to show that for any $N \subset G$, an adapted set $X_N \in \Delta^o\Gamma_{G+}$ does exist. For example, one can take the union S_N of all G-orbits $[G/H]$, $H \subset G$ not containing N, consider the pointed simplicial set $E(S_N)$ given by

$$E(S_N)([n]) = S_N^{n+1}, \tag{29}$$

and let X_N be the cone of the natural map $E(X_N) \to [1]_+$. If for this choice of X_N, one computes the homology of the simplicial category Δ^o by the standard complex, then (28) gives (15) in homological degree 0.

Definition 3.3.4. A simplicial finite pointed G-set $X \in \Delta^o\Gamma_{G+}$ is a *homological sphere* if for any subgroup $H \subset G$, we have $\overline{C}_\bullet(X^H,\mathbb{Z}) \cong \mathbb{Z}[d_H]$ for some integer $d_H \geq 0$.

Lemma 3.3.5 ([5, Proposition 7.18]). *For any homological sphere X, the functor*

$$E \mapsto X \wedge E$$

is an autoequivalence of the category $\mathcal{DM}(G,R)$.

Constructing homological spheres is as easy as constructing adapted sets. For example, for any finite-dimensional real representation V of the group G, we can take a G-invariant triangulation of its one-point compactification S_V and obtain a homological sphere (the dimensions d_H are given by $d_H = \dim_{\mathbb{R}} V^H$). Thus Lemma 3.3.5 produces many autoequivalences of the category $\mathcal{DM}(G,R)$.

3.4 An example.

To illustrate the theory of derived Mackey functors on a concrete example, let us consider the case $G = \mathbb{Z}/p\mathbb{Z}$, the cyclic group of a prime order. Then there are exactly two subgroups in G, the trivial subgroup $\{e\} \subset G$ and G itself, so that up to an isomorphism, O_G has two objects: the free orbit $[G/\{e\}]$ and the trivial orbit $[G/G]$. Thus a G-Mackey functor $E \in \mathcal{M}(G, R)$ is defined by two R-modules, $E^0 = E([G/\{e\}])$ and $E^1 = E([G/G])$. The module E^0 carries the action of the group G. Equivalently, if we denote by $\sigma \in \mathbb{Z}/p\mathbb{Z}$ the generator, then we have an automorphism $\sigma : E^0 \to E^0$ such that $\sigma^p = \mathsf{id}$. Moreover, there one non-trivial map $f : [G/\{e\}] \to [G/G]$. Since $f \circ \sigma = f$, the corresponding maps f_* and f^* induce natural maps

$$V = f_* : (E^0)_\sigma \to E_1, \qquad F = f^* : E^1 = (E^0)^\sigma, \tag{30}$$

where $(E^0)_\sigma$, $(E^0)^\sigma$ stands for coinvariants and invariants with respect to the automorphism σ. The double coset formula (6) then reads as

$$F \circ V = \mathsf{id} + \sigma + \cdots + \sigma^{p-1}. \tag{31}$$

These are the only conditions: the category $\mathcal{M}(G, R)$ is equivalent to the category of pairs $\langle E^0, E^1 \rangle$ of two R-modules equipped with an automorphism $\sigma : E^0 \to E^0$ of order p and two maps (30) satisfying (31).

To obtain a similar description of the category $\mathcal{DM}(G, R)$, choose a resolution $P_\bullet$ of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}[G]$-modules, and denote

$$C_\bullet(G, -) = P_\bullet \otimes_{\mathbb{Z}[G]} -, \qquad C^\bullet(G, -) = \operatorname{Hom}_{\mathbb{Z}[G]}(P_\bullet, -)$$

the homology and cohomology complexes of the group G computed using $P_\bullet$ (for example, one can take the standard periodic resolution, and this would give the standard periodic complexes). For any $R[G]$-module E, we have $H_0(G, E) = E_\sigma$ and $H^0(G, E) = E^\sigma$, and the functorial trace map $\mathsf{id} + \sigma + \cdots + \sigma^{p-1} : E_\sigma \to E^\sigma$ induces a functorial trace map

$$\mathsf{tr} : C_\bullet(G, -) \to C^\bullet(G, -). \tag{32}$$

Then $\mathcal{DM}(G, R)$ is obtained by inverting quasiisomorphisms in the category of pairs $\langle E^0_\bullet, E^1_\bullet \rangle$ of complexes of R-modules equipped with an automorphism $\sigma : E^0_\bullet \to E^0_\bullet$ of order p and two maps

$$C_\bullet(G, E^0_\bullet) \xrightarrow{V} E^1 \xrightarrow{F} C^\bullet(G, E^0_\bullet) \tag{33}$$

whose composition $F \circ V$ coincides with the trace map (32), $F \circ V = \mathsf{tr}$. This description shows clearly why the category $\mathcal{DM}(G, R)$ is different from the derived category $\mathcal{D}(\mathcal{M}(G,R))$. Indeed, objects in $\mathcal{D}(\mathcal{M}(G,R))$ are also represented by pairs $\langle E^0_\bullet, E^1_\bullet\rangle$ and maps σ, F, V, but the homology and cohomology complexes in (33) are replaces with coinvariants $(E^0_\bullet)_\sigma$ and invariants $(E^0_\bullet)^\sigma$. To make these quasiisomorphic to the whole homology complexes, one needs to choose a representative $E^0_\bullet$ that is both h-projective and h-injective as a complex of $R[G]$-modules. For general complex of $R[G]$-modules, such a representative does not exists.

For any object $E \in \mathcal{DM}(G,R)$ represented by a pair $\langle E^0_\bullet, E^1_\bullet\rangle$ as above, the geometric fixed points $\Phi^G(E) \in \mathcal{D}(R)$ can be computed by Lemma 3.3.3. The result is, $\Phi^G(E)$ is quasiisomorphic to the cone $\overline{E}^1_\bullet$ of the map V of (33). This suggests a very useful alternative description of the category $\mathcal{DM}(G,R)$. Namely, recall that the *Tate cohomology complex* $\check{C}^\bullet(G,E_\bullet)$ with coefficients in a complex $E_\bullet$ of $R[G]$-modules is by definition the cone of the trace map (32). Then the map F in (33) induces a natural map

$$\varphi : \overline{E}^1_\bullet \to \check{C}^\bullet(G, E^0_\bullet). \tag{34}$$

Conversely, given $\overline{E}^1_\bullet$ and the map φ, one recovers $E^1_\bullet$ as the cone of the natural map

$$\overline{E}^1_\bullet \xrightarrow{\ \varphi\ } \check{C}^\bullet(G,E^0_\bullet) \longrightarrow C_\bullet(G,E^0_\bullet)[1],$$

this $E^1_\bullet$ comes equipped with the map V of (33), and φ then induces the map F. We see that $\mathcal{DM}(G,R)$ can be obtained by inverting quasiisomorphisms in the category of pairs $\langle E^0_\bullet, \overline{E}^1_\bullet\rangle$ of complexes of R-modules equipped with an automorphism $\sigma : E^0_\bullet \to E^0_\bullet$ of order p and a map φ of (34).

To rephrase this description even further, let $\widetilde{P}_\bullet$ be the cone of the augmentation map $P_\bullet \to \mathbb{Z}$, where $P_\bullet$ is our fixed projective resolution. Then one can always choose $E^0_\bullet$ to be an h-injective complex of $R[G]$-modules, and in this case, $\check{C}^\bullet(G,E^0_\bullet)$ is quasiisomorphic to the sum-total complex of the bicomplex

$$(\widetilde{P}_\bullet \otimes E^0_\bullet)^\sigma.$$

Then the map φ of (34) becomes a σ-invariant map

$$\varphi : \overline{E}^1_\bullet \to \widetilde{P}_\bullet \otimes E^0_\bullet,$$

and the whole data $\langle E^0_{\bullet}, \overline{E}^1_{\bullet}, \sigma, \varphi\rangle$ can be packaged into a single DG comodule over the DG coalgebra $T_{\bullet}(G,R)$ over R of the following form:

$$T_{\bullet}(G,R) = \begin{pmatrix} R[G] & \widetilde{P}_{\bullet} \otimes R \\ 0 & R \end{pmatrix}, \tag{35}$$

where $R[G]$ is the group coalgebra of the group G, and $\widetilde{P}_{\bullet} \otimes R$ is the $R[G]$-comodule obtained from $\widetilde{P}_{\bullet}$. The category $\mathcal{DM}(G,R)$ is then equivalent to the derived category $\mathcal{D}(T_{\bullet}(G,R))$ (that is, the category obtained by inverting quasiisomorphisms in the category of DG comodules over $T_{\bullet}(G,R)$, as in [4, Subsection 1.5.3]).

Using DG coalgebras here is essential – the complex $\widetilde{P}_{\bullet}$ is infinite, and one cannot simply dualize things and interpret $\mathcal{D}(T_{\bullet}(G,R))$ as the derived category of DG modules over a DG algebra. Let us also note that the complex $\widetilde{P}_{\bullet} \otimes R$ is acyclic, so that $T_{\bullet}(G,R)$ is quasiisomorphic to the diagonal DG coalgebra $\overline{T}_{\bullet}(G,R)$ with entries $R[G]$ and R. However, the derived categories $\mathcal{D}(T_{\bullet}(G,R))$ and $\mathcal{D}(\overline{T}_{\bullet}(G,R))$ are different: the latter is the direct sum of $\mathcal{D}(R)$ and $\mathcal{D}(R[G])$, while the former is obtained by gluing these two categories along the gluing functor given by Tate cohomology.

3.5 Maximal Tate cohomology.

To extend the description of derived Mackey functors in terms of geometric fixed points and Tate cohomology to arbitrary finite groups, note that for any group G and derived Mackey functor $E \in \mathcal{DM}(G,R)$, the value $E([G/\{e\}])$ of E at the free orbit $[G/\{e\}] \in \Gamma_G$ is acted upon by $G = \operatorname{Aut}_G([G/\{e\}])$ and gives an object in the derived category $\mathcal{D}(R[G])$. For any subgroup $H \subset G$ with $W_H = N_H/H$, we can apply this observation to the group W_H. Then sending $E \in \mathcal{DM}(G,R)$ to $\Phi^H(E)([W_H/\{e\}]) \in \mathcal{D}(R[W_H])$ gives a natural functor

$$\overline{\Phi}^H : \mathcal{D}(G,R) \to \mathcal{D}(R[W_H]),$$

a restricted version of the geometric fixed points functor of (23). It has been proved in [5, Lemma 7.2] that this functor has a right-adjoint restricted inflation functor

$$\overline{\mathsf{Infl}}^H : \mathcal{D}(R[W_H]) \to \mathcal{D}(G,R). \tag{36}$$

Taking the direct sum of the functors $\overline{\Phi}^H$ over all conjugacy classes of subgroups $H \subset G$, one obtains the functor

$$\overline{\Phi}^{\bullet} : \mathcal{DM}(G, R) \to \prod_{H \subset G} \mathcal{D}(R[W_H]), \tag{37}$$

and one proves the following result.

Lemma 3.5.1 ([5, Lemmas 7.1, 7.3])**.** *The functor $\overline{\Phi}^{\bullet}$ of* (37) *is conservative, and for any $H \subset G$, the restricted inflation functor* (36) *is fully faithful. Moreover, let $\mathcal{DM}_H(G, R) \subset \mathcal{DM}(G, R)$ be the image of the fully faithful embedding $\overline{\mathsf{Infl}}^H$. Then every object $E \in \mathcal{DM}(G, R)$ is an iterated extensions of objects $E_H \in \mathcal{DM}_H(G, R)$, $H \subset G$, and we have* $\operatorname{Hom}(E_H, E_{H'}) = 0$ *unless H is conjugate to a subgroup in H'.*

Thus the category $\mathcal{DM}(G, R)$ admits a semiorthogonal decomposition whose graded pieces $\mathcal{DM}_H(G, R)$ are numbered by conjugacy classes of subgroup $H \subset G$, and we have $\mathcal{DM}_H(G, R) \cong \mathcal{D}(R[W_H])$. For any two subgroups $H' \subset H \subset G$, we then have the gluing functor

$$E^H_{H'} = \overline{\Phi}^H \circ \overline{\mathsf{Infl}}^{H'} : \mathcal{D}(R[H']) \to \mathcal{D}(R[H]). \tag{38}$$

To compute these gluing functors, one introduces in [5, Subsection 7.2] a certain generalization of Tate cohomology that we call *maximal Tate cohomology*. Let us describe it.

Recall that for any subgroup $H \subset G$, the restriction functor

$$r^G_H : \mathcal{D}(R[G]) \to \mathcal{D}(R[H])$$

has a left and right-adjoint induction functor $i^H_G : \mathcal{D}(R[H]) \to \mathcal{D}(R[G])$.

Definition 3.5.2 ([5, Definition 7.4])**.** A $\mathbb{Z}[G]$-module E is *induced* if E is a direct summand of a sum of objects of the form $i^H_G(E_H)$, $H \subset G$ a proper subgroup, E_H a finitely generated $\mathbb{Z}[H]$-module projective over $\mathbb{Z}$. For any $E \in \mathcal{D}^b(\mathbb{Z}[G])$, the *maximal Tate cohomology* $\check{H}^{\bullet}(G, E)$ is given by

$$\check{H}^{\bullet}(G, E) = \operatorname{RHom}^{\bullet}_{\overline{\mathcal{D}}(\mathbb{Z}[G])}(\mathbb{Z}, E),$$

where $\mathcal{D}^b_i(\mathbb{Z}[G]) \subset \mathcal{D}^b(\mathbb{Z}[G])$ is the smallest Karoubi-closed triangulated subcategory containing all induced modules, and

$$\overline{\mathcal{D}}(\mathbb{Z}[G]) = \mathcal{D}^b(\mathbb{Z}[G]) / \mathcal{D}^b_i(\mathbb{Z}[G])$$

is the quotient category.

If $G = \mathbb{Z}/p\mathbb{Z}$, a $\mathbb{Z}[G]$-module is induced if and only if it is finitely generated and projective, so that maximal Tate cohomology coincides with the usual Tate cohomology. In general, they are different. For example, if $G = \mathbb{Z}/n\mathbb{Z}$ is a cyclic group, then $\check{H}^\bullet(G, -) = 0$ unless n is a prime (this is [4, Lemma 7.15 (ii)]).

To compute maximal Tate cohomology, and to generalize it to possibly infinite coefficients, it is convenient to introduce the following.

Definition 3.5.3. A complex $P_\bullet$ of $\mathbb{Z}[G]$-modules is *maximally adapted* if

1. $P_i = 0$ for $i < 0$, $P_0 \cong \mathbb{Z}$, and P_i is induced for any $i \geq 1$, and
2. $r^H_G(P_\bullet)$ is contractible for any proper subgroup $H \subset G$.

Then for any ring R and maximally adapted complex $P_\bullet$, one defines the Tate cohomology object with coefficients in $E \in \mathcal{D}(R[G])$ as

$$\check{C}^\bullet(G, E) = \lim_{\substack{\to \\ l}} C^\bullet(G, E \otimes F^l P_\bullet), \tag{39}$$

where $F^l P_\bullet$ stands for the stupid filtration. It has been proved in [5, Subsection 7.2] that as an object in $\mathcal{D}(R)$, this does not depend on the choice of $P_\bullet$, and for any $E \in \mathcal{D}^b(\mathbb{Z}[G])$, the complex $\check{C}^\bullet(G, E)$ computes the maximal Tate cohomology groups $\check{H}^\bullet(G, E)$ of Definition 3.5.2. If G is a normal subgroup in a larger group G', then for any $E \in \mathcal{D}(R[G'])$ and an appropriate choice of an adapted complex, the expression (39) also defines $\check{C}^\bullet(G, E)$ as an object in $\mathcal{D}(R[G'/G])$.

A full description of the gluing functors (38) in terms of the maximal Tate cohomology objects (39) is given in [5, Proposition 7.10]. Here we only reproduce the answer in the key case $H' = \{e\}$, $H = G$. In this case, we have

$$E^G_{\{e\}} \cong \check{C}^\bullet(G, -).$$

Based on this, in [4, Section 6], one develops a description of the category $\mathcal{DM}(G, R)$ in terms of a certain upper-triangular DG coalgebra $T_\bullet(G, R)$ similar to the coalgebra (35). Unfortunately, this is rather heavy technically (in particular, one cannot get a genuine DG coalgebra and has to settle for its A_∞-version).

A big simplification occurs in the case of the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$, $n \geq 1$. As we have mentioned, $vH^\bullet(\mathbb{Z}/n\mathbb{Z}, -) = 0$ unless n is a prime; moreover, for any two primes p, p', it has been proved in [5, Lemma 9.10] that

$$\check{C}^\bullet(\mathbb{Z}/p\mathbb{Z}, \check{C}(\mathbb{Z}/p'\mathbb{Z}, -)) = 0,$$

so that the compositions of the gluing functors (38) vanish tautologically. Because of this, it has been possible to obtain a reasonably simple description of $\mathcal{DM}(G,R)$. Let us reproduce it (this is [5, Subsection 9.4]).

Fix an integer $n \geq 1$, let $G = \mathbb{Z}/n\mathbb{Z}$, and denote by I_n the groupoid of G-orbits. These are numbered by divisors of n, so that explicitly, we have

$$I_n = \coprod_{d|n} \mathsf{pt}_d,$$

where pt_d stands for the groupoid with one object with automorphism group $\mathbb{Z}/d\mathbb{Z}$. For any prime p, let

$$I_n^p = \coprod_{p|d|n} \mathsf{pt}_d \subset I_n,$$

and let $I_n^\bullet$ be the disjoint union of I_n^p over all primes p. Of course I_n^p is empty unless p divides n. For any p dividing n, we have the natural embedding $i : I_n^p \to I_n$, and we also have a natural projection $\pi : I_n^p \to I_n$ given by the union of quotient maps $\mathsf{pt}_{pm} \to \mathsf{pt}_m$, m a divisor of n/p. Let

$$i, \pi : I_n^\bullet \to I_n \tag{40}$$

be the disjoint union of these functors. Finally, choose a projective resolution $P_\bullet$ of the constant functor $\mathbb{Z} \in \operatorname{Fun}(I_n^\bullet, \mathbb{Z})$, and let $\widetilde{P}_\bullet$ be the cone of the augmentation map $P_\bullet \to \mathbb{Z}$.

Definition 3.5.4. An *R-valued fixed points datum* for I_n is a pair $\langle M_\bullet, \alpha\rangle$ of a complex $M_\bullet$ in the category $\operatorname{Fun}(I_n, R)$ and a map

$$\alpha : \pi^* M_\bullet \to \widetilde{P}_\bullet \otimes i^* M_\bullet,$$

where i and π are the projections (40).

Then I_n-fixed points data form a category, and inverting quasiisomorphisms in this category, one obtains a category $\mathcal{D}_n^\alpha(R)$. It has been shown in [5, Subsection 9.3] that $\mathcal{D}_n^\alpha(R)$ is a triangulated category that does not depend on the coice of a resolution $P_\bullet$. One then proves the following comparison result.

Proposition 3.5.5 ([5, Proposition 9.14 (i)])**.** *For any $n \geq 1$ and ring R, there exiss a natural equivalence of triangulated categories*

$$\mathcal{DM}(\mathbb{Z}/n\mathbb{Z}, R) \cong \mathcal{D}_n^\alpha(R).$$

4 Mackey profunctors.

4.1 Definitions.

Assume now given an infinite group G. Then if one considers arbitrary G-sets, all the theory of Section 2 and Section 3 becomes trivial (for example, the Burnside ring of (12) would be identically zero). If we stick to finite G-sets, the theory works, up to a point. However, finer parts of the theory such as Subsection 3.5 break down.

As it turns out, there is an interesting possibility in between the two extremes. It is based on the following definition.

Definition 4.1.1 ([5, Definition 3.1])**.** A G-set S is *admissible* if

1. for any $s \in S$, its stabilizer $G_s \subset G$ is a cofinite subgroup, and
2. for any cofinite subgroup $H \subset G$, the fixed points set S^H is finite.

Explicitly, any G-set S decomposes into a disjoint union of orbits

$$S = \coprod_{s \in S/G} [G/H_s]. \tag{41}$$

Then S is admissible iff all the subgroups $H_s \subset G$ are cofinite, and for any cofinite subgroup $H \subset G$, at most a finite number of them contain H.

One denotes the category of admissible G-sets by $\widehat{\Gamma}_G$, and one notes that this category has fibered products. Therefore we can consider the category $Q\widehat{\Gamma}_G$ and the embedding (2).

Definition 4.1.2. For any ring R, an object $E \in \mathcal{D}(\widehat{\Gamma}^o_G, R)$ is *additive* if for any admissible G-set S with decomposition (41), the natural map

$$E(S) \to \prod_{s \in S/G} E([G/H_s])$$

is an isomorphism. An *R-valued G-Mackey profunctor* is an object $E \in \operatorname{Fun}(Q\widehat{\Gamma}_G, R)$ whose restriction e^*E is additive.

We denote by

$$\widehat{\mathcal{M}}(G, R) \subset \operatorname{Fun}(Q\Gamma_G, R)$$

the full subcategory spanned by Mackey profunctors. Since infinite sums in the category of R-modules are exact, $\widehat{\mathcal{M}}(G, R)$ is an abelian category.

As in the finite group case, for any cofinite subgroup $H \subset G$, we have the pair of adjoint functors ρ^H, γ^H between $\widehat{\Gamma}_G$ and $\widehat{\Gamma}_H$. The functors preserve fibered products, the corresponding functors $Q(\rho^H)$, $Q(\gamma^H)$ are also adjoint, and the functor $Q(\rho^H)_! \cong Q(\gamma^H)^*$ preserves the additivity condition of Definition 4.1.2, thus sends Mackey profunctors to Mackey profunctors. The corresponding functor

$$\Psi^H = Q(\rho^H)_! \cong Q(\gamma^H)^* : \widehat{\mathcal{M}}(G,R) \to \widehat{\mathcal{M}}(H,R)$$

is the *categorical fixed points* functor. Moreover, if we denote $W = N_H/H$, then the fixed points functor $\varphi^H : \widehat{\Gamma}_G \to \Gamma_W$ also preserves fibered products, and we have an adjoint pair of functors

$$\begin{aligned} \Phi^H &= Q(\varphi^H)_! : \operatorname{Fun}(Q\widehat{\Gamma}_G, R) \to \operatorname{Fun}(Q\Gamma_W, R), \\ \mathsf{Infl}^H &= Q(\varphi^H)^* : \operatorname{Fun}(Q\Gamma_W, R) \to \operatorname{Fun}(Q\widehat{\Gamma}_G, R). \end{aligned} \tag{42}$$

For the same reasons as in the usual case, these preserve additivity, thus induce functors between $\widehat{\mathcal{M}}(G,R)$ and $\mathcal{M}(W,R)$, the *geometric fixed points functor* and the *inflation functor*.

Lemma 2.2 also holds for Mackey profunctors. In particular, for any normal cofinite subgroup $N \subset G$, the inflation functor Infl^N is fully faithful. Any Mackey profunctor E gives rise to a Mackey functor $E_N = \Phi^N E \in \mathcal{M}(W,R)$, and we have a natural surjective map

$$E \to \mathsf{Infl}^N(E_N).$$

For any two cofinite normal subgroups $N \subset N' \subset G$, we have a natural isomorphism

$$\Phi^{N'/N} E_N \cong E_{N'}. \tag{43}$$

It is convenient to axiomatize the situation as follows.

Definition 4.1.3. An *R-valued G-normal system* $\langle E_\bullet \rangle$ is a collection of Mackey functors $E_N \in \mathcal{M}(G/N,R)$, one for each cofinite normal subgroup $N \subset G$, and a collection os isomorphisms (43), one for each pair of cofinite normal subgroups $N \subset N' \subset G$.

Normal systems form an additive R-linear category which we denote by $\mathcal{N}(G,R)$. Sending E to $\langle \Phi^N(E) \rangle$ gives a functor $\Phi : \widehat{\mathcal{M}}(G,R) \to \mathcal{N}(G,R)$. This functor has a right-adjoint

$$\mathsf{Infl} : \mathcal{N}(G,R) \to \widehat{\mathcal{M}}(G,R) \tag{44}$$

sending a normal system $\langle E_N \rangle$ to

$$E = \lim_{\stackrel{N}{\leftarrow}} \mathsf{Infl}^N(E_N), \tag{45}$$

where the limit is taken over all cofinite normal subgroups $N \subset G$, with respect to the natural maps adjoint to the isomorphisms (43). It turns out that the following is true.

Lemma 4.1.4 ([5, Proposition 3.5])**.** *The functor* Infl *of* (44) *is fully faithful. Conversely, for any Mackey profunctor* $E \in \widehat{\mathcal{M}}(G,R)$*, the adjunction map*

$$E \to \mathsf{Infl}(\Phi(E)) = \lim_{\stackrel{N}{\leftarrow}} \mathsf{Infl}^N(\Phi^N(E)) \tag{46}$$

is surjective.

One says that a Mackey profunctor E is *separated* if the surjective map (46) is bijective – or equivalently, if E lies in the image of the fully faithful embedding Infl. One denotes by $\widehat{\mathcal{M}}_s(G,R) \subset \widehat{\mathcal{M}}(G,R)$ the full subcategory spanned by separated Mackey profunctors. This category is additive but not necessarily abelian. However, it is equivalent to $\mathcal{N}(G,R)$, and this allows to transport results about Mackey functors to separated Mackey profunctors. In particular, it has been proved in [5, Lemma 3.9] that the natural embedding $\widehat{\mathcal{M}}_s(G,R) \subset \operatorname{Fun}(Q\widehat{\Gamma}_G, R)$ admits a left-adjoint additivization functor

$$\mathsf{Add} : \operatorname{Fun}(Q\widehat{\Gamma}_G, R) \to \widehat{\mathcal{M}}_s(G,R). \tag{47}$$

Using this functor, one can extend the definition of the geomeric fixed points functor Φ^H to an arbitrary subrgroup $H \subset G$. Indeed, even if $H \subset G$ is not cofinite, the inflation functor Infl^H of (42) sends Mackey profunctors to Mackey profunctors, and it also sends separated Mackey profunctors to separated ones. Then it has a natural left-adjoint given by

$$\Phi^H = \mathsf{Add} \circ Q(\varphi^H)_!.$$

Another application of the addivization functor is tensor products of separated Mackey profunctors; these are defined by the same formula (11) as in the Mackey functor case. The product is associative, commutative and unital. The unit object is the Burnside Mackey profunctor $\widehat{\mathcal{A}}$. Explicitly, it is given by

$$\widehat{\mathcal{A}}([G/H]) = \widehat{A}^H,$$

where the completed Burnside ring $\widehat{A}^H$ is given by (12) with Γ_H replaced by $\widehat{\Gamma}_H$. By virtue of the decomposition (41), we have

$$\widehat{A}^G = \prod_{H \subset G} \mathbb{Z}, \tag{48}$$

where the product is over conjugacy classes of cofinite subgroups $H \subset G$.

4.2 An example.

To illustate the difference between Mackey functors and Mackey profunctors, consider the case $G = \mathbb{Z}$, the infinite cyclic group. Cofinite subgroups in $\mathbb{Z}$ are of the form $n\mathbb{Z} \subset \mathbb{Z}$, $n \geq 1$, so that $\mathbb{Z}$-orbits are numbered by positive integers. The Burnside ring $A^{\mathbb{Z}}$ is given by (13); explicitly, as an abelian group, we have

$$A^{\mathbb{Z}} = \mathbb{Z}[\varepsilon_1, \varepsilon_2, \dots],$$

where the generators ε_n, $n \geq 1$ correspond to $\mathbb{Z}$-orbits $[\mathbb{Z}/n\mathbb{Z}]$. For any $n, m \geq 1$, the product $[\mathbb{Z}/m\mathbb{Z}] \times [\mathbb{Z}/n\mathbb{Z}]$ is the union of copies of the orbit $\mathbb{Z}/\{n, m\}\mathbb{Z}$, where $\{n, m\}$ is the least common multiple of n and m. Since the cardinality of this product is nm, this implies that

$$\varepsilon_n \varepsilon_m = \frac{nm}{\{n, m\}} \varepsilon_{\{n,m\}}, \tag{49}$$

and this completely defines the product in the Burnside ring $A^{\mathbb{Z}}$.

Now, for the completed Burnside ring $\widehat{A}^{\mathbb{Z}}$, the sum (13) is replaced by the product (48). Therefore we have

$$\widehat{A}^{\mathbb{Z}} = \mathbb{Z}\{\varepsilon_1, \varepsilon_2, \dots\},$$

that is, the group of infinite linear combinations of the generators ε_n, $n \geq 1$. The product is still given by (49).

Here is one observation we can make right away: the completed Burnside ring $\widehat{A}^{\mathbb{Z}}$ is isomorphic to the universal Witt vectors ring $\mathbb{W}(\mathbb{Z})$. There is a reason for this coincidence, but it goes beyond the subject of the present paper.

Another observation is the following. For any ring R, any R-valued $\mathbb{Z}$-Mackey functor E is acted upon by the Burnside ring $A^{\mathbb{Z}}$, and any $\mathbb{Z}$-Mackey profunctor E is acted upon by the completed Burnside ring $\widehat{A}^{\mathbb{Z}}$. Assume that R is p-local for some prime p — that is, every integer n

prime to p is invertible in R. Then for any such n, we have a well-defined endomorphism

$$\frac{1}{n}\varepsilon_n : E \to E, \tag{50}$$

and (49) shows that this endomorphism is idempotent. Thus E is naturally equipped with a large family of commuting idempotents.

If E is a $\mathbb{Z}$-Mackey functor, then this is the end of the story. However, if E is a $\mathbb{Z}$-Mackey profunctor, we can replace the commuting idempotents (50) with orthogonal commuting idempotents $\varepsilon_{(n)}$ given by

$$\varepsilon_{(n)} = \frac{1}{n}\varepsilon_n \cdot \prod_{i \text{ does not divide } n} \left(1 - \frac{1}{i}\varepsilon_i\right),$$

where the product is over i prime to p. We have

$$1 = \sum_{n \text{ prime to } p} \varepsilon_{(n)},$$

so that for any $E \in \widehat{\mathcal{M}}(\mathbb{Z}, R)$, we have a canonical decomposition

$$E = \prod_{n \text{ prime to } p} E_{(n)}, \qquad E_{(n)} = \operatorname{Im} \varepsilon_{(n)} \subset E, \tag{51}$$

which is functorial in E, and gives a decomposition of the category $\widehat{\mathcal{M}}(\mathbb{Z}, R)$.

It turns out that the pieces of this decomposition can be described in terms of Mackey profunctors for the group $\mathbb{Z}_p$ of p-adic integers. Namely, by definition, the category $\widehat{\mathcal{M}}(G, R)$ only depends on the profinite completion of the group G. The profinite completion $\widehat{\mathbb{Z}}$ is given by

$$\widehat{\mathbb{Z}} = \prod_l \mathbb{Z}_l,$$

where the product is over all primes l. If we denote by $\mathbb{Z}'_p$ the product of all primes different from p, then $\widehat{\mathbb{Z}} = \mathbb{Z}_p \times \mathbb{Z}'_p$, and we have the geomeric fixed points functor

$$\Phi^{(p)} = \Phi^{\mathbb{Z}'_p} : \widehat{\mathcal{M}}(\mathbb{Z}, R) \cong \widehat{\mathcal{M}}(\widehat{\mathbb{Z}}, R) \to \widehat{\mathcal{M}}(\mathbb{Z}_p, R).$$

In [5, Subsection 9.2], this functor has been refined – for any integer $n \geq 1$ prime to p, one constructs a functor

$$\Phi^{(p)}_{[n]} : \widehat{\mathcal{M}}(\mathbb{Z}, R) \to \widehat{\mathcal{M}}(\mathbb{Z}_p, R[\mathbb{Z}/n\mathbb{Z}])$$

such that $\Phi^{(p)} = \Phi^{(p)}_{(1)}$, and one proves the following result.

Proposition 4.2.1 ([5, Proposition 9.4])**.** *Assume that the ring R is p-local. Then the functor*

$$\prod \Phi^{(p)}_{(n)} : \widehat{\mathcal{M}}(\mathbb{Z}, R) \to \prod_{n \text{ prime to } p} \widehat{\mathcal{M}}(\mathbb{Z}_p, R[\mathbb{Z}/n\mathbb{Z}])$$

is an equivalence of categories, and for any $E \in \widehat{\mathcal{M}}(\mathbb{Z}, R)$, the component $E_{(n)}$ of the decomposition (51) *corresponds to $\Phi^{(p)}_{(n)}(E)$.*

If one identifies the Burnside ring $\widehat{A}^{\mathbb{Z}}$ with the Witt vectors ring $\mathbb{W}(\mathbb{Z})$, then the decomposition (51) corresponds to the p-typical decomposition of Witt vectors; because of this, in [5, Subsection 9.2], (51) is called *p-typical decomposition.*

4.3 Derived version.

The derived counterpart of the theory of Mackey profunctors is largely parallel to the theory of derived Mackey functors of Section 3. Since the category $\widehat{\Gamma}_G$ has fibered products, it can be plugged into the machinery of Subsection 3.1. We thus have the category $S\widehat{\Gamma}_G$ and the category $\mathcal{DS}(\widehat{\Gamma}_G, R)$ for any ring R. The additivity condition of Definition 4.1.2 makes sense of objects in the derived category $\mathcal{D}(\widehat{\Gamma}^o_G, R)$. A *derived R-valued G-Mackey profunctor* is an object $E \in \mathcal{DS}(\widehat{\Gamma}_G, R)$ whose base part $\overline{E}$ of (19) is additive.

Denote the category of derived G-Mackey profunctors by $\widehat{\mathcal{DM}}(G, R)$. This category has a t-structure induced by the standard t-structure on $\mathcal{DS}(\widehat{\Gamma}_G, R)$, and the heart of this t-structure is identified with $\widehat{\mathcal{M}}(G, R)$.

For any cofinite subgroup $H \subset G$, we have the categorical fixed points functor

$$\Psi^H : \widehat{\mathcal{DM}}(G, R) \to \widehat{\mathcal{DM}}(H, R)$$

defined exactly as in (22). Moreover, if we denote $W = N_H/H$, then we have the adjoint pair

$$\Phi^H : \widehat{\mathcal{DM}}(G, R) \to \mathcal{DM}(W, R), \quad \mathsf{Infl}^H : \mathcal{DM}(W, R) \to \widehat{\mathcal{DM}}(G, R)$$

of the geometric fixed points functor Φ^H and the inflation functor Infl^H defined as in (23). The inflation functor is defined even for a subgroup $H \subset G$ that is not cofinite. For a cofinite normal subgroup $N \subset G$, the inflation functor Infl^N is fully faithful (this is [5, Lemma 6.12]). We also have a version of Lemma 3.3.3 (the statement is the same, but one needs to use a slightly stronger notion of an adapted set given in [5, Definition 6.3]).

To proceed further, one needs to introduce a derived counterpart of the notion of a normal system. Simply repeating Definition 4.3.1 does not work since it does not produce a triangulated category – we have to package the same data in a more elaborate way. To do this, let N be the partially ordered set of normal cofinite subgroups $N \subset G$, ordered by inclusion, and let

$$\overline{\Gamma}_G \subset \Gamma_G \times \mathsf{N}^o$$

be the subcategory of pairs $\langle S, N\rangle$ such that N acts trivially on S (we treat the partially ordered set N as a small category in the usual way, and let N^o be the opposite category). For every $N \in \mathsf{N}$, we have a natural functor $\tau_N : \Gamma_{G/N} \to \overline{\Gamma}_G$ sending S to $\langle S, N\rangle$. Moreover, for any pair of cofinite normal subgroups $N \subset N' \subset G$, we have the fixed points functor $\varphi^{N'/N} : \Gamma_{G/N} \to \Gamma_{G/N'}$, and the inclusions $S^{N'/N} \subset S$ glue together to give a map of functors

$$\tau_{N'} \circ \varphi^{N'/N} \to \tau_N. \tag{52}$$

We also have the forgetful functor $\nu : \overline{\Gamma}_G \to \mathsf{N}^o$ sending $\langle S, N\rangle$ to $N \in \mathsf{N}^o$, and to define derived normal systems, we do the S-construction of Subsection 3.1 relatively over N^o – that is, we consider the full subcategory

$$S(\overline{\Gamma}_G/\,\mathsf{N}^o) \subset S\overline{\Gamma}_G$$

formed by diagrams (16) in $\overline{\Gamma}_G$ such that all maps become invertible after applying ν. Then for any $N \in \mathsf{N}^o$, the functor τ_N gives a functor

$$S(\tau_N) : S\Gamma_{G/N} \to S(\overline{\Gamma}_G/\,\mathsf{N}^o),$$

and for a pair $N \subset N' \subset G$, $N, N' \in \mathsf{N}$, the morphism (52) gives a morphism

$$S(\tau_N) \to S(\tau_{N'}) \circ S(\varphi^{N'/N}). \tag{53}$$

Therefore any object $E \in \mathcal{D}(S(\overline{\Gamma}_G/\,\mathsf{N}), R)$ produces a collection of objects

$$E_N = S(\tau_N)^*E \in \mathcal{D}(S\Gamma_{G/N}, R), \qquad N \in \mathsf{N},$$

related by morphisms

$$E_N \to S(\varphi^{N'/N})^*E_{N'}. \tag{54}$$

Definition 4.3.1. An *R-valued derived G-normal system* is an object $E \in \mathcal{D}(S(\overline{\Gamma}_G/\mathsf{N}), R)$ such that for any $N \in \mathsf{N}$, $E_N \in \mathcal{D}(S\Gamma_{G/N}, R)$ is a Mackey functor, and for any $N \subset N' \subset G$, the map $\Phi^{N'/N}E_N \to E_{N'}$ adjoint to the map (54) is an isomorphism.

By construction, derived normal systems form a triangulated category; we denote it by $\mathcal{DN}(G,R)$. For every integer n, we let

$$\mathcal{DN}^{\leq n}(G,R) \subset \mathcal{DN}(G,R)$$

be the full subcategory formed by objects E such that for any $N \in \mathsf{N}$, E_N lies in $\mathcal{DM}^{\leq n}(G/N,R)$. These subcategories do not necessarily give a t-structure on $\mathcal{DN}(G,R)$. However, they are perfectly well-defined.

With these definitions, one constructs a functor

$$\Phi : \widehat{\mathcal{DM}}(G,R) \to \mathcal{DN}(G,R)$$

such that for any $E \in \mathcal{DM}(G,R)$ and $N \in \mathsf{N}$, $\Phi(E)_N$ is canonically identified with $\Phi^N(E)$, and the map (43) is adjoint to the map (54). Then one proves the following derived counterpart of Lemma 4.1.4.

Proposition 4.3.2 ([5, Proposition 8.2])**.** *Assume that the group G is finitely generated. Then for every integer n, the functor Φ induces an equivalence of categories*

$$\Phi : \mathcal{DM}^{\leq n}(G,R) \cong \mathcal{DN}^{\leq n}(G,R).$$

As a corollary of this Proposition, we can consider the union

$$\mathcal{DN}^{-}(G,R) = \bigcup_n \mathcal{DN}^{\leq n}(G,R) \subset \mathcal{DN}(G,R)$$

of all the categories $\mathcal{DN}^{\leq n}(G,R)$, and see that it is naturally equivalent to the full subcategory $\widehat{\mathcal{DM}}^{-}(G,R) \subset \widehat{\mathcal{DM}}(G,R)$ of derived Mackey profunctors bounded from above with respect to the standard t-structure.

As we see, Proposition 4.3.2 is much stronger that Lemma 4.1.4: a derived Mackey profunctor is separated as soon as it is bounded from above. In particular, any Mackey profunctor $E \in \widehat{\mathcal{M}}(G,R) \subset \widehat{\mathcal{DM}}(G,R)$ is separated in the derived sense. What happens is, the equivalence Infl inverse to Φ is given explicitly by

$$\mathsf{Infl}(E) = \lim_{\overleftarrow{N}}{}^{\bullet}\, \mathsf{Infl}^N(E_N),$$

but since the inverse limit is not an exact functor, this is different from (45) – there could be non-trivial contributions from $R^1 \lim_{\leftarrow}$ in the right-hand side. One shows that we in fact have $R^i \lim_{\leftarrow} = 0$ for $i \geq 2$, so for any Mackey profunctor E, the kernel of the canonical surjective map (46) is identified with

$$R^1 \lim_{\stackrel{\leftarrow}{N}} \mathsf{Infl}^N(L^1\Phi^N(E)).$$

Applications of Proposition 4.3.2 are similar to those of Lemma 4.1.4. Firstly, one proves in [5, Lemma 8.3] that the embedding

$$\widehat{\mathcal{DM}}^-(G,R) \subset \mathcal{DS}^-(\widehat{\Gamma}_G, R)$$

admits a left-adjoint additivization functor

$$\mathsf{Add} : \mathcal{DS}^-(\widehat{\Gamma}_G, R) \to \widehat{\mathcal{DM}}^-(G,R). \tag{55}$$

We note that by adjunction, Add is right-exact with respect to the standard t-structures, and on the hearts of the standard t-structures, it induces a functor $\operatorname{Fun}(Q\,\widehat{\Gamma}_G, R) \to \widehat{\mathcal{M}}(G,R)$ right-adjoint to the natural embedding — that is, a refinement of the additivization functor (47). Using this refinement, one extends the product (11) to all Mackey profunctors, not only the separated ones, and one does the same with the geometric fixed points functors Φ^H for arbitrary subgroups $H \subset G$.

In the derived theory, one uses (55) to define the tensor product of derived Mackey profunctors by (24), and one defines the geometric fixed points functor Φ^H with respect to an arbitrary subgroup $H \subset G$ by

$$\Phi^H = \mathsf{Add} \circ S(\varphi^H)_!.$$

This is left-adjoint to the inflation functor Infl^H. We also have a version of Lemma 3.3.5 — namely, [5, Lemma 8.7] — and can prove an analog of Lemma 3.5.1, although the semiorthognal decomposition on $\mathcal{DM}^-(G,R)$ would have an infinite number of terms.

Finally, let us note that in the case $G = \mathbb{Z}$, the infinite cyclic group, we have a complete analog of Proposition 3.5.5. Namely, let

$$I = \coprod_{n \geq 1} \mathsf{pt}_n$$

be the groupoid of $\mathbb{Z}$-orbits, let $I^p \subset I$ be the subcategory spanned by pt_{np}, $n \geq 1$, let $I^\bullet$ be the disjoint union of the categories I^p over all prime p, and let

$$i, \pi : I^\bullet \to I \tag{56}$$

be the natural functors defined in the same way as (40). Choose a projective resolution $P_\bullet$ of the constant functor $\mathbb{Z} \in \mathrm{Fun}(I^\bullet, \mathbb{Z})$, and let $\widetilde{P}_\bullet$ be the cone of the augmentation map $P_\bullet \to \mathbb{Z}$.

Definition 4.3.3. An *R-valued fixed points datum* is a pair $\langle M_\bullet, \alpha \rangle$ of a complex $M_\bullet$ in the category $\mathrm{Fun}(I, R)$ bounded from above, and a map

$$\alpha : \pi^* M_\bullet \to \widetilde{P}_\bullet \otimes i^* M_\bullet,$$

where i and π are the projections (56).

Then just as in the case of a finite cyclic group, I-fixed points data form a category, and inverting quasiisomorphisms in this category, one obtains a category $\mathcal{D}^\alpha(R)$. One shows that $\mathcal{D}^\alpha(R)$ is a triangulated category that does not depend on the choice of a resolution $P_\bullet$, and proves the following result.

Proposition 4.3.4 ([5, Proposition 9.14 (ii)]). *For any ring R, there exiss a natural equivalence of triangulated categories*

$$\mathcal{DM}^-(\mathbb{Z}, R) \cong \mathcal{D}^\alpha(R).$$

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References

[1] T. tom Dieck, *Transformation groups*, De Gruyter, Berlin-New York, 1987.

[2] A.W.M. Dress, *Contributions to the theory of induced representations*, in *Algebraic K-Theory II*, (H. Bass, ed.), Lecture Notes in Math. **342**, Springer-Verlag, 1973; pp. 183–240.

[3] A. Grothendieck, *SGA I, Exposé VI.*

[4] D. Kaledin, *Derived Mackey functors*, Mosc. Math. J. **11** (2011), 723–803.

[5] D. Kaledin, *Mackey profunctors*, `arxiv:1412.3248`.

[6] L.G. Lewis, J.P. May, and M. Steinberger, *Equivariant stable homotopy theory*, with contributions by J. E. McClure, Lecture Notes in Mathematics, **1213**, Springer-Verlag, Berlin, 1986.

[7] H. Lindner, *A remark on Mackey functors*, Manuscripta Math. **18** (1976), 273–278.

[8] J.P. May, *Equivariant homotopy and cohomology theory*, with contributions by M. Cole, G. Comezana, S. Costenoble, A.D. Elmendorf, J.P.C. Greenlees, L.G. Lewis, Jr., R.J. Piacenza, G. Triantafillou, and S. Waner, CBMS Regional Conference Series in Mathematics, **91**. Published for the Conference Board of the Mathematical Sciences, Washington, DC, by the American Mathematical Society, Providence, RI, 1996.

[9] J. Thevenaz and P. Webb, *The structure of Mackey functors*, Trans. AMS **347** (1995), 1865–1961.