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Approximating diffeomorphisms by elements of Thompson's groups F and T *

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Abstract

We show how to approximate diffeomorphisms of the closed interval and the circle by elements of Thompson's groups F and T, respectively. This is relevant in the context of Jones' continuum limit of discrete multipartite systems and its dynamics.

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1 Introduction

Over the past few years, V. F. R. Jones has introduced *discrete* analogues of conformal field theories (CFTs) with the aim of constructing a suitable continuum limit to recover a CFT [2, 3, 4]. In the discrete theory, a finitely generated infinite group known as Thompson's group T takes the role of $\mathrm{Diff}_+(\mathbb{S}^1)$, the group of orientation-preserving diffeomorphisms of the circle. In contrast to diffeomorphisms, the elements of T are piecewise-linear homeomorphisms, which explains the term 'discrete'. The idea has already been applied to physics in the context of holography [5, 1].

The dynamics of the discrete theory is given by (projective) unitary representations of T on an appropriate Hilbert space. While it has been

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shown that most of these representations are topologically discontinuous and thus unphysical [3, 6], interesting exceptions may still exist. The idea – and challenge – is to find a procedure that takes a discrete theory as input and then outputs a continuous theory. Such a procedure would certainly include some kind of limit $g_n \to f$, where $g_n \in T$ and $f \in \mathrm{Diff}_+(\mathbb{S}^1)$.

The purpose of this paper is to clarify how orientation-preserving diffeomorphisms of $\mathbb{S}^1$ can be approximated by elements of Thompsons's group T. This includes a similar description for orientation-preserving diffeomorphisms of the interval $I = [0, 1]$ and Thompson's group F. The corresponding density theorems are certainly known and have been proved for $\mathrm{Homeo}_+(I)$ and $\mathrm{Homeo}_+(\mathbb{S}^1)$ in a much more general setting [7, 8]. The advantage of our work is a direct proof that is hands-on for the present context and can be directly translated into an algorithm to construct approximations, suitable for the computer.

The reader who is specifically interested in computational applications can find a step-by-step outline of the construction in Section 3.1.

2 Main Facts

Recall that the dyadic rationals are all numbers of the form $m/2^k$ with $m \in \mathbb{Z}$ and $k \in \mathbb{N} = \{0, 1, 2, \dots\}$. By a *breakpoint* of a piecewise linear function we mean the points at which it is not differentiable.

Definition 2.1. Thompson's group F is the group of piecewise linear homeomorphisms g of the closed unit interval $I = [0, 1]$ such that

(Th_1) the breakpoints of g and their images are dyadic rationals;

(Th_2) on intervals of differentiability, the derivatives of g are integer powers of 2;

and Thompson's group T is the group of piecewise linear homeomorphisms g of $\mathbb{S}^1$ with these properties.[2]

Let $\mathrm{Diff}^1_+(I)$ denote the group of orientation-preserving C^1-diffeomorphisms of the interval, and similarly for $\mathbb{S}^1$. Our result is stated in

[2]These definitions of F and T differ from, but are equivalent to, the standard reference [9]. In particular, our definition of F is not minimal since it actually suffices to require that only the breakpoints are dyadic rationals. Their images are then automatically dyadic due to property (Th_2) and the fact that 0 is a fixpoint.

terms of the C^0-norm

$$\|f\| = \sup_x |f(x)|.$$

Theorem 2.2. *For every $f \in \mathrm{Diff}^1_+(I)$ and $\epsilon > 0$, there exists $g \in F$ such that $\|f - g\| < \epsilon$. Similarly, if $f \in \mathrm{Diff}^1_+(\mathbb{S}^1)$, then there exists $g \in T$ with this property.*

This statement is known and follows from [8, Thm. A4.1] and [7, Prop. 4.3]. It is actually true for all orientation-preserving homeomorphisms. In Section 3 we will give a direct proof of the theorem in the present context.

The next logical question is whether there is an approximation for the first derivatives of diffeomorphisms. While generally elements of both F and T are not everywhere differentiable, we can define a function

$$d(f,g) = \sup_{x \in \mathbb{S}^1 \setminus B_g} |f'(x) - g'(x)|.$$

that measures the distance between the first derivatives of $f \in \mathrm{Diff}^1_+(\mathbb{S}^1)$ and $g \in T$ wherever g' is defined. Here B_g denotes the set of breakpoints of g. (The definition of d for $\mathrm{Diff}^1_+(I)$ and F is analogous.) We can therefore rephrase the question: Given a diffeomorphism f and $\epsilon > 0$, is there a function g from the appropriate Thompson group such that $d(f,g) < \epsilon$? The answer is that such an approximation is not possible since the set of all integer powers of 2 is very sparse in $(0,1)$. This fact is made precise in the following proposition, which is similar to [10, Thorme III.2.3].

Proposition 2.3. *For every $f \in \mathrm{Diff}^1_+(\mathbb{S}^1)$ which is not a rotation, there exists $\mu > 0$ such that $d(f,g) > \mu$ for all $g \in T$. The same holds when $\mathbb{S}^1$ is replaced by I and T is replaced by F.*

Here the rotations in $\mathrm{Diff}^1_+(\mathbb{S}^1)$ are all elements f with $f'(x) = 1$ for all $x \in \mathbb{S}^1$, which includes the identity. In $\mathrm{Diff}^1_+(I)$, the identity is the only rotation.

3 Approximating Diffeomorphisms

In this section, we describe the approximation procedure that represents a proof of Theorem 2.2. We begin with a few simplifying observations.

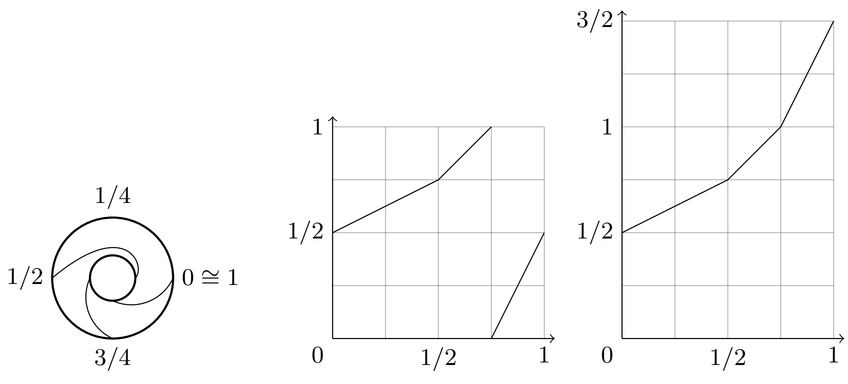

Figure 1: Three representations of the same element of Thompson's group T: (a) as a map $\mathbb{S}^1 \to \mathbb{S}^1$, here drawn by indicating how breakpoints (on the inner circle) are mapped to their images (on the outer circle); (b) the usual representation as a function $[0,1] \to [0,1]$; (c) the representation as a function $[0,1] \to \mathbb{R}$, which we will use – note that it is a homeomorphism onto its image $[1/2, 3/2]$.

The graph of a piecewise linear function can be described by specifying the (finitely many) breakpoints at which the function is not differentiable, and the images of the breakpoints. For a strictly monotone piecewise linear function g, we therefore have a partition of the domain of g by points

$$x_1 < x_2 < \cdots < x_n$$

and a partition of the codomain of g by the points

$$g(x_1) < g(x_2) < \cdots < g(x_n)$$

such that g is the function corresponding to the curve of connected line segments through the points

$$\big(x_1, g(x_1)\big), \big(x_2, g(x_2)\big), \ldots, \big(x_n, g(x_n)\big).$$

In the case of Thompson's groups F and T, the breakpoints have to be at dyadic rationals.

Given any homeomorphism $f\colon \mathbb{S}^1 \to \mathbb{S}^1$, we can identify it with a homeomorphism $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ that satisfies

$$\tilde{f}(x+1) = \tilde{f}(x) + 1.$$

In particular, $\tilde{f}|_{[0,1]}$ is continuous, which will be needed later. An example is shown in Figure 1.

3.1 Outline of the Construction

Before we come to technical details, we present a rough outline of the proof for the case of $\mathrm{Diff}^1_+(I)$ and Thompson's group F. Let $f \in \mathrm{Diff}^1_+(I)$ be given.

1. Divide the domain of f into n small intervals of equal length, where n is a power of 2. Therefore the breakpoints ξ_i of the partition are dyadic rationals.

2. For each breakpoint ξ_i choose a dyadic rational η_i close to the image $f(\xi_i)$.

3. Find a piecewise linear homeomorphism $\gamma_i\colon [\xi_i, \xi_{i+1}] \to [\eta_i, \eta_{i+1}]$ for each $i = 0, \ldots, n-1$ that serves as a *dyadic interpolation* from the point (ξ_i, η_i) to the point (ξ_{i+1}, η_{i+1}), which means that γ_i has breakpoints at dyadic rationals and its slopes are powers of 2 (Section 3.2).

By defining a function $g\colon [0,1] \to [0,1]$ whose values on the interval $[\xi_i, \xi_{i+1}]$ are determined by γ_i, we obtain a homeomorphism $g \in F$ close to f.

3.2 Dyadic Interpolation

Let two distinct points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $\mathbb{R}^2$ be given, with $p_1 < q_1$ and $p_2 < q_2$ and such that all coordinates p_i, q_i are dyadic rational numbers. Then $r = q - p$ also has dyadic rational coordinates r_1 and r_2 which can be written as

$$r_1 = \frac{m_1}{2^{k_1}}, \quad r_2 = \frac{m_2}{2^{k_2}}$$

with $m_1, m_2, k_1, k_2 \in \mathbb{N}$ and $m_1, m_2 > 0$. We proceed as illustrated in Figure 2. Let $(a,b) = (1,2)$ if $m_1 \le m_2$ and $(a,b) = (2,1)$ if $m_1 > m_2$, so that $m_b = \max\{m_1, m_2\}$ and $m_a = \min\{m_1, m_2\}$. Set $d = m_b - m_a$. For the moment, assume $d > 0$. Consider the sequence (c_n) defined by $c_0 = 0$ and

$$c_n = m_a \sum_{i=0}^{n-1} 2^i = m_a(2^n - 1)$$

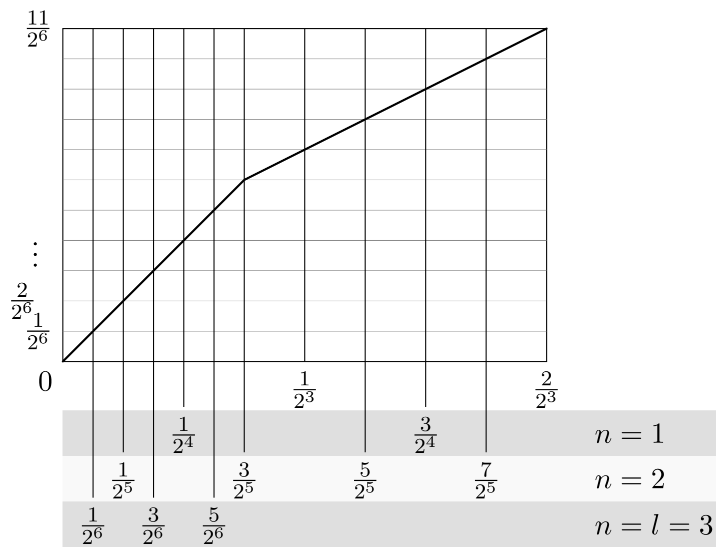

Figure 2: Illustration of how to cut the sides of a dyadic rectangle such that all sides are divided into dyadic partitions with equally many subintervals. In this example, $m_1/2^{k_1} = 11/2^6$ and $m_2/2^{k_2} = 2/2^3$. Since $11 > 2$, we divide the left side of the rectangle into 11 intervals, each of length $1/2^6$. The bottom side is first divided into 2 intervals, each of length $1/2^3$. Then we successively cut all its intervals in half, repeatedly going from left to right, until the bottom side is also divided into 11 intervals. The thick line shows the graph of the piecewise linear function arising from these partitions.

for $n \geq 1$. Let $l \geq 1$ be the smallest integer with $c_l \geq d$. Define a sequence of dyadic numbers $\xi_1, \dots, \xi_d$ by setting

$$\xi_{i+c_n} = \frac{2i-1}{2^{k_a+n}}$$

for all i, n with either $1 \leq i \leq 2^n m_a$ and $0 \leq n \leq l-1$, or $1 \leq i \leq d - c_{l-1}$ when $n = l$. Set

$$X = \left\{ \frac{m}{2^{k_a}} \,\middle|\, 0 \leq m \leq m_a \right\} \cup \{\xi_1, \dots, \xi_d\}$$

for $d > 0$ and

$$X = \left\{ \frac{m}{2^{k_a}} \,\middle|\, 0 \leq m \leq m_a \right\}$$

for $d = 0$. We arrange the $m_a + d = m_b$ elements of X in increasing order and denote them $x_1^a \leq \dots \leq x_{m_b}^a$. They are the breakpoints of a standard dyadic partition of $[0, m_a/2^{k_a}]$ into m_b intervals. Furthermore, set $x^b_m = m/2^{k_b}$ for $0 \le m \le m_b$. The points

$$p_1 + x^1_1,\ p_1 + x^1_2,\ \ldots,\ p_1 + x^1_n$$

and

$$p_2 + x^2_1,\ p_2 + x^2_2,\ \ldots,\ p_2 + x^2_n$$

form standard dyadic partitions dividing the two intervals $[p_1, p_2]$ and $[q_1, q_2]$, respectively, into equally many subintervals. To these partitions corresponds a piecewise linear function. By construction, it is bijective, has breakpoints only at dyadic rationals, and only slopes wich are powers of 2.

3.3 Finding Dyadic Rationals

Let $0 < p < q$ be given. Since the dyadic rationals are dense in $\mathbb{R}$, one can always find a dyadic number in the open interval (p, q). For an example, let

$$\overline{\text{ceil}}(x) = \min\{n \in \mathbb{Z} \mid n > x\} = \begin{cases} x + 1 & \text{if } x \in \mathbb{Z}, \\ \lceil x \rceil & \text{otherwise.} \end{cases}$$

Set

$$\begin{aligned} k &= \max\left\{0, \overline{\text{ceil}}(-\log_2(q-p))\right\}, \\ m &= \overline{\text{ceil}}(2^k p). \end{aligned}$$

Then $m, k \in \mathbb{N}$, and $m/2^k \in (p, q)$ is a dyadic rational.

3.4 The Construction

We proceed with the construction of approximations, which then proves Theorem 2.2. Let $f \in \text{Diff}^1_+(I)$ and $\epsilon > 0$ be given, and assume $\epsilon < 1$ without loss of generality. Set $S = \max_{x \in I} f'(x)$ and note that $S \ge 1$. Let $\Delta = \lceil -\log_2 \frac{\epsilon}{3S} \rceil \in \mathbb{N}$ and $n = 2^\Delta$, and note that $\Delta \ge 1$. Set

$$\xi_i = i/n, \quad i = 0, \ldots, n.$$

(This implies that $\xi_0 = f(\xi_0) = 0$ and $\xi_n = f(\xi_n) = 1$.) Moreover, set

$$\delta = \min\{\epsilon/2, (f(\xi_n) - f(\xi_{n-1})/2)\}$$

and note that the interval

$$I_i = (\max\{f(\xi_{i-1}) + \delta, f(\xi_i)\}, f(\xi_i) + \delta)$$

is non-empty and a subset of $(0,1)$ for $i = 1,\dots,n-1$. We pick a dyadic rational $\eta_i \in I_i$ for each $i = 1,\dots,n-1$. Let $\eta_0 = 0$ and $\eta_n = 1$, and define the function $g\colon [0,1] \to [0,1]$ by setting

$$g(x) = \gamma_i(x) \tag{1}$$

for $x \in [\xi_i, \xi_{i+1}]$ and $i = 0,\dots,n-1$, where γ_i is a dyadic interpolation from the point (ξ_i, η_i) to the point (ξ_{i+1}, η_{i+1}). From the definitions of γ, $\{\xi_i\}$ and $\{\eta_i\}$ it is clear that $g \in F$. Furthermore, for all $i = 0,\dots,n-1$ and $x \in [\xi_i, \xi_{i+1}]$, consider the sequence of statements

$$\begin{align}
|g(x) - f(x)| &\le g(\xi_{i+1}) - f(\xi_i) \tag{2}\\
&< f(\xi_{i+1}) - f(\xi_i) + \epsilon/2 \tag{3}\\
&= \frac{f(\xi_{i+1}) - f(\xi_i)}{\xi_{i+1} - \xi_i}(\xi_{i+1} - \xi_i) + \epsilon/2 \tag{4}\\
&< S2^{-\Delta} + \epsilon/2 \tag{5}\\
&< \epsilon/3 + \epsilon/2 < \epsilon. \tag{6}
\end{align}$$

(2) holds since f and g are strictly increasing and $g(\xi_i) > f(\xi_i)$. For (3), recall that $g(\xi_{i+1}) = \eta_{i+1} < f(\xi_{i+1}) + \delta$. (4) to (6) are obvious. We have thus found $g \in F$ with $\max_{x\in[0,1]}|f(x) - g(x)| < \epsilon$.

If instead $f \in \mathrm{Diff}^1_+(\mathbb{S}^1)$, f corresponds to a function $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ with $\mathrm{im}(\tilde{f}) = [u, u+1]$ for some $u \in \mathbb{R}$ and such that $\tilde{f}\colon [0,1] \to [u, u+1]$ is a diffeomorphism (as explained above). Define S, Δ, n, ξ_i and I_i as above, but with $\delta = \min\{\epsilon/2, (\tilde{f}(\xi_1) - \tilde{f}(\xi_0))/2\}$. Choose $\eta_i \in I_i$ for $i = 1,\dots,n-1$ as before. Let η_0 be a dyadic rational in the interval $(\tilde{f}(\xi_0) + \delta, \tilde{f}(\xi_1))$ and set $\eta_n = \eta_0 + 1$. This ensures that

$$\max\{\tilde{f}(\xi_{n-1}) + \delta, \tilde{f}(\xi_n)\} < \eta_n.$$

Now we can define a function $\tilde{g}\colon [0,1] \to \mathbb{R}$ as in (1). It follows that (2) to (6) hold, and that $g \in T$ upon taking the quotient $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. This concludes the proof of Theorem 2.2. □

4 C^1-Discreteness

Finally, we show that it is not possible to go beyond C^0-approximation. Note that the proof is also valid in the more general case when $\mathrm{Diff}^1_+(I)$

and $\mathrm{Diff}_+^1(\mathbb{S}^1)$ are replaced by the sets of all differentiable bijections of I or $\mathbb{S}^1$, respectively, whose inverses are also differentiable.

Proof of Proposition 2.3. Let $g \in T$ and $f \in \mathrm{Diff}_+^1(\mathbb{S}^1)$. We will identify f and g with functions on the interval $[0,1]$ as before. Let $x_0 \in [0,1] \setminus B_g$. The two powers of 2 closest to $f'(x_0)$ are given by

$$2^{\lfloor \log_2 f'(x_0) \rfloor} \leq f'(x_0) \leq 2^{\lceil \log_2 f'(x_0) \rceil}.$$

If $f'(x_0)$ is not a power of 2, the inequalities are strict and therefore

$$d(f,g) \geq \min\Big\{ \big| f'(x_0) - 2^{\lfloor \log_2 f'(x_0) \rfloor} \big|, \big| f'(x_0) - 2^{\lceil \log_2 f'(x_0) \rceil} \big| \Big\} > 0.$$

The case that $f'(x_0)$ is not a power of 2 for some $x_0 \in [0,1] \setminus B_g$ occurs for all differentiable $f \in \mathrm{Diff}_+^1(\mathbb{S}^1)$ except for rotations. For if f is not a rotation, there exists $x_1 \in [0,1]$ with $f'(x_1) = c \neq 1$. By the mean value theorem, there also exists $x_2 \in [0,1]$ with $f'(x_2) = 1$. Without loss of generality, assume $c < 1$ and $x_1 < x_2$. Then by Darboux's theorem, $[c,1] \subset f'([x_1,x_2])$. Since B_g is finite, $[c,1] \setminus f'(B_g) \subset \mathrm{im}(f')$ surely contains points which are not powers of 2.

It is clear that $\mathrm{Diff}_+^1(I)$ and F are a special case of this argument, which concludes the proof. □

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