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A graph-theoretic viewpoint for discrete Morse theory

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Abstract

A well known theorem of discrete Morse theory states that a discrete vector field is acyclic if and only if it is a gradient vector field for a discrete Morse function f. In this paper we give a simple proof using a well known theorem in graph theory. We do the same for another well known result in discrete Morse theory that states that in a simplicial complex endowed with a discrete gradient vector field, if two critical cells of the same dimension are such that there exists a unique gradient path between them, we can find a new vector field for which these two cells are not critical and every other critical cell remains critical in the new field.

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1 Introduction

In this paper we shall give a graph-theoretic view point of two well known theorems in discrete Morse theory.

The first one is a characterization of discrete gradient vector fields. This theorem has been proved in [1] (page 94), where this graph theoretic view point is also mentioned. This graph-theoretic view point simplifies the proof considerably and uses a well known result in graph theory which we also shall prove.

The second theorem gives a condition for cancelling two critical cells in a discrete gradient vector field. It is also proved in [1] (page 110) but, to the best of our knowledge, it has not been proved using graph-theory tools. This result is very useful since critical cells play an important role in discrte Morse theory.

In both cases I will give proofs of abstract theorems in graph theory and then apply them for discrete Morse theory. I will use the usual graph-theory notation, given a digraph D, $V(D)$ shall denote the set of vertices and $A(D)$ the set of arrows.

2 Graph theory

Definition 2.1. Given a digraph D, $x, y \in V(D)$, an xy-path is a sequence of vertices $(x = x_1, x_2, ..., x_n = y)$ such that for every $i = 1, ..., n-1$ there exists an arrow $(x_i, x_{i+1}) \in A(D)$. The length of the path is n and a cycle is a closed path in the sense that $x_1 = x_n$

By an acyclic digraph we mean a digraph with no cycles. In particular an acyclic digraph has no loops and no symmetric arrows, as there would be cycles of length one and two respectively.

Lemma 2.2. *Let D be an acyclic digraph and suppose $\gamma = (x_0, ..., x_n)$ is the only x_0x_n-path in D. If W is the digraph obtained by inverting every arrow in γ. Then W is acyclic.*

Proof. Proceeding by contradiction, suppose that W has a cycle C. Let Γ be the x_nx_0-path in W. Clearly C has at least one arrow in Γ. Let $(x_m, x_{m-1}, ..., x_k)$ be a segment of C contained in Γ with the property that neither of the two arrows (x_{m+1}, m) and (x_k, x_{k-1}) are in $A(C)$.

Let P be the segment of C disjoint from γ that starts at the vertex x_k and ends at a vertex x_i for some $i = 0, 1, ..., n$.

1. If $i < k$ then $P \cup (x_i, x_{i+1}, ..., x_k)$ is a cycle in the digraph D, a contradiction.
2. If $i > k$ then $(x_0, x_1, ..., x_k) \cup P \cup (x_i, x_{i+1}, ..., x_n$ is another x_0x_n-path, a contradiction.

Hence W is acyclic. □

Lemma 2.3. *A finite acyclic digraph D has at least one vertex $v \in V(D)$ with $\delta^+(v) = 0$ where $\delta^+(v) = |\{x \in V(D) : (v, x) \in A(D)\}|$.*

Proof. Since D is finite and acyclic there exists a longest path in D. The last vertex of this path can not have any outward arrows. □

Theorem 2.4. *A finite digraph D is acyclic if and only if there exists a function $f : V(D) \to \mathbb{N} \cup \{0\}$ which decreases along directed paths.*

Proof. Suppose D is acyclic. Given a vertex v, let $p(v)$ denote the lenght of the largest path in D starting from v. Define the sets $V_i = \{v \in V(D) : p(v) = i\}$. Since D is finite, $\bigsqcup_{i=0}^{n} V_i = V(D)$ for a suficiently large n. From the previous lemma we know that V_0 is non empty. We define $f : V(D) \to \mathbb{N} \cup \{0\}$ given by $f(x) = i$ for all $x \in V_i$.

We must now prove that f decreases along paths. Suppose γ is a path and let (x, y) be an arrow in γ. If $p(x) \leq p(y)$, denote the largest path starting from y by P_y and the largest path starting from x by P_x. Then P_y is longer than P_x but $P_y \cup (x, y)$ is a longer path starting from x, which is a contradiction.

If such a function f exists and $\{x_0, x_1, ..., x_n = x_0\}$ is a cycle then $f(x_0) > f(x_1) > ... > f(x_n) = f(x_0)$ which is impossible. □

3 Discrete Morse theory

Definition 3.1. Let X be a set and K a collection of subsets of X. We say that the pair (X, K) is a simplicial complex if $\tau \in K$ and $\nu \subset \tau$ implies $\nu \in P$. The elements of K are called simplexes and the dimension of a simplex τ is its cardinality minus one.

Given a simplicial complex we shall denote by σ^p that the dimention of a simplex σ is p. We will denote that σ is a face of τ by $\sigma < \tau$.

Definition 3.2. A discrete Morse function on a simplicial complex X is a function $f : K(X) \to \mathbb{R}$, where $K(X)$ denotes the set of simplexes of X, such that given a simplex σ,

$$|\{\tau \in K(X) : \sigma^p > \tau^{p-1}, f(\sigma) \leq f(\tau)\}| \leq 1$$

and

$$|\{\nu \in K(X) : \sigma^p < \nu^{p+1} : f(\sigma) \geq f(\nu)\}| \leq 1.$$

A discrete Morse function can be defined on an CW-complex but for our purposes we shall only consider simplicial complexes.

Definition 3.3. A discrete vector field on a simplicial complex X is a collection of pairs of simplexes $\{(\sigma, \tau) : \sigma < \tau, dim\tau - dim\sigma = 1\}$ such that every simplex is in at most one pair.

Given a discrete Morse function f, we can obtain a discrete vector field called the gradient vector field of f.

Definition 3.4. The gradient vector field of a discrete Morse function is the vector field consisting precisely of the pairs $\sigma^p < \tau^{p+1}$ for which $f(\tau) \geq f(\sigma)$.

In general we say that a discrete vector field is gradient if it is the gradient vector field of a discrete Morse function.

Definition 3.5. Given a simplicial complex X, we can associate a digraph to it called the Hasse diagram. The veritices are the simplexes of X. The set of arrows is $\{(\tau, \sigma) : \sigma^p < \tau^{p+1}\}$.

When X has a discrete vector field we can indicate which pairs belong to the vector field in the Hasse diagram by inverting the corresponding arrow. We call this the modified Hasse diagram.

Definition 3.6. The simplexes that do not belong to any pair of the discrete vector field V are called the critical simplexes of V.

We shall now make an observation about the modified Hasse diagram D of a discrete vector field. When we have a gradient vector field V associated to the discrete Morse function f, notice that $(\alpha, \beta) \in A(D)$ if and only if $|dim\alpha - dim\beta| = 1$ and one of the following holds:

- $\beta > \alpha$, with $f(\beta) \leq f(\alpha)$.
- $\alpha > \beta$, with $f(\alpha) > f(\beta)$.

This means that a discrete Morse function does not increse along paths in the modified Hasse diagram. We shall use this observation in the following theorem.

Definition 3.7. Given a discrete vector field W, a W-path of dimention p is a sequence of p-simplexes $\nu_1, \nu_2, ..., \nu_k$ such that $\nu_i < W(\nu_{i-1})$ for $i = 1, ..., k$, where $(\nu_i, W(\nu_i)) \in W$. We say that the lenght of the path is k and that the path is closed if $\nu_1 = \nu_k$.

Theorem 3.8. *A discrete vector field W on a finite simplicial complex X is gradient if and only if it has no closed paths.*

Proof. Note that in particular a closed W-path is a cycle in the modified Hasse diagram D. By theorem 2.4, the modified Hasse diagram is acyclic if and only if there exists a funcion $f : X \to \mathbb{N} \cup \{0\}$ which decreases along paths. We thus only need to show that the function we constructed in 2.4 is indeed a discrete Morse function with gradient vector field W. Suppose $\sigma^{p-1} < \tau^p$ and $\nu^{p-1} < \tau^p$ are simplexes such that $f(\sigma) > f(\tau)$ and $f(\nu) > f(\tau)$. Since f does not increase along paths, in particular it can not increase along arrows. Recalling the construction of f in theorem 2.4 we see that f decreases along arrows. This means that $(\sigma, \tau), (\nu, \tau) \in A(D)$ but this implies that τ would belong to two pairs in W which is a contradiction since W is a discrete vector field.

Similarly if $\tau^p < \alpha^{p+1}$ and $\tau^p < \beta^{p+1}$ are simplexes such that $f(\tau) \geq f(\alpha)$ and $f(\tau) \geq f(\beta)$ we reach a contradiction. Hence f is a discrete Morse function.

Consider the pairs (ν, τ) such that $|\dim(\nu) - \dim(\tau)| = 1$ and one of the following holds:

- $\nu < \tau$, with $f(\nu) \geq f(\tau)$.
- $\tau < \nu$, with $f(\nu) \leq f(\tau)$.

Note that these are precisely the pairs of W and therefore f has discrete gradient vector field W. □

Theorem 3.9. *Let V be a discrete gradient vector field. Let α and β be two critical simplexes such that $dim\alpha = dim\beta - 1$. Suppose there exists a unique path from β to α in the modified Hasse diagram. Then there exists a discrete gradient vector field W on X for which the set of critical simplexes is: $\{\tau \in X - \{\alpha, \beta\} : \tau \text{ is critical for } V\}$.*

Proof. Let γ be the unique path from β to α in the modified Hasse diagram G of V. We shall define W by constructing its modified Hasse diagram D. Let D be the digraph obtained from G by reversing every arrow in γ. From lemma 2.2 we know that no cycles are created in D. This means that W is also a discrete gradient vector field. Now let us look at the critical simplexes of W. Notice that every simplex outside of γ is critical for W if and only if it is critical for V. For the simplexes in γ different from α and β, which were all non-critical for V, they remain non-critical for W. As for α and β, since in D one of their incident arrows has been reversed they are not critical for W. □

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