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Computations of hitting time densities for the generalized Cox-Ingersoll-Ross diffusion

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Abstract

We give explicit formulæ for the density function of first hitting time of the so-called generalized Cox-Ingersoll-Ross process. In fact, we treat the several cases of the diffusion depending on the values of the parameters. To find the density function we use the eigenvalues and eigenfunctions associated with the infinitesimal operator. It turns out that a very important tool in this analysis is the so-called Kummer equation, where we use the known solutions; this allows us to compute the eigenfunctions in terms of the confluent hypergeometric functions.

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1 Introduction

As described in [8], the changes of the so-called membrane potential between two neurons in the human brain can be modeled using an Itô's diffusion

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t,$$

see also [4]. It turns out that this model can be a good approximation of a real phenomenon, according to [8]. One of main interests in these applications is what people in neuroscience call the interspike intervals, which can be seen as random variable of the form

$$\tau = \inf\{t : X_t \geq f(t)\},$$

where f is a deterministic time function. The importance of this random variable relies on the hypothesis that the flow of information in the nervous

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system is encoded in the timing of spikes. For this kind of phenomenon, a model considered in the literature is the diffusion

$$dX_t = (\mu - aX_t)dt + \sigma\sqrt{X_t - S}dB_t,$$

where $\mu,\ a,\ \sigma,\ S$ are constants.

Coincidentally, this model is also used in financial mathematics to model interest rates. In fact, according to the theory of pricing, the price of a so-called zero-coupon bond is give by the expectation

$$E\left[e^{-\int_0^t X_s ds}\right],$$

see e.g. [2]. Again, knowing information of the first time when X_t surpasses a function is of importance in this context. When $S = 0$, the process X is called the Cox-Ingersoll-Ross process (CIR), see also [6] for more details. In the context of financial mathematics, specifically in the so-called risk theory, an important paradigm is modelling the time of default of a bond based on the so-called intensity models, where it is used a function $\lambda(x)$ to study the random time

$$\tau = \inf\{t : \int_0^t \lambda(X_s)ds \geq E\},$$

for some constant E, see e.g. [7].

These are some reasons that motivate us to work with a diffusion that is a generalization of the CIR process. Precisely, the solution of

$$dX_t = (\mu - aX_t)dt + \sigma\sqrt{X_t - S}dB_t.$$

In particular, we study the first hitting time of the process X, defined as

$$\tau_y := \inf\{t > 0 : X_t = y\},$$

where y is a given constant.

Such task of finding the density has been done before using theory of spectral decompositions, in particular we use results in [10]. The case $S = 0$ is well known and study, see e.g. [11] . Notice that when S is not zero one might try to apply a space transformation and use Itô's lemma to go back to the case $S = 0$. However, in this paper we do not do that. Instead do that, we work directly with the differential equations and the spectral decomposition, this helps to see the connection with the so-called Kummer differential equation, which is an important tool for the whole analysis. This way of work has the benefit to see how the spectral theory works when dealing hitting times, and can be applied to other diffusions.

Let us mention how the paper is organized. In the coming Section 2 we present the basic theory that we use along the paper. Basic concepts such as the speed measure, the scale function and the killing measure are introduced. We also recall the classification of the end-points in the state space.

In Section 3 we will state the result of V. Linetsky [10] and we present some tables which summarizes our results. In the section 4 we study the classification of the end-point S. Sections 5, 6 and 7 presents the solution ψ of the equation $L\psi + \lambda\psi = 0$, and we provide the proofs of the results.

Finally, in Section 8 we study the first hitting time density of the process Y that is solution of the stochastic differential equation

$$dY_t = (\beta - bY_t)dt + \sigma\sqrt{s - Y_t}dB_t.$$

Such process is the reflected analogous of X. To this end, we will apply the Itô's formula to recycle the formulas obtained for X. To finish, in the Section 9 we present a numerical illustration.

2 Preliminaries

Let $\{X_t : t \geq 0\}$ be a one-dimensional diffusion whose state space is some interval $I \subseteq \mathbb{R}$ with end-points e_1 and e_2.

Every diffusion has three basic characteristics that determine the process: speed measure, scale function and killing measure, see [3] for more details. We consider the special case when the three basic characteristics are absolutely continuous with respect to Lebesgue measure in the interior of I, i.e.

$$(1) \quad m(dx) = m(x)dx, \quad k(dx) = k(x)dx, \quad s(x) = \int^x s'(y)dy, \quad x \in (e_1, e_2).$$

Then the infinitesimal generator associated is the following

$$(2) \qquad Lf(x) = \frac{1}{2}a^2(x)f''(x) + b(x)f'(x) - c(x)f(x), \quad x \in (e_1, e_2).$$

The functions a, b, c are called the infinitesimal parameters of X, and are related to m, k, s through the following formulas

$$s'(x) = e^{-\displaystyle\int^x \frac{2b(y)}{a^2(y)}dy}, \quad m(x) = \frac{2}{a^2(x)s'(x)}, \quad k(x) = \frac{2c(x)}{a^2(x)s'(x)}.$$

The speed measure, the scale function and the killing measure determine the behavior of the diffusion in the interior of the state space I. However the behavior of the diffusion at the boundary points is characterized by boundary conditions.

In [3] it is presented a classification of the end-points of I according to the behavior of the diffusion in the neighborhood of these end-points. To explain this, let z be fixed such that $e_1 < z < e_2$. According with (1) we have

$$(3) \qquad m((x, z)) := \int_x^z m(y)dy, \text{ and } s(x) := \int_z^x s'(y)dy.$$

Now define

$$A := \int_{e_1}^{z} [m((x,z)) + k((x,z))]\, s'(x)dx,$$
$$B := \int_{e_1}^{z} [s(z) - s(x)]\,(m(x) + k(x))dx.$$

Then the end-point e_1 it is classified in the following manner (for e_2 is similar, see [12])

i. The end-point e_1 is called **exit** if

(4) $$A < \infty \text{ and } B = \infty.$$

ii. The end-point e_1 is called **entrance** if

(5) $$A = \infty \text{ and } B < \infty.$$

iii. The end-point e_1 is called **regular** if

(6) $$A < \infty \text{ and } B < \infty.$$

In this case, one additionally has the following subclassification:

- if $m(\{e_1\}) = k(\{e_1\}) = 0$ then e_1 is called **regular reflecting**,
- if $m(\{e_1\}) < \infty$, and $k(\{e_1\}) = \infty$ then e_1 is called **regular killing**,
- if $0 < m(\{e_1\}) < \infty$, and $k(\{e_1\}) = 0$ then e_1 is called **regular sticky**,
- if $m(\{e_1\}) = 0$, and $k(\{e_1\}) > 0$ then e_1 is called **regular elastic**,
- if $m(\{e_1\}) = \infty$, and $k(\{e_1\}) \geq 0$ then e_1 is called **regular absorbing**.

iv. The end-point e_1 is called **natural** if

(7) $$A = \infty \text{ and } B = \infty.$$

Remark 2.1. Since in this paper m, k are absolutely continuous with respect to Lebesgue measure, then the boundary condition at a regular boundary can only be reflecting or killing, for more details see [12]. Also, in this paper we do not consider the case when e_1 is natural, because in our examples this case is not presented.

Note that if $k((e_1, e_2)) = 0$, if we use the previous classification and the sacle function s, it is known that for $e_1 < x < y$

$$(8) \quad P_x(\tau_y < \infty) = \begin{cases} 1, & \text{if } e_1 \text{ is entrance or regular reflecting;} \\ \dfrac{\int_{e_1}^x s'(z)dz}{\int_{e_1}^y s'(z)dz}, & \text{if } e_1 \text{ is exit or regular killing,} \end{cases}$$

where $\tau_y := \inf\{t > 0 : X_t = y\}$; see for instance [10].

3 Spectral expansion for first hitting time density

Let us present the spectral decomposition theorem of V. Linetsky found in [10]; see also [9]. Consider a diffusion X which is solution of the stochastic differential equation

$$(9) \qquad dX_t = (\mu - aX_t)dt + \sigma\sqrt{X_t - S}\, dB_t,$$

where $\mu, a, S \in \mathbb{R}$ and $\sigma > 0$. It is known that the infinitesimal operator is

$$(10) \qquad Lf(x) = \frac{\sigma^2(x-S)}{2} f''(x) + (\mu - ax)f'(x), \quad x \in (S, \infty),$$

acting on twice differentiable functions $f : (S, \infty) \to \mathbb{R}$. To specify completely the process X, we also need to specify the behavior of X at S; this fact is important for the following theorem.

Theorem 3.0.1. (Linetsky [10]) *Let X be a diffusion that is solution of $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$ and whose state space I has the end-points e_1 and e_2. Define $I^y := [e_1, y]$ if e_1 is regular reflecting, and $I^y := (e_1, y]$ in any other case. Fix $X_0 = x$ and $y \in I$ such that $e_1 < x < y < e_2$, and suppose that e_1 is either regular, entrance or exit. For $\lambda \in \mathbb{C}$ and $x \in I^y$, let $\psi(x, \lambda)$ be the unique non trivial solution (up to a multiple independent of x) of the Sturm-Liouville equation $L\psi + \lambda\psi = 0$,with boundary condition at e_1 given by*

$$(11) \qquad \lim_{x \to e_1^+} \psi(x, \lambda) = 0 \;\; or \;\; \lim_{x \to e_1^+} \frac{\psi_x(x, \lambda)}{s'(x)} = 0.$$

Then the spectral expansion of $P_x(t < \tau_y < \infty)$, with $e_1 < x < y$ takes the form

$$(12) \qquad P_x(t < \tau_y < \infty) = -\sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(x, \lambda_n)}{\lambda_n \psi_\lambda(y, \lambda_n)},$$

where $0 < \lambda_1 < \lambda_2 < \lambda_3 \cdots$ are the simple positive zeros of $\psi(y, \lambda)$, i.e., $\psi(y, \lambda_n) = 0$. Note that each λ_n depends on y.

Remark 3.0.2. The function $\psi(x,\lambda)$ appearing in the previous theorem is square-integrable with respect to m in a neighborhood of e_1; and $\psi(x,\lambda)$ and $\psi_x(x,\lambda)$ are continuous in x and λ in $I^y \times \mathbb{C}$ and entire in $\lambda \in \mathbb{C}$ for each $x \in I^y$ fixed. For more details see [9].

Remark 3.0.3. In our case the state space is (S,∞), this implies that using the notation of the Theorem 3.0.1 we have that $e_1 := S$ and $e_2 := \infty$.

If we apply the identity

$$P_x(\tau_y < \infty) = P_x(\tau_y < t) + P_x(t < \tau_y < \infty), \tag{13}$$

by Theorem 3.0.1 and (8) we obtain that for $e_1 < x < y$

$$P_x\left(\tau_y \le t\right) = P_x(\tau_y < \infty) + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(x,\lambda_n)}{\lambda_n \psi_\lambda(y,\lambda_n)}. \tag{14}$$

An important ingredient for the methodology is solving the equation $L\psi + \lambda\psi = 0$. It turns out that the boundary condition for ψ at $e_1 = S$ depends on the classification of S (see [3] for details):

i. If e_1 is exit or regular killing then the boundary condition is

$$\lim_{x \to e_1^+} \psi(x,\lambda) = 0.$$

ii. If e_1 is entrance or regular reflecting then the boundary condition is

$$\lim_{x \to e_1^+} \frac{\psi_x(x,\lambda)}{s'(x)} = 0,$$

where $\psi_x(x,\lambda) := \frac{\partial}{\partial x}\psi(x,\lambda)$.

Remark 3.0.4. When $e_1 < y < x < e_2$, the first hitting time problem is treated similarly. In this case we have $I^y := [y, e_2]$ if e_2 is regular reflecting, and $I^y := [y, e_2)$ in any other case. It is also known that

$$P_x\left(\tau_y < \infty\right) = \begin{cases} 1, & \text{if } e_2 \text{ is entrance or regular reflecting;} \\ \dfrac{\int_x^{e_2} s'(z)dz}{\int_y^{e_2} s'(z)dz}, & \text{if } e_2 \text{ is exit or regular killing.} \end{cases} \tag{15}$$

Then the spectral expansion of $P_x(t < \tau_y < \infty)$, with $y < x < e_2$ is

$$P_x(t < \tau_y < \infty) = -\sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\phi(x,\lambda_n)}{\lambda_n \phi_\lambda(y,\lambda_n)}, \tag{16}$$

where the solution $\phi(x,\lambda)$ of $Lu + \lambda u = 0$ is square-integrable with respect to m in a neighborhood of e_2 and satisfying the appropriate boundary condition at e_2, i.e.

i. $\lim_{x \to e_2^-} \phi(x, \lambda) = 0$ if e_2 is exit or regular killing, or

ii. $\lim_{x \to e_2^-} \frac{\phi_x(x, \lambda)}{s'(x)} = 0$ if e_2 is entrance or regular reflecting.

Also, $\phi(x, \lambda)$ and $\phi_x(x, \lambda)$ are continuous in x and λ in $I^y \times \mathbb{C}$ and entire in $\lambda \in \mathbb{C}$ for each $x \in I^y$ fixed. The λ_n are all the simple positive zeros of $\phi(y, \lambda)$.

Table 1. Nature of the end-point S

Parameters	S	Boundary Condition
I. $-1 < \frac{2aS - 2\mu}{\sigma^2} < 0$ with $a \neq 0$	Regular reflecting	$\lim_{x \to S^+} \frac{\psi_x(x, \lambda)}{s'(x)} = 0$
II. $\frac{2aS - 2\mu}{\sigma^2} \geq 0$ with $a \neq 0$	Exit	$\lim_{x \to S^+} \psi(x, \lambda) = 0$
III. $\frac{2aS - 2\mu}{\sigma^2} \leq -1$ with $a \neq 0$	Entrance	$\lim_{x \to S^+} \frac{\psi_x(x, \lambda)}{s'(x)} = 0$
IV. $a = 0$ and $-1 < \frac{-2\mu}{\sigma^2} < 0$	Regular killing	$\lim_{x \to S^+} \psi(x, \lambda) = 0$

Table 2. Solution ψ

Parameters	Solution of $L\psi + \lambda\psi = 0$
I. $-1 < \frac{2aS - 2\mu}{\sigma^2} < 0$ with $a \neq 0$	$\psi(x, \lambda) = F\left(\frac{-\lambda}{a}, \frac{2\mu - 2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)$
II. $\frac{2aS - 2\mu}{\sigma^2} \geq 0$ with $a \neq 0$	$\psi(x, \lambda) = \left(\frac{2a(x-S)}{\sigma^2}\right)^{1 - \frac{2\mu - 2aS}{\sigma^2}} F\left(\frac{-\lambda}{a} - \frac{2\mu - 2aS}{\sigma^2} + 1, 2 - \frac{2\mu - 2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)$
III. $\frac{2aS - 2\mu}{\sigma^2} \leq -1$ with $a \neq 0$	$\psi(x, \lambda) = F\left(\frac{-\lambda}{a}, \frac{2\mu - 2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)$
IV. $a = 0$ and $-1 < \frac{-2\mu}{\sigma^2} < 0$	$\psi(x, \lambda) = \left(\frac{2(x-S)}{\sigma^2}\right)^{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)} \cdot J_{\left(1 - \frac{2\mu}{\sigma^2}\right)}\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right)$

We will apply Theorem 3.0.1 to find the first hitting time density of the process X that is solution of (9). We summarize our results in the following tables. Table 1 shows the nature of the end-point S (second column) and the type of boundary condition (third column), both depending on certain regions for the parameters μ, a, S and σ (first column).

Table 2 shows the solution of the equation (11) and Table 3 shows precisely the formula (14). Note that in the end there are four types: I, II, III, IV. The proofs of the first two columns of Table 1 are give in Section 4. Section 5 deals with cases I and III of these tables. Section 6 deals with the case II, and the Section 7 presents the case IV.

Let us give the notation used in the tables:

- $F(a,b,x) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}$, where $(a)_n := a \cdot (a+1) \cdots (a+n-1)$.
- F_λ represents the derivative of F with respect to λ.
- $J_v(x) := \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{v+2n}}{n!\Gamma(v+n+1)}$ the Bessel function.
- $J_{v,n}$ represents a positive zero of the Bessel function J_v.

Here Γ is the Gamma function.

Table 3. Spectral decomposition

Parameters	Spectral decomposition for the first hitting time density
I. $-1 < \frac{2aS-2\mu}{\sigma^2} < 0$ with $a \neq 0$	$P_x(\tau_y \le t) = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)}$
II. $\frac{2aS-2\mu}{\sigma^2} \geq 0$ with $a \neq 0$	$P_x(\tau_y \le t) = \frac{\int_S^x s'(z)dz}{\int_S^y s'(z)dz} + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(x,\lambda_n)}{\lambda_n \psi_\lambda(y,\lambda_n)}$
III. $\frac{2aS-2\mu}{\sigma^2} \leq -1$ with $a \neq 0$	$P_x(\tau_y \le t) = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)}$
IV. $a = 0$ and $-1 < \frac{-2\mu}{\sigma^2} < 0$	$P_x(\tau_y \le t) = \left(\frac{x-S}{y-S}\right)^v - 2\left(\frac{x-S}{y-S}\right)^{\frac{v}{2}} \sum_{n=0}^{\infty} e^{-\frac{\sigma^2 J_{v,n} t}{8(y-S)}} \frac{J_v\left(J_{v,n}\sqrt{\frac{x-S}{y-S}}\right)}{J_{v,n} \cdot J_{1+v}(J_{v,n})}$

4 Classification of the end-point S

We consider the process X that is solution of (9) with state space give by the interval $I = [S, \infty)$ if S is regular reflecting or $I = (S, \infty)$ in any other case. We suppose that $X_0 = x > S$ and we consider $y \in I$ fixed such that $x < y$. To study this process we first classified the end-point S.

Proposition 4.0.1. *The end-point S has the following classification:*

i. If $-1 < \frac{2aS-2\mu}{\sigma^2} < 0$ *then S is regular.*

ii. If $\frac{2aS-2\mu}{\sigma^2} \geq 0$ *then S is exit.*

iii. If $\dfrac{2aS-2\mu}{\sigma^2} \le -1$ *then* S *is entrance.*

Proof. We prove the first case, the other two cases are similar. Suppose that

$$-1 < \frac{2aS-2\mu}{\sigma^2} < 0,$$

and let z be a fixed value such that $S < z < \infty$. To verify that S is regular, we calculate the following integrals to see that they are finite:

$$\begin{aligned}
&\int_S^z m((x,z))s(x)dx \\
&\quad = \int_S^z \left(\int_x^z 2\sigma^{-2} e^{\frac{-2ay}{\sigma^2}} (y-S)^{\frac{2\mu-2aS}{\sigma^2}-1} dy \right) e^{\frac{2ax}{\sigma^2}} (x-S)^{\frac{2aS-2\mu}{\sigma^2}} dx \\
&\quad \le M \int_S^z \left(\int_S^z (y-S)^{\frac{2\mu-2aS}{\sigma^2}-1} dy \right) (x-S)^{\frac{2aS-2\mu}{\sigma^2}} dx \\
&\quad = M \int_S^z (y-S)^{\frac{2\mu-2aS}{\sigma^2}-1} dy \cdot \int_S^z (x-S)^{\frac{2aS-2\mu}{\sigma^2}} dx \\
&\quad < \infty,
\end{aligned}$$

where M is a constant. In similar way we obtain

$$\begin{aligned}
&\int_S^z s(x)m(x)dx \\
&\qquad = \int_S^z \left(\int_x^z e^{\frac{2ay}{\sigma^2}} (y-S)^{\frac{2aS-2\mu}{\sigma^2}} dy \right) 2\sigma^{-2} e^{\frac{-2ax}{\sigma^2}} (x-S)^{\frac{2\mu-2aS}{\sigma^2}-1} dx \\
&\qquad < \infty.
\end{aligned}$$

We used, in both cases, the fact that $-1 < \dfrac{2aS-2\mu}{\sigma^2} < 0$, to obtain that the integrals are finite. □

Corollary 4.0.2. *If* $a = 0$ *and* $-1 < \frac{-2\mu}{\sigma^2} < 0$ *then the end-point* S *is regular.*

The condition $a \neq 0$ presented in tables shows up in the following sections.

5 First hitting time density for the cases I and III

According to Theorem 3.0.1, for the cases I and III (in Tables 1, 2 and 3) we have to find a function $\psi(x,\lambda)$ such that it satisfies

$$\text{(17)} \quad \frac{\sigma^2(x-S)}{2}\psi''(x) + (\mu - ax)\psi'(x) + \lambda\psi(x) = 0, \quad \text{and} \quad \lim_{x\to S^+} \frac{\psi_x(x,\lambda)}{s'(x)} = 0.$$

In turn we have the following proposition.

Proposition 5.0.1. *If $a \neq 0$, then the function*

$$\psi(x,\lambda) = F\left(\frac{-\lambda}{a}, \frac{2\mu - 2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right), \tag{18}$$

is solution of (17), with

$$F(a,b,x) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{x^n}{n!},$$

where $(a)_n := a \cdot (a+1) \cdots (a+n-1)$.

Proof. Consider $z = \frac{\sigma^2(x-S)}{2}$, and define $g(z) := f\left(\frac{2z}{\sigma^2} + S\right)$. From (17) we obtain the equation

$$zg''(z) + \left(\frac{2\mu}{\sigma^2} - \frac{4az}{\sigma^4} - \frac{2aS}{\sigma^2}\right) g'(z) + \frac{4\lambda}{\sigma^4} g(z) = 0.$$

Now consider $w = \frac{4az}{\sigma^4}$ and define $h(w) := g\left(\frac{\sigma^4 w}{4a}\right)$. Since $a \neq 0$, we arrive at:

$$wh''(w) + (\frac{2\mu - 2aS}{\sigma^2} - w)h'(w) - \left(\frac{-\lambda}{a}\right) h(w) = 0.$$

The previous equation is called the Kummer equation (see [15]), and according with [15, p.2] the solution is

$$h(w) = F\left(\frac{-\lambda}{a}, \frac{2\mu - 2aS}{\sigma^2}, w\right).$$

Returning to the variable x we obtain the result. □

Applying the Proposition 5.0.1 and the formula (14), we obtain the following theorem.

Theorem 5.0.2. *Let X be the process that is solution of (9) with $S < x < y$, $a \neq 0$, such that S is regular reflecting or entrance. Then the first hitting time distribution of the process X is*

$$P_x\left(\tau_y \leq t\right) = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)}, \tag{19}$$

where the derivative $F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu - 2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)$ is

$$\frac{-1}{a}\sum_{k=0}^{\infty} \frac{\left(\frac{-\lambda_n}{a}\right)_k}{\left(\frac{2\mu-2aS}{\sigma^2}\right)_k} \phi\left(\frac{-\lambda_n}{a} + k\right) \frac{\left(\frac{2a(y-s)}{\sigma^2}\right)^k}{k!} + \frac{1}{a}\phi\left(\frac{-\lambda_n}{a}\right) F\left(\frac{-\lambda_n}{a}, \frac{2\mu - 2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right), \tag{20}$$

and $\phi(z) := \dfrac{\Gamma'(z)}{\Gamma(z)}$, *where* Γ *is the Gamma function. See [10] for more details.*

From the Theorem 5.0.2 we obtain a formula for the first hitting time density for the cases I and III

$$
\begin{aligned}
(21) \qquad P_x(\tau_y \in dt) &= -\sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)} \\
&= \sum_{n=1}^{\infty} e^{-\lambda_n t} \lambda_n c_n,
\end{aligned}
$$

where

$$
(22) \qquad c_n := \frac{-F\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)}.
$$

Remark 5.0.3. To be able to carry out a numerical procedure we propose a naive but effective approximation of λ_n and c_n. First consider the following formula found in [15],

$$
\begin{aligned}
(23) \qquad F(a,b,x) = {} & \pi^{\frac{-1}{2}} \Gamma(b) e^{\frac{x}{2}} \left[x\left(\frac{b}{2}-a\right)\right]^{\frac{1}{4}-\frac{b}{2}} \\
& \cos\left(2\sqrt{x\left(\frac{b}{2}-a\right)} - \frac{b\pi}{2} + \frac{\pi}{4}\right)\left\{1 + O\left(|\, a\,|^{\frac{-1}{2}}\right)\right\}.
\end{aligned}
$$

Since the λ_n are the zeros of $\psi(y,\lambda)$, with y fixed, we will use the formula (23) to find an approximation of λ_n for n large given by

$$
(24) \qquad \lambda_n \approx \left[\frac{\sigma^2}{2a(y-S)}\left(\frac{\pi(\mu-aS)}{2\sigma^2} + \frac{n\pi}{2} - \frac{3\pi}{8}\right)^2 - \frac{\mu-aS}{\sigma^2}\right] \times a.
$$

Then applying again the formula (23) we obtain an approximation for c_n in (22) with n large given by

$$
\begin{aligned}
c_n \quad \approx \quad & \frac{(-1)^{n+1} 2\pi\left(n + \frac{\mu-aS}{\sigma^2} - \frac{3}{4}\right) \cdot e^{\frac{2a(x-y)}{\sigma^2}}}{\pi^2\left(n + \frac{\mu-aS}{\sigma^2} - \frac{3}{4}\right)^2 - \frac{4\mu-4aS}{\sigma^2} \cdot \frac{2a(y-S)}{\sigma^2}} \cdot \left(\frac{x-S}{y-S}\right)^{\frac{1}{4}-\frac{\mu-aS}{\sigma^2}} \\
\times \quad & \cos\left(\pi\left(n + \frac{\mu-aS}{\sigma^2} - \frac{3}{4}\right)\sqrt{\frac{x-S}{y-S}} - \frac{\pi(\mu-aS)}{\sigma^2} + \frac{\pi}{4}\right).
\end{aligned}
$$

6 First hitting time density for the case II

For the case II, in the tables of section 3, we have to find a function $\psi(x,\lambda)$ such that

$$L\psi + \lambda\psi = 0 \text{ and } \lim_{x\to S^+} \psi(x,\lambda) = 0. \tag{25}$$

We have the following proposition:

Proposition 6.0.1. *If $a \neq 0$, then the function $\psi(x,\lambda)$ defined by the formula*

$$\left(\frac{2a(x-S)}{\sigma^2}\right)^{1-\frac{2\mu-2aS}{\sigma^2}} F\left(\frac{-\lambda}{a} - \frac{2\mu{-}2aS}{\sigma^2} + 1, 2 - \frac{2\mu{-}2aS}{\sigma^2}, \frac{2a(x{-}S)}{\sigma^2}\right),$$

is solution of (25).

Proof. The proof is similar to the Proposition 5.0.1. With the same changes of variable we obtain the Kummer equation. □

Applying the formula (14) and (8) we arrive at

$$P_x(\tau_y \le t) = \frac{\int_S^x s'(z)dz}{\int_S^y s'(z)dz} + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(x,\lambda_n)}{\lambda_n \psi_\lambda(y,\lambda_n)}, \tag{26}$$

where $s'(z) = e^{\frac{2az}{\sigma^2}} (z-S)^{\frac{2aS-2\mu}{\sigma^2}}$. Therefore the first hitting time density $P_x(\tau_y \in dt)$ for the case II is

$$-\sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{(x-S)^{1-\frac{2\mu-2aS}{\sigma^2}} F\left(\frac{-\lambda_n}{a} - \frac{2\mu-2aS}{\sigma^2} + 1, 2 - \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{(y-S)^{1-\frac{2\mu-2aS}{\sigma^2}} F_\lambda\left(\frac{-\lambda_n}{a} - \frac{2\mu-2aS}{\sigma^2} + 1, 2 - \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)}.$$

To find an approximation of the λ_n (with n large), we use again the formula (23), thus

$$\lambda_n \approx -a \times \left[y\left(1 - \frac{\mu - aS}{\sigma^2}\right) - \left\{\frac{\pi}{8} - \frac{\pi}{2}\left(\frac{\mu - aS}{\sigma^2} - n\right)\right\}^2 + \frac{2\mu - 2aS}{\sigma^2} - 1\right].$$

We apply again the formula (23) if we wish to approximate c_n.

7 First hitting time density for the case IV

Note that in previous two sections we consider $a \neq 0$. Now we present an example of a particular situation when $a = 0$. In this case the process X is the solution of

$$dX_t = \mu dt + \sigma\sqrt{X_t - S}dB_t.$$

We consider only the case when $-1 < \frac{-2\mu}{\sigma^2} < 0$ (The other cases in Proposition 4.0.1 with $a = 0$ are similar). From Corollary 4.0.2 we have that the end-point S is regular, and we now assume that S is killing. Therefore we have to find a function $\psi(x,\lambda)$ such that it satisfies

$$\frac{\sigma^2(x-S)}{2}\psi''(x) + \mu\psi'(x) + \lambda\psi(x) = 0 \text{ and } \lim_{x\to S^+}\psi(x,\lambda) = 0. \tag{27}$$

Proposition 7.0.1. *The function*

$$\psi(x,\lambda) = \left(\frac{2(x-S)}{\sigma^2}\right)^{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)} \cdot J_{\left(1-\frac{2\mu}{\sigma^2}\right)}\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right), \tag{28}$$

is solution of (27), where $J_v(x) := \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{v+2n}}{n!\Gamma(v+n+1)}$.

Proof. It follows using the formula in Table 15 of [13]. □

Remark 7.0.2. For y fixed such that $S < x < y$, we find the λ_n such that $\psi(y,\lambda_n) = 0$ in the following manner, let $J_{v,n}$ be the positive zeros of the Bessel function J_v, where $v := 1 - \frac{2\mu}{\sigma^2}$. Then by (28) the values λ_n must satisfies the equation

$$2\sqrt{\frac{2\lambda_n(y-S)}{\sigma^2}} = J_{v,n}. \tag{29}$$

Thus

$$\lambda_n = \frac{\sigma^2 J_{v,n}^2}{8(y-S)}. \tag{30}$$

Lemma 7.0.3. *Let ψ be the function in (28), then*

$$\psi_\lambda(y,\lambda_n) = -\left(\frac{2(y-S)}{\sigma^2}\right)^{\frac{3}{4}-\frac{\mu}{\sigma^2}} \cdot \frac{2\sqrt{2}\sqrt{y-S}}{\sigma \cdot J_{v,n}} \cdot J_{2-\frac{2\mu}{\sigma^2}}\left(J_{v,n}\right). \tag{31}$$

Proof. We first compute the derivative of ψ with respect to λ, where ψ is (28). Then we arrive at the following expression for $\psi_\lambda(x,\lambda)$:

$$\left(\frac{2(x-S)}{\sigma^2}\right)^{\frac{v}{2}} \sqrt{\frac{2(x-S)}{\sigma^2}} \cdot \frac{1}{2\sqrt{\lambda}} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k(v+2k)\left(\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right)^{v+2k-1}}{k!\cdot\Gamma(k+v+1)}. \tag{32}$$

On the other hand, notice that

$$J_v'(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n(v+2n)\left(\frac{x}{2}\right)^{v+2n-1}}{n!\Gamma(v+n+1)}. \tag{33}$$

Then by evaluating function in (33) at $\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}$, then (32) reads as

$$\psi_\lambda(x,\lambda) = \left(\frac{2(x-S)}{\sigma^2}\right)^{\frac{v}{2}+\frac{1}{2}} \cdot \frac{1}{\sqrt{\lambda}} \cdot J'_v\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right). \tag{34}$$

Now we use the following identity found in [10]:

$$J'_v(z) = -J_{v+1}(z) + \frac{v}{z}J_v(z). \tag{35}$$

Applying the identity (35) we arrive at

$$\begin{aligned}\psi_\lambda(x,\lambda) &= \left(\frac{2(x-S)}{\sigma^2}\right)^{\frac{v}{2}+\frac{1}{2}} \cdot \frac{1}{\sqrt{\lambda}} \cdot J'_v\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right)\\ &= \frac{\left(\frac{2(x-S)}{\sigma^2}\right)^{\frac{v}{2}+\frac{1}{2}}}{\sqrt{\lambda}}\left[\frac{\sigma v}{2\sqrt{2\lambda(x-S)}} \cdot J_v\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right)\right.\\ &\qquad\left. - J_{v+1}\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right)\right].\end{aligned}$$

Using (30), and the fact that $J_v(J_{v,n}) = 0$, we obtain

$$\begin{aligned}&\psi_\lambda(y,\lambda_n)\\ &\quad= \left(\frac{2(y-S)}{\sigma^2}\right)^{\frac{v}{2}+\frac{1}{2}} \frac{2\sqrt{2(y-S)}}{\sigma \cdot J_{v,n}}\left[\frac{\sigma v}{2\sqrt{2\lambda_n(y-S)}} J_v\left(J_{v,n}\right) - J_{v+1}\left(J_{v,n}\right)\right]\\ &\quad= \left(\frac{2(y-S)}{\sigma^2}\right)^{\frac{v}{2}+\frac{1}{2}} \frac{2\sqrt{2(y-S)}}{\sigma \cdot J_{v,n}} \cdot \left[-J_{v+1}\left(J_{v,n}\right)\right].\end{aligned}$$

This completes the proof. □

Note that applying the Lemma 7.0.3 we have

$$\frac{\psi(x,\lambda_n)}{\lambda_n\psi_\lambda(y,\lambda_n)} = \frac{-2}{J_{v,n}} \cdot \left(\frac{x-S}{y-S}\right)^{\frac{v}{2}} \cdot \frac{J_v\left(J_{v,n}\sqrt{\frac{x-S}{y-S}}\right)}{J_{1+v}\left(J_{v,n}\right)}. \tag{36}$$

By joining everything, we obtain the following theorem.

Theorem 7.0.4. *Let X be the process that is solution of*

$$dX_t = \mu dt + \sigma\sqrt{X_t - S}dB_t.$$

Suppose that $X_0 = x$ and $S < x < y$, for y fixed. If $-1 < \frac{-2\mu}{\sigma^2} < 0$ and S is killing, then

$$P_x(\tau_y \le t) = \left(\frac{x-S}{y-S}\right)^v - 2\cdot\left(\frac{x-S}{y-S}\right)^{\frac{v}{2}} \cdot \sum_{n=0}^{\infty} e^{-\frac{\sigma^2 J_{v,n} t}{8(y-S)}} \frac{J_v\left(J_{v,n}\sqrt{\frac{x-S}{y-S}}\right)}{J_{v,n} \cdot J_{1+v}\left(J_{v,n}\right)}. \tag{37}$$

Remark 7.0.5. When $\sigma = 2$, $\mu = 2v + 2$ and $S = 0$, then the process X is the Squared Bessel process. Also note that if τ_y^R :=inf$\{t > 0 : R_t = y\}$ where R represents the Bessel process, then

$$\begin{aligned} P_x\left(\tau_y^R \leq t\right) &= P_{x^2}\left(\inf\{s > 0 : R_s^2 = y^2\} \leq t\right) \\ &= P_{x^2}\left(\inf\{s > 0 : X_s = y^2\} \leq t\right) \\ &= P_{x^2}\left(\tau_{y^2}^X \leq t\right). \end{aligned}$$

Therefore using the formula (37), we can recover the formula for the first hitting time density of the Bessel process when $S = 0$ is killing. Indeed, in [10, p.391] one can see such formula:

$$P_x(\tau_y^R \leq t) = \left(\frac{x}{y}\right)^{2v} - 2\left(\frac{x}{y}\right)^{v} \sum_{n=0}^{\infty} e^{\frac{-J_{v,n}^2 t}{2y^2}} \frac{J_v\left(\frac{x}{y} J_{v,n}\right)}{J_{v,n} J_{1+v}(J_{v,n})}.$$

8 Spectral expansion for the reflected generalized Cox-Ingersoll-Ross process

In this section we will consider another process Y that is solution of

$$dY_t = (\beta - bY_t)dt + \sigma\sqrt{S - Y_t}\, dB_t, \tag{38}$$

with state space $I = (-\infty, S]$ if S is regular reflecting, or $I = (-\infty, S)$ if S is not regular reflecting. The infinitesimal generator is given by

$$Lf(x) = \frac{\sigma^2(S - x)}{2} f''(x) + (\beta - bx)f'(x). \tag{39}$$

We want to find the density of ζ_y :=inf $\{t > 0 : Y_t = y\}$. To this end, we will use the formula of the first hitting time density of the process X that is solution of (9). We first present three tables with the results of this section. Similar to the Tables 1, 2, and 3, Table 4 shows the nature of the end-point S and the type of boundary condition, Table 5 shows the solution of equation (11), and Table 6 shows the formula (14).

Table 4. Nature of the end-point S

Parameters	S	Boundary Condition
I′. $-1 < \frac{2\beta - 2bS}{\sigma^2} < 0$ with $b \neq 0$	Regular reflecting	$\lim_{x \to S^+} \frac{\psi_x(x,\lambda)}{s'(x)} = 0$
II′. $\frac{2\beta - 2bS}{\sigma^2} \geq 0$ with $b \neq 0$	Exit	$\lim_{x \to S^+} \psi(x,\lambda) = 0$
III′. $\frac{2\beta - 2bS}{\sigma^2} \leq -1$ with $b \neq 0$	Entrance	$\lim_{x \to S^+} \frac{\psi_x(x,\lambda)}{s'(x)} = 0$

Table 5. Solution ψ

Parameters	Solution of $L\psi + \lambda\psi = 0$
I′. $-1 < \frac{2\beta-2bS}{\sigma^2} < 0$ with $b \neq 0$	$\psi(x,\lambda) = F\left(\frac{-\lambda}{b}, \frac{2\beta}{\sigma^2}, \frac{2b(x-S)}{\sigma^2}\right)$
II′. $\frac{2\beta-2bS}{\sigma^2} \geq 0$ with $b \neq 0$	$\psi(x,\lambda) = \left(\frac{2b(x-S)}{\sigma^2}\right)^{1-\frac{2bS-2\beta}{\sigma^2}} F\left(\frac{-\lambda}{b} - \frac{2bS-2\beta}{\sigma^2} + 1, 2 - \frac{2bS-2\beta}{\sigma^2}, \frac{2b(x-S)}{\sigma^2}\right)$
III′. $\frac{2\beta-2bS}{\sigma^2} \leq -1$ with $b \neq 0$	$\psi(x,\lambda) = F\left(\frac{-\lambda}{b}, \frac{2\beta}{\sigma^2}, \frac{2b(x-S)}{\sigma^2}\right)$

Table 6. Spectral decomposition

Parameters	Spectral decomposition for the first hitting time density
I′. $-1 < \frac{2\beta-2bS}{\sigma^2} < 0$ with $b \neq 0$	$P_x(\zeta_y \leq t) = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{b}, \frac{2bS-2\beta}{\sigma^2}, \frac{2b(S-x)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{b}, \frac{2bS-2\beta}{\sigma^2}, \frac{2b(S-y)}{\sigma^2}\right)}$
II′. $\frac{2\beta-2bS}{\sigma^2} \geq 0$ with $b \neq 0$	$P_x(\zeta_y \leq t) = \frac{\int_S^{2S-x} s'(z)dz}{\int_S^{\overline{y}} s'(z)dz} + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(2S-x,\lambda_n)}{\lambda_n \psi_\lambda(\overline{y},\lambda_n)}$
III′. $\frac{2\beta-2bS}{\sigma^2} \leq -1$ with $b \neq 0$	$P_x(\zeta_y \leq t) = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{b}, \frac{2bS-2\beta}{\sigma^2}, \frac{2b(S-x)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{b}, \frac{2bS-2\beta}{\sigma^2}, \frac{2b(S-y)}{\sigma^2}\right)}$

In order to analyze process Y, we use the Itô's formula to contruct a new process Z, which will allow us to use the results of previous sections. Suppose that $Y_0 = x$ and let y be fixed such that $S > x > y$. Consider the function $g(x,t) := -x + 2S$. Applying the Itô's formula we arrive at

$$dZ_t = (2bS - \beta - bZ_t)dt + \sigma\sqrt{Z_t - S}\, dB_t, \text{ where } Z_t = -Y_t + 2S. \tag{40}$$

This process is (9) with $\mu := 2bS - \beta$ and $a := b$. Note that $Z_0 = 2S - x$ and define $\overline{y} := 2S - y$. Then using the Proposition 4.0.1 for the end-point S, we obtain the following proposition.

Proposition 8.0.1. *The end-point S obeys the following classification*

i. If $-1 < \frac{2\beta - 2bS}{\sigma^2} < 0$ *then S is regular.*

ii. If $\frac{2\beta - 2bS}{\sigma^2} \geq 0$ *then S is exit.*

iii. If $\frac{2\beta - 2bS}{\sigma^2} \leq -1$ *then S is entrance.*

Proof. Notice that the nature of the end-point S for process Y is the same as for process Z. Then we can use Proposition 4.0.1. □

For the cases I′ and III′ in Proposition 8.0.1 we have to find a function $\psi(x,\lambda)$ such that

$$\begin{aligned}\frac{\sigma^2(x-S)}{2}\psi''(x)+(2bS-\beta-bx)\psi'(x)+\lambda\psi(x)=0, \quad \text{and}\\ \lim_{x\to S^+}\frac{\psi_x(x,\lambda)}{s'(x)}=0.\end{aligned}\tag{41}$$

In turn we have the following proposition.

Proposition 8.0.2. *If $b\neq 0$, then the function*

$$\psi(x,\lambda)=F\left(\frac{-\lambda}{b},\frac{2bS-2\beta}{\sigma^2},\frac{2b(x-S)}{\sigma^2}\right),\tag{42}$$

is solution of (41).

Proof. Is similar to Proposition 5.0.1. With the same changes of variable to obtain the Kummer equation. □

If we define $\eta_y:=\inf\{t>0: Z_t=y\}$, then

$$P_x(\zeta_y\le t)=P_{2S-x}(\eta_{\overline{y}}\le t).\tag{43}$$

Then applying the formula (14) we arrive at

$$\begin{aligned}P_x\left(\zeta_y\le t\right)&=P_{2S-x}(\eta_{\overline{y}}\le t)\\ &=1+\sum_{n=1}^{\infty}e^{-\lambda_n t}\frac{F\left(\frac{-\lambda_n}{b},\frac{2bS-2\beta}{\sigma^2},\frac{2b(S-x)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{b},\frac{2bS-2\beta}{\sigma^2},\frac{2b(S-y)}{\sigma^2}\right)}.\end{aligned}\tag{44}$$

Remark 8.0.3. To find an approximation for λ_n (with n large), we use the formula (23)

$$\lambda_n\approx\left[\frac{\sigma^2}{2b(\overline{y}-S)}\left(\frac{\pi(bS-\beta)}{2\sigma^2}+\frac{n\pi}{2}-\frac{3\pi}{8}\right)^2-\frac{bS-\beta}{\sigma^2}\right]\times b.\tag{45}$$

Remark 8.0.4. For the case II′ in the Proposition 8.0.1, we have to find a function $\psi(x,\lambda)$ such that $L\psi+\lambda\psi=0$ and

$$\lim_{x\to S^+}\psi(x,\lambda)=0.$$

If $b\neq 0$, we obtain that the solution $\psi(x,\lambda)$ is given by the expression

$$\left(\frac{2b(x-S)}{\sigma^2}\right)^{1-\frac{2bS-2\beta}{\sigma^2}}F\left(\frac{-\lambda}{b}-\frac{2bS-2\beta}{\sigma^2}+1,2-\frac{2bS-2\beta}{\sigma^2},\frac{2b(x-S)}{\sigma^2}\right).$$

Then applying the formula (14) and (8) we arrive at

$$(46)\quad P_x(\zeta_y \le t) = P_{2S-x}(\eta_{\overline{y}} \le t) = \frac{\int_S^{2S-x} s'(z)dz}{\int_S^{\overline{y}} s'(z)dz} + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(2S-x, \lambda_n)}{\lambda_n \psi_\lambda(\overline{y}, \lambda_n)},$$

where $s'(z) = e^{\frac{2bz}{\sigma^2}}(z-S)^{\frac{2\beta-2bS}{\sigma^2}}$. To find an approximation for λ_n (with n large), we use again the formula (23). Then we have

$$(47)\qquad \lambda_n \approx -b \times \left[\overline{y}\left(1 - \frac{bS-\beta}{\sigma^2}\right) - \left\{ \frac{\pi}{8} - \frac{\pi}{2}\left(\frac{bS-\beta}{\sigma^2} - n\right) \right\}^2 + \frac{2bS-2\beta}{\sigma^2} - 1 \right].$$

9 Numerical example

In this section, using the package Wolfram Mathematica 10.1, we give a illustration of a numerical approximation for the case III of Tables 1, 2 and 3. Consider the process X that is solution of

$$(48)\qquad dX_t = (3 - 2X_t)dt + \sqrt{X_t - 1}\, dB_t.$$

Suppose that $X_0 = x = \frac{9}{8}$ and $y = \frac{5}{4}$. In this case, applying the Proposition 4.0.1, we have that the end-point $S = 1$ is entrance because

$$(49)\qquad \frac{2aS - 2\mu}{\sigma^2} = \frac{2 \cdot 2 \cdot 1 - 2 \cdot 3}{1} = -2 < -1.$$

Then the function ψ that we consider is

$$(50)\qquad \psi(x, \lambda) = F(-4\lambda,\ 1,\ 2x - 1).$$

Using the formula (23) we obtain an approximation for the function ψ, and we also have approximations for the eigenvalues λ_n ($\psi(y, \lambda) \approx 0$ with $y = \frac{5}{4}$) for $n = 1, 2, 3 ....$ The approximation of the graph of ψ is drawn in Figure 1.

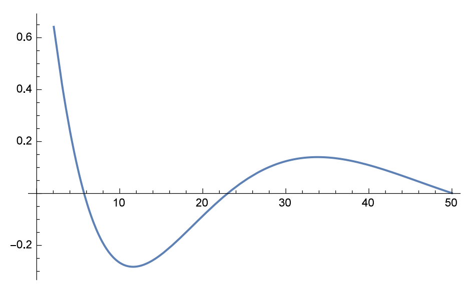

Figure 1: Approximation graph ψ.

Now we give an approximation for c_n using the formula (25) to have one picture of an approximation the first hitting time density. For our estimation,

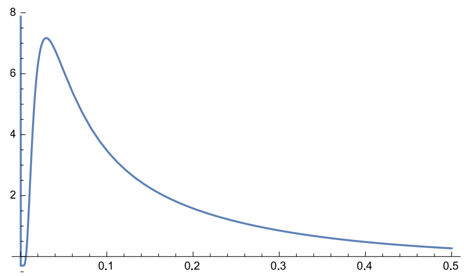

Figure 2: Approximation first hitting time density.

we have truncated the serie (21) at the first 100 terms. The graph of this approximation is presented in the Figure 2.

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