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On an algebraic invariant for monomials

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Abstract

The interaction between Algebraic Geometry and Combinatorics can be very fruitful. One important structure in both areas is the set $M_{d,n}$ of monic monomials of degree d in n variables over a field K. Here we consider $\tau_{d,n}$, the minimum cardinality of a subset T of $M_{d,n}$ such that every element in $M_{d-1,n}$ divides at least one monomial in T. This algebraic invariant has been defined and studied before and it is link to two radical different conjectures: the 1-dimensional ideal generating conjecture and the Stanley's conjecture on the h-vector of a matroid. However, explicit computations of non-trivial cases of the invariant have only been done for $d = 3$ and all n and for $n = 3$ and all d. Here we compute the invariant in the cases for $d = 4$ and all n, for $n = 4$ with d even and for $d = 5$ with n odd. Our approach is combinatorial and confirms the general formula conjectured before by the first author of this work.

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1 Introduction

Examples of the interaction of Combinatorics and Algebraic Geometry by combining a graph structure with an algebraic structures are abundant: for example, the affirmative answer to the conjecture of Read and

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Rota-Heron-Welsh by Huh in [6] or Baker and Norine graph-theoretic analogue of the classical Riemann-Roch theorem in [2]. Here we consider the set $M_{d,n}$ of monic monomials of degree d in $K[X_0, \dots, X_{n-1}]$, for a field K, and the graph $G_{d,n}$ that has as its vertices the set $M_{d,n}$, and two monomials m and m' are adjacent if $X_i m = X_j m'$ for $i \neq j$. We are mainly interested in the algebraic invariant $\tau_{d,n}$ that is the minimum cardinality of a subset T of $M_{d,n}$ such that every monomial of degree $d-1$ in $M_{d-1,n}$ divides at least one monomial in T.

The graph $G_{d,n}$ and the invariant $\tau_{d,n}$ have been studied by Geramita et al. [5] in the context of Algebraic Geometry relating $\tau_{d,n}$ with the 1-dimensional ideal generating conjecture where explicit computations of non-trivial cases for $\tau_{d,n}$ were done for $d = 3$ and all n and for $n = 3$ and all d. Also, $G_{d,n}$ and $\tau_{d,n}$ have been studied by Merino et al. [7] in Combinatorics relating $\tau_{d,n}$ with Stanley's conjecture on the h-vector of a matroid. The invariant $\tau_{d,n}$ was introduced independently in both papers with different but equivalent definitions. Here we compute the invariant $\tau_{d,n}$ in the cases for $d = 4$ and all n, for $n = 4$ with d even and for $d = 5$ with n odd. Our approach is combinatorial and confirms the general formula conjectured in [7].

2 The graph $G_{d,n}$

We start by giving some basic notions of graph theory. A subset T of a graph G is called a *clique* if any two distinct vertices of T are adjacent; the trivial graph with one vertex contains only one clique with one vertex. A clique is a *maximal clique* if it is not contained in any larger clique and it is a *maximum clique* if it is of maximum size. An r-colouring of a graph is a function from the vertices of the graph to the set of r colours, $\{1, 2, \dots, r\}$, such that adjacent vertices have different colours. For more on graph theory, see [3]. Now, we explain some of the combinatorial structure of $G_{d,n}$. The cardinality of the vertex set of $G_{d,n}$ is $\binom{d+n-1}{n-1}$. We define the *standard colouring* of $G_{d,n}$ as the function that assigns to $X_0^{a_0} \cdots X_{n-1}^{a_{n-1}}$ the color $0a_0 + \ldots + (n-1)a_{n-1}$ *mod n*. In fact, the standard colouring of $G_{d,n}$ is an n-colouring, thus any clique in $G_{d,n}$ contains at most n elements. Standard colouring was independently defined in [5, 7].

We describe the vertex set of the two types of maximum cliques in $G_{d,n}$. The set of monomials obtained by multiplying a fixed monomial of degree $d-1$ by each variable X_i, and the corresponding clique is called *upward*. The set of monomials $\{m/X_i\}$ obtained from a monomial m of degree $d+1$ divisible by all variables, and the corresponding clique is called *downward*. Figure 1 shows the graph $G_{4,3}$, upward cliques are triangles pointing up, and downward cliques are triangles pointing down.

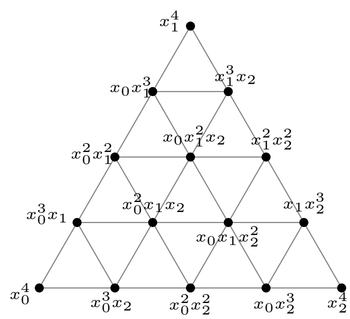


Figure 1: **The graph** $G_{4,3}$**.**

The maximum cliques in $G_{d,n}$ correspond to monomials in $G_{d-1,n}$ and $G_{d+1-n,n}$. A bijection between the set of upward cliques of $G_{d,n}$ and the monomials in $G_{d-1,n}$ comes directly from the definition of upward clique. A bijection between the set of downward cliques in $G_{d,n}$ with the monomials in $G_{d+1-n,n}$ is obtained from the definition of a downward clique and the graph $G'_{d+1-n,n}$ which is the induced subgraph of monomials which are divisible by $X_0 \cdots X_{n-1}$ in $G_{d+1,n}$.

Proposition 2.1. *Take integers $d, n \geq 1$ and $0 \leq i \leq n-1$. Let T_i be the chromatic class of i in the standard colouring of $G_{d,n}$, then it holds:*

i) T_i covers the graph $G_{d-1,n}$, i. e. for any monomial m' of degree $d-1$ there exist a monomial $m \in T_i$ such that $m' \leq m$ $(m'|m)$.

ii) For any $m_1,\ m_2 \in T_i$ define the sets $M_1 = \{m \in G_{d-1,n} : m \leq m_1\}$ and $M_2 = \{m \in G_{d-1,n} : m \leq m_2\}$ as the covered monomials of m_1 and m_2 respectively, then M_1 and M_2 are disjoints; we refer to this property saying that any covering induced by a chromatic class is formed by disjoint sets.

Proof. To prove the first claim take any monomial m' of degree $d-1$ and consider the associated upward clique C in $G_{d,n}$ when multiplying m' by $X_0, \dots, X_{n-1}$ respectably, observe that $m' \leq m$ for each $m \in C$. Since the standard colouring is an n-colouring, C must have one monomial with color i for each $0 \leq i \leq n-1$.

To prove the second claim take $0 \leq i \leq n-1$. Let $m_1, m_2 \in T_i$ be any monomials, suppose that there exist $m_3 \in G_{d-1,n}$ such that $m_3 \in M_1 \cap M_2$, i. e. $m_3 \leq m_1$ and $m_3 \leq m_2$. Then there exist $0 \leq j_1, j_2 \leq n-1$ such that $m_1 = m_3 X_{j_1}$ and $m_2 = m_3 X_{j_2}$, thus $X_{j_1} m_2 = X_{j_0} m_1$ and m_1 is adjacent to m_2 which is a contradiction since they have the same color. $\square$

3 The invariant $\tau_{d,n}$

The invariant $\tau_{d,n}$ was define in [5], where it is denoted $\tau_n(d)$, as the minimum cardinality of a set of vertices T of $G_{d,n}$ such that every upward clique contains a vertex of T. Clearly both definitions are equivalent. The following result appears in that paper.

Theorem 3.1. *For all $n \geq 1$ and $d \geq 1$ we have that $\tau_{d,3} = \lfloor \frac{1}{3}\binom{d+2}{2} \rfloor$ and $\tau_{3,n} = \lfloor \frac{1}{n}\binom{2+n}{n-1} \rfloor = \lceil (n^2+3n)/6 \rceil$.*

We give a brief explanation of the interest of this invariant in Algebraic Geometry. For $R = K[X_0, \dots, X_{n-1}]$, we say that $J \subset R$ is a *monomial ideal* if it is generated by monomials. When J is a monomial ideal its number of generators $\nu(J)$ is finite, and J has a structure of partial ordered set by setting $m \leq m'$ if $m|m'$. In this case the set of generators of J is the set of minimal monomials in J. When the ring $B = R/J$ is finite (equivalently $\sqrt{J} = (X_0, \dots, X_{n-1})$) the number of maximal monomials not in J is the *Cohen-Macaulay Type* of B, denoted $r(B)$. The invariants $\nu(J)$ and $r(B)$ have been studied in the general case where J is not a monomial ideal and some conjectures has been made about its values, see [5].

The invariant τ was used in Theorem 4.7 of [5] to bound a range of integer values where the 1-dimensional ideal generating conjecture can be proved using the lifting of monomial ideals. For a detailed treatment, see [5].

3.1 A plausible formula for $\tau_{d,n}$

Now, we explain the interest of the invariant τ in Combinatorics. We start by given the definitions of multicomplex and O-sequence.

Definition 3.2. A multicomplex is a set $T \subset R$ of monomials such that for every pair of monomials m and m' such that $m \leq m'$ and $m' \in T$ then $m \in T$. A multicomplex is called *pure* if all its maximal elements have the same degree.

Definition 3.3. A sequence $(h_0, h_1, \ldots)$ is called an O-sequence if there is a multicomplex T with exactly h_i monomials of degree i. It is called a *pure* O-sequence if there is a pure multicomplex T with exactly h_i monomials of degree i.

In Combinatorics, the invariant $\tau_{d,n}$ has been used in two different problems. The value of $\tau_{d,n}$ is used in the proof of Stanley's conjecture for paving matroids, see [7]. Stanley's conjecture states that the h-vector of a matroid is a pure O-sequence, see [8]. Also, it was conjectured in [7] that $\tau_{d,n}$ equals the number of binary aperiodic necklaces with n white beads and d black beads, denoted $L_2(n,d)$, and both of these number equal the minimum cardinality of a chromatic class of the standard colouring of $G_{d,n}$, denoted $\overline{f}(d,n)$. For reference we write this in the following statement.

Conjecture 3.4. *For all integers $d, n \geq 1$,*

$$L_2(d,n) = \overline{f}(d,n) = \tau_{d,n}.$$

The next lemma is well known, it gives us a formula to count the number of binary aperiodic necklaces with n white beads and d black beads.

Proposition 3.5. *For all integers $n, d \geq 1$,*

$$L_2(d,n) = \frac{1}{d+n} \sum_{k|(d+n,d)} \mu(k) \binom{d+n/k}{d/k}.$$

We give a proof for the first equality of Conjecture 3.4. We use the formula in the previos lemma and an explicit formula for the number of solution for the system

$$\sum_{j=0}^{n-1} j\lambda_j \equiv k \text{ (mod n)};\ \sum_{i=0}^{n-1} \lambda_i = m,$$

that is denoted by $a_k(n,m)$. The following two results are in [4].

Theorem 3.6. *For any integers k, n, m,*

$$a_k(n,m) = \frac{1}{n+m} \sum_{d|(n,m)} c_d(k) \binom{n/d + m/d}{n/d};$$

in particular

$$a_k(n,m) = a_k(m,n).$$

The value $c_d(k)$ is the sum of the k-th powers of the d-th primitive roots of unity that is known as Ramanujan's sum.

Proposition 3.7. *For any n, m and k,*

$$a_k(n,m) = \sum_{d'|(n,m,k)} a_1(n/d', m/d').$$

Proposition 3.7 tells us that the chromatic class of 1 (with the standard colouring) is minimal in terms of cardinality, and $a_k(n,m)$ is the same as the number of monomials in $G_{m,n}$ which has color k; therefore $\overline{f}(d,n)$ equals $a_1(d,n)$. Also, It is well known that $c_d(1) = \mu(d)$, so the formula in Theorem 3.6 coincides with the formula in Proposition 3.5; thus the first equality of the Conjecture 3.4 is true, this leds the next Proposition.

Proposition 3.8. *For all $d, n \geq 1$,*

$$L_2(d,n) = \overline{f}(d,n).$$

4 The values $\tau_{3,n}$ and $\tau_{d,3}$

Our approach to finding $\tau_{3,n}$ and $\tau_{d,3}$ uses the combinatorial interpretation of the function $\overline{f}(d,n)$. Observe that Proposition 2.1 tells us that any chromatic class in $G_{d,n}$ induces a covering for the graph $G_{d-1,n}$, and Proposition 3.8 gives us a way to prove the second equality of Conjecture 3.4. In fact, we only need to show that the covering induced by the chromatic class of 1 uses the minimum number of elements among all coverings of $G_{d-1,n}$. We call this covering *the standard covering*.

Let us fix the notation before going forward. The vertices in $G_{d,n}$ which are divisible by $X_0 \cdots X_{n-1}$ are called *internal vertices* and the rest of the vertices are called *frontier vertices*. Also, we refer to the covering elements as vertices and the covered elements as monomials.

Next, we present two results which can be proved using Theorem 3.1 but that were obtained independently, and we include them here as examples of the technique we are using. The proofs are not necessarily smaller than in [5].

Theorem 4.1. *For all $d \geq 1$,*

$$L_2(d,3) = \overline{f}(d,3) = \tau_{d,3}.$$

Proof. Observe that for each vertex m in $G_{d,3}$ the monomials that are covered by m form one of the next tree configurations: a single point ($m = X_i^d$) or a line ($m = X_i^{d-k}X_j^k$) in the frontier of $G_{d-1,3}$, or a downward triangle which are the downwards cliques. So the problem reduces to find a covering for $G_{d-1,3}$ using elements of these types.

It is easy to see that the standard covering follows a pattern which is shown in Figure 2, to continue the pattern to the next level $d+1$ we just have to add one more level to the right of the triangle.

Observe that the subgraph $G'_{d-3,3}$ induced by the monomials in $G_{d,3}$ which are divisible by $X_0X_1X_2$ is isomorphic to $G_{d-3,3}$, more over, when restricting the standard covering to this graph its easy to see that this cover correspond to the standard covering in $G_{d-3,3}$. Therefore we proceed by induction according with the congruence modulo 3.

First we show that the standard covering use the least possible number of downward cliques to cover the internal vertices of $G_{d,3}$. Assume by induction that it holds for $d-3$ and take any covering $\mathcal{C}$ of $G_{d,3}$, consider the subgraph $G'_{d-3,3}$ defined above, by induction hypothesis the covering $\mathcal{C}'$ induced by $\mathcal{C}$ when restricting to $G'_{d-3,3}$ cannot have

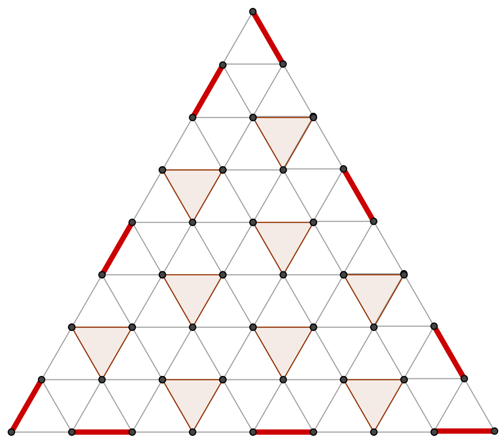
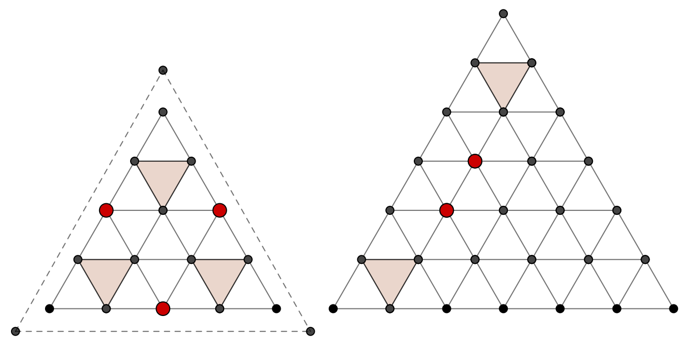


Figure 2: **The Standard Covering for $n = 3$.**

less downward cliques that the standard covering of $G_{d-3,3}$ and it only remains to cover some monomials in the frontier of $G'_{d-3,3}$, if there are k remaining monomials we need at least $\lceil k/2 \rceil$ elements to cover them since there is no vertex in $G_{d,3}$ which covers 3 monomials in the frontier of $G'_{d-3,n}$, this is exactly what the standard covering does, so the covering $\mathcal{C}$ cannot have less downward cliques than the standard covering. The base cases are trivial to verify that they use the minimum possible number of downward cliques to cover $G_{d,3}$.

Next we prove that for the cases $d \equiv 1, 2 \, mod \, 3$ the standard covering uses the maximum number of downward cliques without intersection between them. Figure 3 shows that this is the case for every $d \neq 5$ (optimality for d=5 has to be verified by inspection), we observe that if the pattern of the standard covering is not used then the highlighted points cannot be covered without using a downward clique intersecting another one used before.

The cases where $d \equiv 0$ mod 3 has a configuration that uses exactly one extra downward clique than the standard covering which is shown



a) **b)**

Figure 3: **Coverings with disjoints Downward Cliques.**

in Figure 4, but this configuration isolates the vertices of the form X_i^d so this result in using one element for each of this vertices and uses more elements than the standard covering.

An easy induction argument shows that for $d \equiv 1, 2$ mod 3, if a covering has $l \geq 1$ additional downward cliques than the standard covering then it has at least l intersection, and for $d \equiv 0$ mod 3, if a covering has $l \geq 2$ additional downward cliques then it has at least l intersection, in both cases the standard covering cannot be improved by augmenting the number of downward cliques. To finish the proof observe that the remaining m vertices that are not covered by downward cliques are covered as best as possible since they are covered by $m/2$ disjoint lines when m is even and $(m-1)/2$ disjoints lines and one point when m is odd. □

Theorem 4.2. *For all $n \geq 1$,*

$$\tau_{3,n} = \overline{f}(3, n) = L_2(3, n).$$

Proof. We proceed as before showing that the standard covering is optimal using the least possible number of elements to cover $G_{2,n}$. The

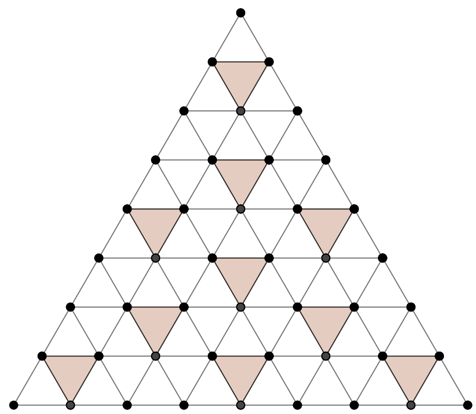


Figure 4: **Configuration with maximum number of disjoints Downward Cliques for $d \equiv 0$ mod 3.**

vertices of the form x_i^2 can be covered only by monomials of the form X_i^3 or $X_i^2X_j$ $(i \neq j)$, observe that the standard covering uses at most one vertex of the form X_i^3 since the congruence $3i \equiv 1$ modulo n has at most one solution, see Figure 5. After covering the vertices X_i^2 it only remains to cover vertices of the form X_iX_j $(i \neq j)$, say there are k of them, observe that vertices in $G_{3,n}$ can cover at most tree monomials in $G_{2,n}$, then, to cover the remaining monomials we need at least $\lceil k/3 \rceil$ elements. But these remaining monomials are covered by the standard covering with vertices of the form $X_iX_jX_k$ $(i, j, k$ all distinct), also we know that the standard covering uses disjoints elements (Proposition 2.1 *ii)*), so it uses exactly $\lceil k/3 \rceil$ elements, which is the best possible. $\square$

5 The case $\tau_{d,4}$

We compute $\tau_{d,4}$, i. e. we have 4 variables and monomials with degree d. We need to cover the graph $G_{d-1,4}$ by four type of structures: points at the corners of the biggest tetrahedra, formed by the monomials $\{X_0^{d-1}$,

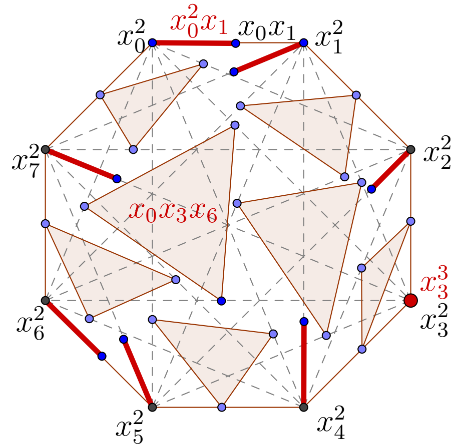


Figure 5: **Standard covering for** $G_{2,8}$**.**

X_1^{d-1}, X_2^{d-1}, X_3^{d-1}}; segments on lines between two points along the edges of the biggest tetrahedra; triangles, pointing down, on the faces on the biggest tetrahedra; and tetrahedrons, pointing down, in the inner part. To achieve our goal, we examine the vertices in $G_{d,4}$ and find out how they cover monomials in $G_{d-1,4}$. This is similar to the case of $\tau_{d,3}$; we find the pattern of the standard covering by analysing Figure 6, where the standard covering is shown, and continuing the covering to the next level. We have the following result.

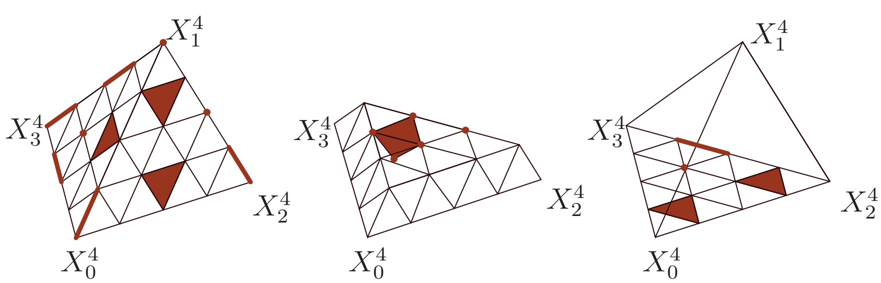


Figure 6: **Standard covering for** $G_{4,4}$**.**

Theorem 5.1. *For all $d \geq 1$ even,*

$$\tau_{d,4} = \overline{f}(d,4) = L_2(d,4).$$

Proof. The pattern of the standard covering can be obtained by inspection, but it is hard to proof the theorem this way, so instead we use counting arguments. First, we refer as internal monomials the monomials in $G_{d,4}$ which are divisible by $X_0X_1X_2X_3$, we call edge monomials the monomials in $G_{d,4}$ which has at most 2 variables and we call face monomials to the rest of the monomials in $G_{d,4}$ which are the monomials with exactly 3 variables.

It easy to check that the total monomials in $G_{d,4}$ is $(d^3+6d^2+11d+6)/6$, the number of internal monomials is $I_d = (d^3-6d^2+11d-6)/6$, the number of face monomials is $F_d = (2d^2-6d+4)$ and the number of edge monomials is $E_d = 6d-2$.

The minimum number of tetrahedrons needed to cover the internal monomials is achieved by the standard covering, this follows from an easy induction argument, so we only consider configuration with at least this number of tetrahedrons.

Consider the configuration shown in Figure 7, this configuration is obtained by considering the graph $G_{d,4}$ as a pyramid, the base case is when $d=3$, the configurations has only one tetrahedra on the first floor. Assume we have constructed the configuration for $d=2k+1$, to construct the configuration for $d=2(k+1)+1$ add two floors to the bottom of the pyramid, and add the configuration $k(k+1)/2$ disjoint tetrahedrons on the first floor as shown in the first part of Figure 7, here we assume d is even, since this configuration does not work for d odd. An easy induction argument shows that this configuration uses the maximum number of disjoints tetrahedrons to cover the internal monomials. Also, this configuration uses exactly $(d^3-6d^2+11d-6)/6$ tetrahedrons, 2 more than the standard covering, so this configuration covers $(d^3-6d^2+11d-6)/2$ face monomials and the rest of face monomials has to be covered by triangles and we need at least $(d^3-6d^2+11d-6)/6$ of them, observe that this can be done with disjoint triangles. This leaves all edge monomials uncovered, and the best we can do (if possible) is use $E_d/2 = 3d-1$ disjoint lines to cover them. In total we have used $(d^3-6d^2+11d-6)/6+(d^3-6d^2+11d-6)/6+3d-1 = (d^3-6d^2+20d-9)/3$

elements to cover $G_{d,4}$. Using the formula given by Proposition 3.5 and the first equality of the Conjecture 3.4 we see that the number of elements in the standard covering of $G_{d,4}$ is $\overline{f}(d,4) = (d^3 + 6d^2 + 8d)/24$ which is smaller, so this configuration does not improve the standard covering.

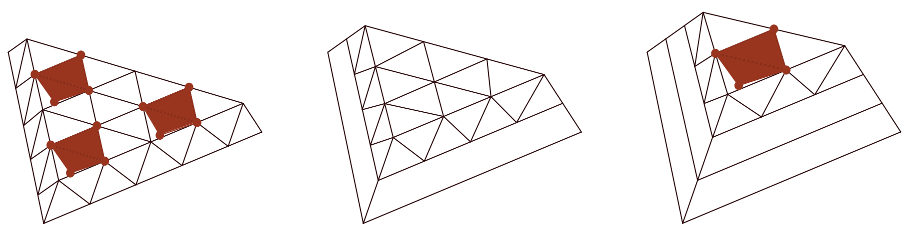


Figure 7: **Maximal configuration of disjoint tetrahedrons for $G_{5,4}$.**

Now consider the configuration with exactly 1 more tetrahedra than the standard covering, in fact the induced covering of the chromatic class of 2 satisfies this, since this configuration is unique (up to symmetries) the best way to cover the remaining monomials is achieved by the chromatic class of 2 it self, but we know that this covering has more elements than the standard covering, so this configuration with exactly one more tetrahedra does not improve the standard covering. Now consider configurations with $l \geq 3$ additional tetrahedrons than the standard covering, we can use induction to see that this configurations has at least l intersection of the tetrahedrons, so the number of elements can not be improved. To finish the proof observe that the standard covering covers the remaining monomials optimally since it uses disjoints elements. □

6 The case $\tau_{4,n}$

The case for $d = 4$ and all n is our more general result.

Theorem 6.1. *For all $n \geq 1$,*

$$\tau_{4,n} = \overline{f}(4,n) = L_2(4,n).$$

Proof. We proof that the standard covering is optimal. Consider separately the cases when n is odd or even. When n is even the congruences

$4i \equiv 1$ modulo n and $2(i+j) \equiv 1$ modulo n do not have solutions, so vertices of the form X_i^4 and $X_i^2 X_j^2$ $(i \neq j)$ do not belong to the chromatic class of 1. This implies that a vertex X_i^3 is covered by X_i^4 and a vertex $X_i^2 X_j$ is covered by $X_i^2 X_j X_k$ $(i,\ j,\ k$ all distinct), this guarantees that vertices in $G_{4,n}$ with at most 2 variables are covered optimally since each vertex is covered by a monomial that maximise the elements covered and the standard covering is formed of disjoints sets. The remaining vertices are of the form $X_i X_j X_k$ $(i,\ j,\ k$ all distinct), but they are covered optimally by monomials $X_i X_j X_k X_l$ $(i,\ j,\ k,\ l$ all distinct) since the sets are disjoints.

When n is odd the congruence $4i \equiv 1$ modulo n has exactly one solution, and congruence $2(i+j) \equiv 1$ modulo n can have several solutions. In order to proof that the standard covering is optimal we argument that a minimal covering of $G_{4,n}$ has to satisfy certain constrain about its cardinality and then we show that the standard covering achieves this constrain and so is optimal. First observe that a vertex X_i^3 can be covered only by X_i^4 or $X_i^3 X_j$, the second covering two monomials, for convenience we assume that an optimal covering uses all but one vertices $X_i^3 X_j$ and one vertex $X_{k_0}^4$. Now we need to cover the vertices of the form $X_i^2 X_j$, these vertices can be covered by $X_i^2 X_j^2$ or $X_i^2 X_j X_k$ ($X_i^3 X_j$ are already used), we need to maximise the number of covered vertices so we maximise the number of monomials $X_i^2 X_j X_k$ used. A monomial $X_i^2 X_j X_k$ covers $X_i^2 X_j$, $X_i^2 X_k$ and $X_i X_j X_k$, so we observe that for each $i \neq k_0$ there are $n-2$ vertices $X_i^2 X_j$ that are not covered yet, among these, we can cover $n-3$ disjointly (since $n-2$ is odd) by monomials $X_i^2 X_j X_k$, then we would need at least $(n-1)(\frac{n-3}{2})$ elements (see Figure 8).

For $i = k_0$ we need at least $\frac{n-1}{2}$ elements. This give us $\frac{n^2-3n+2}{2}$ elements. Among all vertices $X_i^2 X_j$ we covered at most n^2-3n+2, so there are at least $2\binom{n}{2} - (n^2-3n+2) = 2(n-1)$ left. To cover these $2(n-1)$ vertices we need at least $n-1$ elements since now we only can use monomials of the form $X_i^2 X_j^2$. Next we need to cover the vertices $X_i X_j X_k$, we have at least $\binom{n}{3} - \frac{n^2-3n+2}{2} = \frac{1}{6}(n^3-6n^2+11n-6)$ vertices and these vertices need to be covered by at least $\frac{1}{24}(n^3-6n^2+11n-6)$ monomials of the form $X_i X_j X_k X_l$.

Resuming all arguments above, any optimal covering of $G_{4,n}$ need at least n elements to cover the vertices X_i^3, $\frac{n^2-3n+2}{2} + \frac{n-1}{2}$ to cover the vertices $X_i^2 X_j$ (this elements cover at most n^2-3n+2 vertices of

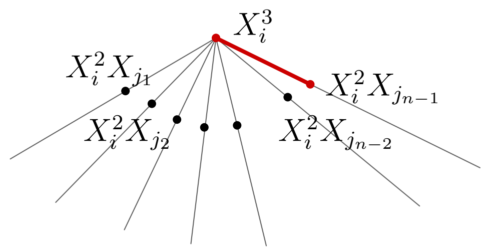


Figure 8: **Vertices X_i^3 and $X_i^2X_{j_{n-1}}$ are covered by $X_i^3X_{j_{n-1}}$.**

the form $X_iX_jX_k$) and $\frac{1}{24}(n^3-6n^2+11n-6)$ elements to cover the remaining vertices of the form $X_iX_jX_k$. Adding these numbers give us at least $\frac{1}{24}(n^3+6n^2+11n+6)$ elements. Using the formula for $\overline{f}(d,n)$ given by Lemma 3.5 we have that the standard covering has

$$\frac{1}{n+4}\sum_{d|(n+4,4)}\mu(d)\binom{n/d+4/d}{4/d}$$

elements, since n is odd, d only takes the valor 1, so the standard covering has exactly

$$\frac{1}{n+4}\binom{n+4}{4}=\frac{1}{24}(n^3+6n^2+11n+6)$$

elements which is the minimum in any optimal covering, thus the result follows. □

7 The case $\tau_{5,n}$ for an odd integer n

We compute $\tau_{5,n}$, i.e. when there are n variables and the monomials have degree 5. Again, we analyze the way monomials of degree 5 cover the graph $G_{4,n}$.

First we divide the vertices of $G_{4,n}$ into 3 disjoint sets: $\mathcal{K}_n$, $\mathcal{T}_n^1$ and $\mathcal{T}_n^2$. The set $\mathcal{K}_n$ has all monomials of $G_{4,n}$ with at most 2 variables. $\mathcal{K}_n$ induces a subgraph isomorphic to a subdivision of K_n, that is, between each pair of monomials $\{X_i^4, X_j^4\}$ there is a path form by the monomials $X_i^3X_j$, $X_i^2X_j^2$ and $X_iX_j^3$. The set $\mathcal{T}_n^1$ has all monomials of the form $X_i^2X_jX_k$ with i, j k distinct, and set $\mathcal{T}_n^2$ has all monomials of the form $X_iX_jX_kX_m$ with i, j, k and m distinct. There are also edges between $\mathcal{K}_n$ and $\mathcal{T}_n^1$, and between $\mathcal{T}_n^1$ and $\mathcal{T}_n^2$.

Figure 9 shows how vertices of degree 5 covers the graph $G_{4,n}$. There are 7 types of vertices of degree 5: Vertices of the form X_i^5, $X_i^4X_j$, $X_i^3X_j^2$, $X_i^3X_jX_k$, $X_i^2X_j^2X_k$, $X_i^2X_jX_kX_l$ and $X_iX_jX_kX_lX_m$; we call these vertices of type 1, 2, ..., and 7 respectively.

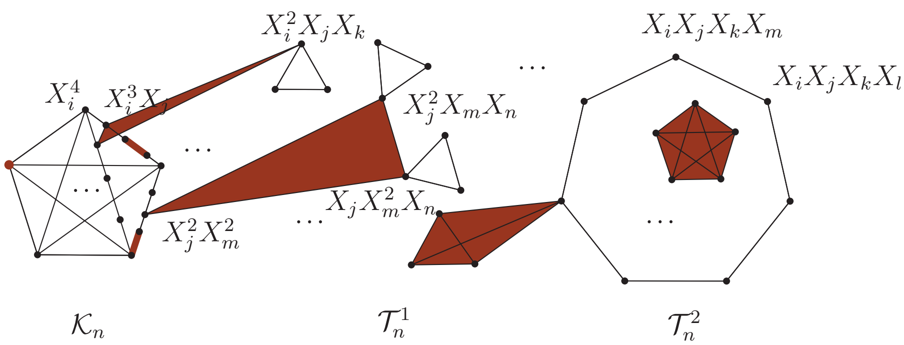


Figure 9: **Vertices of degree 5 covering monomials in $G_{4,n}$.**

Observe that each type of vertices covers monomials in $G_{5,n}$ in a certain manner. Vertices of type 1, 2 and 3 cover monomials in $\mathcal{K}_n$, vertices of type 4 and 5 cover monomials in both $\mathcal{K}_n$ and $\mathcal{T}_n^1$, vertices of type 6 cover monomials in both $\mathcal{T}_n^1$ and $\mathcal{T}_n^2$, while vertices of type 7 only cover monomials in $\mathcal{T}_n^2$; this behaviour is shown in Figure 9. We use this behaviour to show that when n is odd the Conjecture 3.4 holds.

Theorem 7.1. *For all $n \geq 1$ odd,*

$$\tau_{5,n} = \overline{f}(5, n) = L_2(5, n).$$

Proof. We give some cardinality constrain about any optimal covering of $G_{4,n}$ and show that the standard covering reaches this number, thus showing that it is optimal.

First consider the case $n \equiv 0$ modulo 5. Observe that monomials of the form X_i^4 can be covered only by vertices of type 1 and 2, observe that without loss of generality we can assume that any optimal covering of $G_{4,n}$ uses vertices of type 2. Now we need to cover monomials of the

form $X_i^3X_j$; for each $0 \leq i \leq n-1$ there are $n-2$ adjacent monomials to X_i^4 that are not covered yet, these monomials can be covered with vertices of type 3 and 4, since type 4 vertices covers exactly one more monomial than type 3 vertices the best we can do is use as much as possible vertices of type 4 without intersecting any monomial already covered or having intersections between them; observe that if there exist an intersection then by counting the resulting covering will not improve the number of elements. So we use $(n-3)/2$ ($n-2$ is odd) vertices of type 4 for each i, thus using $n(n-3)/2$ vertices of type 4 in total. Now we have for each i exactly one monomial $X_i^3X_j$ not covered, the best we can do is use a vertex of type 3 (without intersection), so we use n vertices of type 3 in total. After using vertices of type 3 and 4 we have $\binom{n}{2}-n$ monomials of the form $X_i^2X_j^2$ not covered yet, the best we can do is use vertices of type 5 to cover these remaining monomials in $\mathcal{K}_n$, so we use in total $\binom{n}{2}-n$ vertices of type 5.
Now we have covered $\mathcal{K}_n$ in such a way that we maximize the number of monomials covered in $\mathcal{T}_n^1$ and minimizing the elements used, since we are assuming that our covering is formed by disjoint sets. The next step is cover the remaining monomials in $\mathcal{T}_n^1$, the best we can do is use vertices of type 6 and assuming that the covering is disjoint. We have $(n^3-3n^2+2n)/2-(n^2-3n)/2-2((n^2-n)/2-n) = \frac{n^3-6n^2+11n}{2}$ monomials not covered in $\mathcal{T}_n^1$, so we use $\frac{n^3-6n^2+11n}{6}$ vertices of type 6 to cover all of them (if possible) and these vertices covers $\frac{n^3-6n^2+11n}{2}$ monomials in $\mathcal{T}_n^2$. To finish it only remains to cover the $\frac{n^4-6n^3+11n^2-6n}{24} - \frac{n^3-6n^2+11n}{2} = \frac{n^4-10n^3+35n^2-50n}{24}$ monomials left in $\mathcal{T}_n^2$, the best we can do is use $\frac{n^4-10n^3+35n^2-50n}{120}$ vertices of type 7. Observe that all our arguments above need the constructed covering (if it exists) to be disjoint in order to guarantee that this covering can not be improved and thus be optimal, to finish the proof we only need to show that the standard covering achieves this number of elements since the standard covering is disjoint by Proposition 2.1. The numbers of vertices used is n, $n(n-3)/2$, n, $\binom{n}{2}-n$, $\frac{n^3-6n^2+11n}{6}$ and $\frac{n^4-10n^3+35n^2-50n}{120}$ of type 2, 4, 3, 5, 6 ans 7 respectably, adding this number we have $\frac{n^4+10n^3+35n^2+50n}{120}$ elements in total, using the formula for $\overline{f}(n,5)$ given by the first equality of Conjecture 3.4 we have $\overline{f}(n,5) = \frac{n^4+10n^3+35n^2+50n}{120}$ and the result follows.

Now we consider the case $n \not\equiv 0$ modulo 5, the main difference between this case and the former is that the standard covering use exactly one vertex of type 1 to cover a monomial of the form X_i^4 since in this case the congruence $5i \equiv 1$ modulo n has exactly one solution, but we can use the same arguments since the covering using only one vertex of type 1 cannot be improved by the number of elements used, again assuming the constructed covering is disjoint. For this case we use 1 vertex of type 1, $n-1$ vertices of type 2, $(n-1)(n-3)/2+(n-1)/2$ vertices of type 4, $n-1$ vertices of type 3 and $\binom{n}{2}-(n-1)$ vertices of type 5 to cover all monomials in $\mathcal{K}_n$, in total they are n^2-n+1, as before, this covering maximises the number of monomials covered in $\mathcal{T}_n^1$. Now we have $(n^3-3n^2+2n)/2-(n^2-n+1) = \frac{n^3-6n^2+11n-6}{2}$ monomials in $\mathcal{T}_n^1$ not covered yet, so we use $\frac{n^3-6n^2+11n-6}{6}$ vertices of type 6. To finish the covering we have $\frac{n^4-6n^3+11n^2-6n}{24} - \frac{n^3-6n^2+11n-6}{6} = \frac{n^4-10n^3+35n^2-50n+24}{24}$ monomials not covered yet in $\mathcal{T}_n^2$, so we use $\frac{n^4-10n^3+35n^2-50n+24}{120}$ vertices of type 7. Adding all vertices used to construct this covering yields $\frac{n^4+10n^3+35n^2+50n+24}{120}$ in total; to complete the proof we only need to see that the standard covering reaches this number of a optimal covering, this is straightforward from the first equality of Conjecture 3.4 and the formula for $\overline{f}(5,n)$. □

8 Conclusions

We compute the algebraic invariant $\tau_{d,n}$ for the cases $d=4$ and all n, for $n=4$ with d even and for $d=5$ with n odd. These values extend the previous knowledge about τ given in [5]. Also, our result confirms the general formula conjectured in [7] for these particular cases. We suggest that the computed values can be used as an initial step to compute the relevant bounds for the 1-dimensional ideal generating conjecture.

The independence number and the minimum cardinality of an upward clique cover of $G_{d,n}$ has been studied in [1]. They find an exact value for the independence number for $G_{d,4}$ and compare this number with the number of 2x2 non negative integer matrices such that the sum of its entrances is equal to d. Also, they present an algorithm to bound the minimum cardinality of an upward clique cover.

The fact that $\tau_{d,n} = \tau_{n,d}$ for the cases $d = 1,2,3$, for all n, and $d=4$, for n even, hints to the possibility that the equality is true for all d and n. We find this a tantalizing problem. If true, it may imply that the collection of graphs $\{G_{d,n}\}$ has an involution which works as a duality operation. However, we were unable to find such an operation, and even a bijective proof of this equality in the case $d=3$ and $d=4$ seems difficult to generalize to all values of n and d. Notice that Conjecture 3.4 implies $\tau_{d,n} = \tau_{n,d}$.

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