

The ideal-valued index of fibrations with total space a G_2 flag manifold*

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Abstract

By using the cohomology of the G_2 -flag manifolds $G_2/U(2)_\pm$, and their structure as a fiber bundle over the homogeneous space $G_2/SO(4)$, we compute the \mathbb{Z}_2 Fadell-Husseini index of such fiber bundles for the \mathbb{Z}_2 action given by complex conjugation. Also, considering the tautological bundle γ over $\tilde{G}_4(\mathbb{R}^7)$, we compute the \mathbb{Z}_2 Fadell-Husseini index of the pullback bundle of $s\gamma$ along the composition of the embedding between $G_2/SO(4)$ and $\tilde{G}_3(\mathbb{R}^7)$, the map that takes the orthogonal complement of a subspace, and the fiber bundle $G_2/U(2)_\pm \rightarrow G_2/SO(4)$. Here $s\gamma$ means the associated sphere bundle of the bundle γ . Furthermore, we derive a general formula for the n -fold product bundle $s\gamma^n$ for which we make the same computations. We finish our work with an application of our computations to a problem concerning discrete geometry.

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1 The exceptional Lie group G_2 .

In this section we will recall three equivalent definitions of the real form of the exceptional Lie group G_2 . General references for the upcoming discussion include [4] and [14].

1.1 Octonions algebra

Given a normed division algebra A , the Cayley-Dickinson construction creates a new algebra A' with elements $(a, b) \in A^2$ and conjugation $(a, b)^* = (\bar{a}, -b)$. The addition in A' is done component-wise, and multiplication goes like

$$(a, b)(c, d) = (ac - d\bar{b}, \bar{a}d + cb),$$

where juxtaposition indicates multiplication in A . An equivalent way to define the new algebra A' is to add an independent square root of -1 , i , that multiplies the second named element on each pair (a, b) . Now the conjugation in A' uses the original conjugation of A and $i^* = -i$. Then the construction becomes an algebra of elements $a + ib$ for some $a, b \in A$.

Starting with \mathbb{R} , the complex numbers are defined via the Cayley-Dickinson construction to be elements $a + ib$, with a and b real numbers. Similarly, the quaternions are generated as a real algebra by $\{1, i, j, k\}$, subject to the relations $i^2 = j^2 = k^2 = ijk = -1$. Having this in mind, we get the following definition

Definition 1.1. The octonions are generated as a quaternionic algebra via the Cayley-Dickinson construction as $\mathbb{H} \oplus \mathbb{H}[l]$, where l denotes an independent square root of -1 .

Even if the Cayley-Dickinson construction is an excellent method to produce other normed division algebras, we lost very nice properties in the process. The multiplication in \mathbb{O} turns out to be non-commutative and non-associative. As a real vector space, \mathbb{O} is generated by $\{1, i, j, k, l, li, lj, lk\} = \{1, e_1, \dots, e_7\}$, where e_1, e_2, \dots, e_7 are imaginary units, which square to -1 , switch sign under complex conjugation and anticommute. They span the seven-dimensional real subspace which we will denote as the purely imaginary part of the octonions, $\text{Im } \mathbb{O}$.

Denote by \cdot the product furnishing the octonions with the structure of real division algebra. The product \cdot determines a cross product in \mathbb{O} ,

expressed as

$$x \times y := \frac{1}{2}x \cdot y - y \cdot x,$$

which contains all the non-trivial algebraic information about the octonions, and turns $\text{Im } \mathbb{O}$ into an algebra. As will be discussed below, the algebra $(\text{Im } \mathbb{O}, \times)$ provides a new definition of the group G_2 .

1.2 The exceptional Lie group G_2 : three equivalent definitions

Now that we are familiar with the \mathbb{R} -algebra \mathbb{O} , we are ready to introduce the main ingredient of this work.

Definition 1.2. The exceptional Lie group G_2 is defined to be the group of all \mathbb{R} -algebra automorphisms of the octonions.

We will consider an alternative but equivalent definition of G_2 . Given two orthogonal and unit imaginary octonions x and y , the cross product $x \times y$ is orthogonal to both of them, and the subalgebra generated by $\{1, x, y, x \times y\}$ is isomorphic to \mathbb{H} . If we consider another unit imaginary octonion z orthogonal to the subspace generated by $\{1, x, y, x \times y\} \cong \mathbb{H}$, then the subalgebra over the quaternions $\mathbb{H} \oplus \mathbb{H}[z]$ is isomorphic to the octonions. This has the consequence that an element $g \in G_2$ can be characterized by prescribing three unit imaginary octonions $\{x, y, z\}$, with z orthogonal to x , y and $x \times y$. The element g is then the unique automorphism of the imaginary part of the octonions, which sends $\{x, y, z\}$ to $\{i, j, l\}$. Then the exceptional Lie group G_2 is defined to be the automorphism group of the algebra $(\text{Im } \mathbb{O}, \times)$.

On $\text{Im } \mathbb{O}$ one can define a cross product 3-form on the generators e_i, e_j, e_k as

$$\phi(e_i, e_j, e_k) = f^{ijk},$$

where $e_i \times e_j = \sum_k f^{ijk} e_k$. If we consider the dual basis $\{w^i := e_i^*\}$ of the generators $\{e_i\}$ given above, then the 3-form ϕ is given by

$$\phi = w^{123} - w^{145} - w^{167} - w^{246} + w^{257} - w^{347} - w^{356},$$

where the notation w^{ijk} denotes the wedge product $e^i \wedge e^j \wedge e^k$. Notice that ϕ encodes the multiplicative structure of cross product in the imaginary part of the octonions. Then, using the above considerations, we

obtain a third equivalent definition of the exceptional Lie group G_2 as follows:

$$G_2 = \{g \in GL(\text{Im } \mathbb{O}) \mid g^* \phi = \phi\}.$$

The following result summarizes the previous discussion about the real form of G_2 , besides some of its properties presented by Bryant in [11, Theorem 1].

Theorem 1.3. *The real form of the group G_2 has dimension 14. It is simple, simply connected, and is isomorphic to one and hence to all of the following Lie groups:*

1. *The group of all \mathbb{R} -algebra automorphisms of the octonions.*
2. *The automorphism group of the subalgebra $(\text{Im } \mathbb{O}, \times)$.*
3. *The subgroup of $SO(7)$ which preserves the cross product 3-form.*

2 G_2 flag manifolds

The flag manifolds on G_2 have received recently a lot of attention from several viewpoints. From riemannian geometry ([11], [25]), algebraic topology relevant to global analysis [1], complex and Kähler geometry ([19], [23]), and to the classical computations of their characteristic classes via representation theory by Borel and Hirzebruch ([10], [16]). The combination of the above facts, excellently produced in [19], arose our interest in the subject and gave us the idea to calculate the Fadell-Husseini index of some fiber bundles over $G_2/SO(4)$ with total space a G_2 -flag manifold.

In this section we introduce some flag manifolds associated to the exceptional Lie group G_2 . Since we are particularly interested on the G_2 -flag manifolds which fiber over $G_2/SO(4)$, we are going to start introducing the G_2 -homogeneous space $G_2/SO(4)$ in order to define the fiber bundles for which we want to calculate their Fadell-Husseini indices.

2.1 The space of associative subspaces $G_2/SO(4)$.

Now that we have introduced the exceptional Lie group G_2 with some equivalent definitions, we are ready to study the G_2 -homogeneous space that we are going to use.

Definition 2.1. A 3-dimensional subspace $\xi \subset \text{Im } \mathbb{O}$ is said to be *associative* if it is the imaginary part of a subalgebra isomorphic to the quaternions.

Notice that the subspace ξ acquires a canonical orientation from that of the quaternions. On the other hand, in terms of the 3-form defined in 1.2, we can also define an associative subspace as the 3-dimensional real subspace of $\mathbb{R}^7 \cong_{\mathbb{R}} \text{Im } \mathbb{O}$ in which the 3-form ϕ agrees with the volume form $\text{Vol}(\xi)$.

The group G_2 stabilizes 1, and since the scalar product is the real part of the octonionic multiplication \cdot , it acts by isometries on $\text{Im } \mathbb{O}$. Since also preserves the vector product, it preserves orientation so that

$$G_2 \subset SO(\text{Im } \mathbb{O}) \cong SO(7).$$

Every associative subspace ξ admits an orthonormal basis $\{e_1, e_2, e_3\}$ with $e_1 \times e_2 = e_3$. Then, given such a triple $\{e_1, e_2, e_3\}$, there exists a unique automorphism of $\text{Im } \mathbb{O}$ taking the ordered triple $\{i, j, l\}$ to it. This induces a transitive action of G_2 on the Grassmannian of associative 3-dimensional subspaces of $\text{Im } \mathbb{O} \cong \mathbb{R}^7$, with stabilizer $SO(4)$. Then $G_2/SO(4)$ is the set of all associative 3-dimensional subspaces of $\text{Im } \mathbb{O} \cong \mathbb{R}^7$. Finally, since every associative subspace has a canonical orientation, there is an embedding of $G_2/SO(4)$ in the 3-dimensional oriented Grassmannian manifold $G_3(\mathbb{R}^7)$, with $\mathbb{R}^7 \cong \text{Im } \mathbb{O}$.

Borel and Hirzebruch studied in [10] the cohomology with \mathbb{F}_2 -coefficients and the Stiefel-Whitney classes of $G_2/SO(4)$. They proved the following result:

Proposition 2.2. *The homogeneous space of associative 3-dimensional real subspaces, denoted by $G_2/SO(4)$, is an 8-dimensional manifold for which $H^*(G_2/SO(4); \mathbb{F}_2)$ is generated by two elements u_2, u_3 of degrees 2 and 3 respectively, with the relations*

$$u_2^3 = u_3^2 \quad \text{and} \quad u_3 u_2^2 = 0.$$

Also, the Stiefel-Whitney classes of $G_2/SO(4)$ are non-zero only in dimensions 0, 4, 6 and 8.

On the other hand, Shi and Zhou in [24, Section 10], Thung in [19, Section 2], and Akbulut and Kalafat in [1, Theorem 10.3], studied the integral cohomology of $G_2/SO(4)$, and the relations between the corresponding generators to write it as a truncated polynomial algebra.

Proposition 2.3. *The integral cohomology of $G_2/SO(4)$ is a truncated polynomial algebra generated by two elements a and b , of degree 3 and 4 respectively, subject to the relations*

$$\{2b = 0, b^3 = 0, a^3 = 0, ab = 0\}.$$

2.2 The six dimensional sphere $G_2/SU(3)$.

We are going to start with a G_2 -flag manifold that, even if it do not fibre over $G_2/SO(4)$, it is a good first example of this type of spaces.

Remember that, by Theorem 1.3 part 2, G_2 can be characterized by triples $\{x, y, z\}$ of mutually orthogonal unitary vectors in $\text{Im } \mathbb{O}$, where z is also orthogonal to $x \times y$. This means that the six dimensional sphere, which can be seen as the unitary vectors within the seven dimensional real subspace $\text{Im } \mathbb{O}$, carries a transitive action of G_2 .

To determine the isotropy group of an element $l \in S^6$, consider the subspace V defined as the orthogonal complement of l inside $\text{Im } \mathbb{O}$, and the complex structure on V given by the left multiplication by l . The identification with \mathbb{C}^3 , equipped with its standard Hermitian scalar product, induces a scalar product on V defined as

$$\langle v, w \rangle_V = (v, w) + l(v, lw),$$

where $(-, -)$ denote the standard real product. Since $g \in (G_2)_l$ preserves $(-, -)$, it also preserves $\langle -, - \rangle_V$, and $(G_2)_l \subset U(V) \cong U(3)$. Calculating the determinant of $g \in (G_2)_l$, can be proved that actually the isotropy group is isomorphic to $SU(3)$. We have obtained the following result:

Proposition 2.4. *There is a transitive action of G_2 on S^6 with isotropy group isomorphic to $SU(3)$. In particular, S^6 is diffeomorphic to the quotient $G_2/SU(3)$.*

2.3 The space of complex coassociative 2-planes $G_2/U(2)_+$.

We will consider now the complexification of the purely imaginary subspace of the octonions, which is isomorphic to the complex vector space \mathbb{C}^7 , in symbols $\text{Im } \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C} \cong_{\mathbb{C}} \mathbb{C}^7$.

Definition 2.5. We call a complex 2-dimensional subspace $W \subset \mathbb{C}^7$ *coassociative* if $v \times w = 0$ for all $v, w \in W$

Notice that the coassociative subspace W is automatically isotropic. Also, the associative 3-form $\phi \otimes \text{id}$ defines a complex 3-form $\phi_{\mathbb{C}}$ on the complex space \mathbb{C}^7 . So, in terms of the 3-form defined in 1.2, a complex vector subspace $W \subset \mathbb{C}^7$ is called coassociative if the complexified form $\phi_{\mathbb{C}}$ vanishes on W . We will consider coassociative complex lines (1-dimensional complex subspaces) and coassociative planes (2-dimensional complex subspaces).

On the other hand, let J be an orthogonal complex structure on ξ^{\perp} with $(1, 0)$ -space W , where ξ is an associative subspace in $\text{Im } \mathbb{O}$, and W is the eigenspace associated to the eigenvalue i of J . Choose now an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of ξ^{\perp} with $J(e_1) = e_2$ and $J(e_3) = e_4$. We say that J , or the corresponding $(1, 0)$ -space W , is positive (resp. negative) according as the basis of ξ^{\perp} is positive (resp. negative).

The following result, which connects the notion of complex-coassociative and positive subspaces, is proved in [25, Lemma 2.2, page 293].

Lemma 2.6. *Let ξ be an associative 3-dimensional subspace in $\text{Im } \mathbb{O}$, and let $W \subset \xi^{\perp} \otimes \mathbb{C}$ be a maximally isotropic subspace. Then W is positive if and only if it is complex coassociative.*

Notation 2.7. *The space of complex coassociative 2-planes in $\text{Im } \mathbb{O} \otimes \mathbb{C}$ will be denoted by $G_2/U(2)_+$.*

The reason for the notation comes from lemma 2.6. Moreover, there is a map

$$\rho_1 : G_2/U(2)_+ \rightarrow G_2/SO(4),$$

given by $W \mapsto (W \oplus \bar{W})^{\perp}$, which exhibits $G_2/U(2)_+$ as the total space of a locally trivial smooth fibration with fiber $\mathbb{C}P^1 \approx S^2$. By lemma 2.6 the fiber of any $\xi \in G_2/SO(4)$ is all positive maximally isotropic subspaces of $\xi^{\perp} \otimes \mathbb{C}$, equivalently, all positive orthogonal complex structures on ξ^{\perp} . Actually, by [25, section 4.4] there is an identification of $G_2/U(2)_+$ with a quaternionic twistor space of $G_2/SO(4)$.

As stated in [19, Prop. 3], the cohomology of $G_2/U(2)_+$ with integral coefficients is generated by classes $g_i \in H^{2i}(G_2/U(2)_+; \mathbb{Z})$, for $i = 1, \dots, 5$, and the multiplicative structure is determined by the relations

$$g_1^2 = 3g_2, g_1g_2 = 2g_3, g_2^2 = 2g_4, g_1g_4 = g_5.$$

The cohomology with coefficients in \mathbb{F}_2 will be calculated in subsection 3.3.

2.4 The space of complex non-coassociative 2-planes $G_2/U(2)_-$.

Notation 2.8. *The space of all 2-planes in $\text{Im } \mathbb{O} \otimes \mathbb{C}$ which are not complex coassociative will be denoted by $G_2/U(2)_-$.*

Similarly to $G_2/U(2)_+$, the reason for the notation comes from lemma 2.6. A 2-dimensional real subspace W , which is a $(1,0)$ -space for a negative complex structure in ξ^\perp , is said to be negative and is still a maximally isotropic subspace of $\xi^\perp \otimes \mathbb{C}$.

Alternatively, we can think the space $G_2/U(2)_-$ as follows. Let Q^5 be the complex quadric $\left\{ [z_1, \dots, z_7] \in \mathbb{C}P^6 \mid \sum_{i=1}^7 z_i^2 = 0 \right\}$ consisting of all 1-dimensional isotropic subspaces of $\text{Im } \mathbb{O} \otimes \mathbb{C}$. There is a G_2 -equivariant isomorphism from Q^5 to $G_2/U(2)_-$ given by $\ell \mapsto \ell^\perp \cap \ell^a$, with inverse $W \mapsto W \times W$, where $\ell^a = \{x \in \text{Im } \mathbb{O} \otimes \mathbb{C} \mid x \times \ell = 0\}$ is the annihilator of ℓ .

Since $\tilde{G}_2(\mathbb{R}^7)$ is also diffeomorphic to the complex quadric Q^5 , we can identify every element in $G_2/U(2)_-$ as an oriented plane P with oriented, orthonormal basis $\{x, y\}$. Then there is a well defined map $\rho_6 : G_2/U(2)_- \rightarrow S^6$ which sends $P \mapsto x \times y := xy$. Actually, the oriented plane P can be identified, via the almost complex structure of S^6 , with a complex line in $T_{xy}S^6$. On the other hand, given an element $k \in S^6$, there is a oriented orthonormal basis $\{x, y\}$ of a complex line in T_kS^6 that satisfies $k = xy$. This means that the fiber over an element $v \in S^6$ is precisely $\mathbb{P}(T_vS^6)$. This exhibits $G_2/U(2)_-$ as $\mathbb{P}(TS^6)$ with ρ_6 as the base point projection. We can also think of ρ_6 as a fibration with fibers diffeomorphic to $\mathbb{C}P^2$.

Finally, the isomorphism $TS^6 \cong T^*S^6$ as real vector bundles induces a diffeomorphism $\mathbb{P}(TS^6) \cong \mathbb{P}(T^*S^6)$. The following result resumes all the diffeomorphic definitions of $G_2/U(2)_-$. See [19, Prop. 8].

Proposition 2.9. *The following 10-dimensional manifolds are all diffeomorphic to each other:*

1. *The space of all 2-planes in $\text{Im } \mathbb{O} \otimes \mathbb{C}$ which are not complex coassociative.*
2. *The Grassmannian $\tilde{G}_2(\mathbb{R}^7)$ of oriented 2-planes in \mathbb{R}^7 .*

3. The complex quadric

$$Q^5 = \left\{ [z_1, \dots, z_7] \in \mathbb{C}P^6 \mid \sum_{i=1}^7 z_i^2 = 0 \right\}$$

4. The projectivization of the tangent and cotangent bundle $\mathbb{P}(TS^6)$ and $\mathbb{P}(T^*S^6)$ for any almost complex structure in S^6 .

The cohomology with integral coefficients of $G_2/U(2)_-$ is calculated in [19, Prop. 11]. It is the quotient of a polynomial algebra with generators x in degree 6, and y in degree 2, satisfying the relations $x^2 = 0$ and $y^3 = -2x$. Unlike $G_2/U(2)_+$, using the bundle ρ_6 we can calculate the cohomology ring of $G_2/U(2)_-$ with coefficients in \mathbb{F}_2 as follows:

Proposition 2.10. *The cohomology of $G_2/U(2)_-$ with coefficients in \mathbb{F}_2 is given by*

$$H^*(G_2/U(2)_-; \mathbb{F}_2) = \mathbb{F}_2[x, y] / \langle x^2, y^3 \rangle,$$

where $\deg(y) = 2$, $\deg(x) = 6$.

Proof. Consider the fiber bundle

$$\mathbb{C}P^2 \hookrightarrow G_2/U(2)_- \rightarrow S^6$$

and the corresponding Serre spectral sequence. Since S^6 is simply connected, there are no local coefficients and the E_2 -term is given by

$$E_2^{p,q} = H^p(S^6; H^q(\mathbb{C}P^2; \mathbb{F}_2)).$$

Let us recall the cohomology of the fiber $\mathbb{C}P^2$ and the base space S^6 as follows:

$$H^*(S^6; \mathbb{F}_2) = \mathbb{F}_2[x] / \langle x^2 \rangle \quad \text{and} \quad H^*(\mathbb{C}P^2; \mathbb{F}_2) = \mathbb{F}_2[y] / \langle y^3 \rangle.$$

For an illustration of the E_2 -term see Figure 1. Then, since all the possible differentials are trivial, $E_2^{p,q} \cong E_\infty^{p,q}$ and

$$H^n(G_2/U(2)_-; \mathbb{F}_2) = \mathbb{F}_2[x, y] / \langle x^2, y^3 \rangle.$$

□

5								
4	y^2						xy^2	
3								
2	y						xy	
1								
0	\mathbb{Z}_2						x	
	0	1	2	3	4	5	6	7

$H^*(S^6; \mathbb{F}_2)$

Figure 1: $E_2^{p,q} = H^p(S^6; H^q(\mathbb{C}\mathbb{P}^2; \mathbb{F}_2)) \Rightarrow H^*(G_2/U(2)_-; \mathbb{F}_2)$.

Similar to $G_2/U(2)_+$, there exists a map

$$\rho_2: G_2/U(2)_- \rightarrow G_2/SO(4),$$

given by $W \mapsto (W \oplus \bar{W})^\perp$, which exhibits $G_2/U(2)_-$ as the total space of a locally trivial smooth fibration with fiber $\mathbb{C}P^1 \approx S^2$. By lemma 2.6, the fiber at any $\xi \in G_2/SO(4)$ consist of all negative maximally isotropic subspaces of $\xi^\perp \otimes \mathbb{C}$, or equivalently, all negative orthogonal complex structures on ξ^\perp . Let us describe now the induced homomorphism ρ_2^* .

Lemma 2.11. *The induced homomorphism in cohomology, denoted by ρ_2^* , with \mathbb{F}_2 coefficients, maps the class u_2 to y .*

Proof. Let us start analyzing the Gysin exact sequence associated to the sphere bundle ρ_2 ,

$$\begin{aligned} \dots \rightarrow H^{i-3}(G_2/SO(4); \mathbb{F}_2) &\xrightarrow{\simeq w_3} H^i(G_2/SO(4); \mathbb{F}_2) \rightarrow \\ &\xrightarrow{\rho_2^*} H^i(G_2/U(2)_-; \mathbb{F}_2) \rightarrow H^{i-2}(G_2/SO(4); \mathbb{F}_2) \rightarrow \dots \end{aligned}$$

Since $H^1(G_2/SO(4); \mathbb{F}_2) = 0$, by the exactness of the sequence,

$$\rho_2^*: H^4(G_2/SO(4); \mathbb{F}_2) \rightarrow H^4(G_2/U(2)_-; \mathbb{F}_2)$$

is a monomorphism. This means that $\rho_2^*(u_2^2) = y^2$ and consequently $\rho_2^*(u_2) = y$. □

2.5 The full flag manifold $G_2/U(1) \times U(1)$.

Given a complex isotropic line $\ell \subset \text{Im } \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$, consider the annihilator ℓ^a , that is the subspace of $\text{Im } \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ described as

$$\{x \in \text{Im } \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C} \mid x \times \ell = 0\}.$$

Notice that this is a complex 3-dimensional isotropic subspace of $\text{Im } \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$. Since the maximal torus in G_2 is of rank 2, the complete flag manifold $G_2/U(1) \times U(1)$ is a smooth complex manifold of dimension 6 that can be described as follows. The space of pairs (ℓ, D) where ℓ is a complex isotropic line, D is a 2-plane containing ℓ , and both are contained in ℓ^a . For every pair (ℓ, D) in $G_2/U(1) \times U(1)$, we write $D = \ell \oplus q$, where q is the orthogonal complement of ℓ . Then we get a fibration

$$\rho_3: G_2/U(1) \times U(1) \rightarrow G_2/SO(4)$$

given by $(\ell, D) \mapsto \xi$, where $\xi \otimes_{\mathbb{R}} \mathbb{C} = q \oplus \bar{q} \oplus (q \times \bar{q})$. The map ρ_3 factors through a fibration over $G_2/U(2)_{\pm}$, which sends (ℓ, D) to the positive (resp. negative) maximally isotropic subspace of $\xi^{\perp} \otimes_{\mathbb{R}} \mathbb{C}$ which contains ℓ , and ρ_1 (resp. ρ_2) defined above.

$$(1) \quad \begin{array}{ccccc} & & G_2/U(1) \times U(1) & & \\ & \swarrow \rho_4 & \downarrow \rho_3 & \searrow \rho_5 & \\ G_2/U(2)_+ & & G_2/SO(4) & & G_2/U(2)_- \cong Q_5 \\ & \searrow \rho_1 & \swarrow \rho_2 & & \searrow \rho_6 \\ & & & & S^6 \end{array}$$

To summarize, all the maps ρ_i in the commutative diagram 1 are fibrations, with fiber diffeomorphic to $\mathbb{C}P^1 \cong S^2$, except ρ_3 and ρ_6 . Furthermore, ρ_1 , ρ_2 and ρ_3 are fiber bundles over $G_2/SO(4)$. In the case of ρ_6 the fiber is diffeomorphic to $\mathbb{C}P^2$. This means that, using the map ρ_5 , we can also calculate the cohomology of $G_2/U(1) \times U(1)$ with \mathbb{F}_2 coefficients.

Proposition 2.12. *The cohomology of $G_2/U(1) \times U(1)$ with \mathbb{F}_2 coefficients is given by*

$$H^n(G_2/U(1) \times U(1); \mathbb{F}_2) = \begin{cases} \mathbb{Z}_2, & \text{if } n = 0, 1, 2. \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } n = 2, 4, 6, 8, 10. \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Consider the fibration

$$S^2 \hookrightarrow G_2/U(1) \times U(1) \xrightarrow{\rho_5} G_2/U(2)_-,$$

and the corresponding Serre spectral sequence, where the cohomology of the fiber is described as $H^*(S^2; \mathbb{F}_2) = \mathbb{F}_2[z]/z^2$. Since $\pi_1(G_2/U(2)_-)$ is trivial, there are no local coefficients and the E_2 -term is given by

$$E_2^{p,q} = H^p(G_2/U(2)_-; H^q(S^2; \mathbb{F}_2)).$$

For an illustration of the E_2 -term see Figure 2. The rest of the proof is similar to the one in 2.10, since all the possible differentials are trivial. \square

$H^*(S^2; \mathbb{F}_2)$	3											
	2	z		$z \otimes y$		$z \otimes y^2$		$z \otimes x$		$z \otimes xy$		$z \otimes xy^2$
	1											
	0	\mathbb{Z}_2		y		y^2		x		xy		xy^2
		0	1	2	3	4	5	6	7	8	9	10
		$H^*(G_2/U(2)_-; \mathbb{F}_2)$										

Figure 2: $E_2^{p,q} = H^p(G_2/U(2)_-; H^q(S^2; \mathbb{F}_2)) \Rightarrow H^*(G_2/U(1) \times U(1); \mathbb{F}_2)$.

3 Calculations of Fadell-Husseini Index.

We start by recalling the definition of the Fadell-Husseini index.

3.1 The Fadell-Husseini Index

Let G be a finite group, and let R be a commutative ring with unit. For a G -equivariant map $p: X \rightarrow B$, the *Fadell-Husseini index* of p with coefficients in R is defined to be the kernel ideal of the following induced map

$$\begin{aligned} \text{Index}_G^B(p; R) &= \ker (p_*: H^*(EG \times_G B; R) \longrightarrow H^*(EG \times_G X; R)) \\ &= \ker (p_*: H_G^*(B; R) \longrightarrow H_G^*(X; R)). \end{aligned}$$

Here $H_G^*(\cdot)$ stands for the Borel cohomology defined as the Čech cohomology of the Borel construction $EG \times_G \cdot$.

The most important property of the Fadell-Husseini index is the *monotonicity*. If $p: X \rightarrow B$ and $q: Y \rightarrow B$ are G -equivariant maps, and $f: X \rightarrow Y$ is a G -equivariant map such that $p = q \circ f$, then

$$\text{Index}_G^B(p; R) \supseteq \text{Index}_G^B(q; R).$$

The monotonicity of the index is usually the key ingredient to verify the existence of equivariant maps. It is common to relate the Fadell-Husseini index with spectral sequences since, under certain conditions, by [20, Theorem 5.19] the Serre spectral sequence compute the kernel of induced homomorphisms in cohomology.

If B is a point, we simplify notation and write $\text{Index}_G^B(p; R) = \text{Index}_G(X; R)$. From here on we will be considering \mathbb{F}_2 coefficients. For more details see [15]

3.2 Dold's argument

In order to compute the Fadell-Husseini index of some of the fibrations exposed in section 2, we present a result of Albrecht Dold [13] which allow us to deduce some differentials of the Serre spectral sequence associated to a particular kind of fiber bundles. Let us start with the general argument.

Let $E \xrightarrow{\pi} B \xleftarrow{\pi'} E'$ be vector bundles of fiber-dimension n , and m , respectively over the same paracompact space B , and let $f: S(E) \rightarrow E'$ be an odd map, that is ($f(-x) = -f(x)$), where $S(E)$ is the total space of the sphere bundle associated to π , such that

$$\begin{array}{ccc} S(E) & \xrightarrow{f} & E' \\ & \searrow s\pi & \swarrow \pi' \\ & & B \end{array}$$

commutes. Let us define $Z_f = \{x \in S(E) \mid f(x) = 0\}$, where 0 stands for the zero section of π' , and the projection maps

$$S(E) \rightarrow \bar{S}(E) = S(E)/\mathbb{Z}_2 \quad \text{and} \quad Z \rightarrow \bar{Z} = Z/\mathbb{Z}_2,$$

where we are considering the antipodal action.

Cohomology H^* is understood in the Čech sense with mod-2 coefficients, and $H^*(B)[t]$ is the polynomial ring over $H^*(B)$ in one indeterminate t of degree 1. Since the antipodal action is fixed point free in $S(E)$ and Z , the projection maps $S(E) \rightarrow \bar{S}(E)$ and $Z \rightarrow \bar{Z}$ are 2-sheeted covering maps. Their characteristic classes are denoted by u , $u | \bar{Z}$, respectively, which can be replaced by the indeterminate t and obtain an homomorphism of $H^*(B)$ -algebras

$$\sigma: H^*(B)[t] \longrightarrow H^*(\bar{S}(E)) \longrightarrow H^*(\bar{Z}),$$

given by $t \mapsto u \mapsto u | \bar{Z}$. Dold proved the following result.

Proposition 3.1. *If $q(t) \in H^*(B)[t]$ is such that $\sigma(q(t)) = 0$, then*

$$q(t)W(\pi'; t) = W(\pi; t)q'(t),$$

for some $q'(t) \in H^*(B)[t]$, where $W(\pi; t) = \sum_{j=0}^n w_j(\pi) \otimes t^{n-j}$.

The last theorem means that, under the last conditions, $W(\pi; t)$ divides $q(t)W(\pi'; t)$. We show the effectiveness of this theorem in the following remark.

Remark 3.2. *Under the same hypothesis above, consider the fiber bundles*

$$\begin{array}{ccc} S^{n-1} \hookrightarrow S(E) & & \{0\} \hookrightarrow B \times \{0\} \\ & \downarrow s\pi & \downarrow \text{proj}_1 \\ & B & B \end{array}$$

where \mathbb{Z}_2 acts antipodally on $S(E)$ and trivial on B . Let $f: S(E) \rightarrow B \times \{0\}$ be a \mathbb{Z}_2 -equivariant map given by $f(e) = (\pi(e), 0)$, such that the following diagram commutes:

$$\begin{array}{ccc} SE & \xrightarrow{f} & B \times \{0\} \\ & \searrow s\pi & \swarrow \text{proj}_1 \\ & & B \end{array}$$

Since $Z_f = S(E)$ and $W(\text{proj}_1, t) = 1$, if we consider $q(t)$ as the image of the transgression map $d_n^{0, n-1}$ of the Serre spectral sequence associated to the sphere bundle

$$S^{n-1} \hookrightarrow EZ_2 \times_{Z_2} E \rightarrow BZ_2 \times B,$$

by proposition 3.1,

$$q(t) = d_n^{0,n-1}(z) = \sum_{j=0}^n w_j(\pi) \otimes t^{n-j},$$

where $H^*(S^{n-1}) = \mathbb{F}_2[z]/\langle z^2 \rangle$.

We are going to use Dold's argument in some of the computations that we present.

3.3 The Fadell-Husseini index of ρ_1 and ρ_2 .

Following the idea presented in section 3.2, we are going to start studying the Stiefel-Whitney classes of ρ_1 and ρ_2 to deduce their corresponding Serre spectral sequences.

Since $G_2/SO(4)$ is a quaternionic Kähler manifold, and the bundles ρ_1 and ρ_2 are twistor fibrations over $G_2/SO(4)$, we can think of each ρ_i as the associated sphere bundle of a 3 rank vector bundle $\eta: H \rightarrow G_2/SO(4)$ with non-trivial Euler class.

In this way, we are considering the Stiefel-Whitney classes of the vector bundle η . For more detail about the bundle $s\eta$ see [25] and [21].

Since we already know the cohomology of $G_2/U(2)_-$, we are going to start with the characteristic classes of ρ_2 .

Lemma 3.3. *The Stiefel-Whitney classes of the fiber bundle*

$$\rho_2: G_2/U(2)_- \longrightarrow G_2/SO(4)$$

are non-zero in dimensions 0, 2 and 3.

Proof. Consider the Gysin exact sequence applied to the sphere bundle ρ_2

$$\begin{aligned} \rightarrow H^{i-3}(G_2/SO(4); \mathbb{F}_2) &\xrightarrow{\simeq w_3} H^i(G_2/SO(4); \mathbb{F}_2) \\ &\xrightarrow{\rho_2^*} H^i(G_2/U(2)_-; \mathbb{F}_2) \rightarrow H^{i-2}(G_2/SO(4); \mathbb{F}_2) \rightarrow \end{aligned}$$

Since $H^3(G_2/U(2)_-; \mathbb{F}_2) = 0$, we get an epimorphism

$$H^0(G_2/SO(4); \mathbb{F}_2) \twoheadrightarrow H^3(G_2/SO(4); \mathbb{F}_2)$$

which, by proposition 2.2, can be seen as $\mathbb{Z}_2 \rightarrow \langle u_3 \rangle$. Hence, $w_3(\rho_2) = u_3$. Considering the definition of the Stiefel-Whitney classes, this means that

$$\phi^{-1} \circ Sq^3(th) \neq 0,$$

where th is the Thom class associated to ρ_2 . Then, since

$$Sq^3 = Sq^1 \circ Sq^2,$$

$Sq^2(th) \neq 0$ and consequently from the definition $w_2 \neq 0$. Is not hard to see that $w_2(\rho_2) = u_2$. \square

Considering the Gysin exact sequence associated to ρ_1 , by proposition 2.2, we get that $H^3(G_2/U(2)_+; \mathbb{F}_2)$ is also trivial. Then, using the same arguments in lemma 3.3, ρ_1 has the same Stiefel-Whitney classes than ρ_2 .

Now that we know the Stiefel-Whitney classes of ρ_1 , we can calculate the cohomology of $G_2/U(2)_+$ with \mathbb{F}_2 coefficients as follows. Consider the Serre spectral sequence associated to ρ_1 , with E_2 -term given by

$$E_2^{p,q} = H^p(G_2/SO(4); H^q(S^2; \mathbb{F}_2)).$$

For an illustration of the E_2 -term see Figure 3. Notice that, for the same reason stated in 2.10, there are no local coefficients. Then, applying the Gysin sequence to ρ_1 [20, Example 5.C], and by lemma 3.3, we get that

$$d_3^{0,2}(z) = w_3(\rho_1) = u_3,$$

where $H^2(S^2; \mathbb{F}_2) = \langle z \rangle$. Finally, using the Leibniz rule, $d_3^{0,2}$ determines the complete cohomology of $G_2/U(2)_+$. We conclude then the following result.

Corollary 3.4. *The G_2 flag manifolds $G_2/U(2)_+$ and $G_2/U(2)_-$ have isomorphic cohomology rings when we consider \mathbb{F}_2 coefficients. Moreover, the result in lemma 2.11 applies in the same way to ρ_1^* .*

From now on, since ρ_1 and ρ_2 have the same topological properties considering \mathbb{F}_2 coefficients, we will not make any distinction between them and we will write $G_2/U(2)_\pm$ and ρ_j . As shown in [19], the similarity mentioned in corollary 3.4 does not happens if we consider the cohomology rings with integer coefficients.

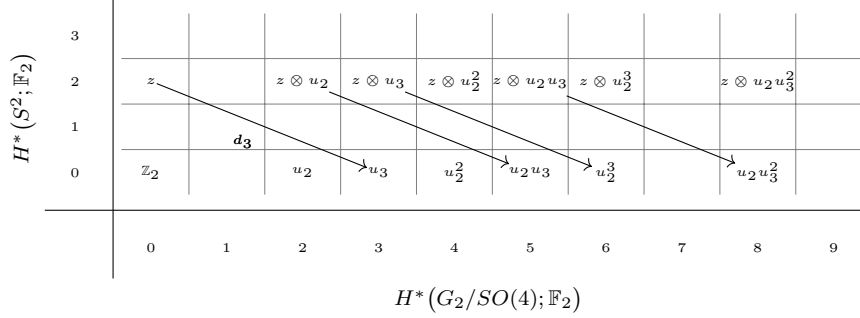


Figure 3: $E_2^{p,q} = H^p(G_2/SO(4); H^q(S^2; \mathbb{F}_2)) \Rightarrow H^*(G_2/U(2)_+; \mathbb{F}_2)$.

Consider the action of \mathbb{Z}_2 on $G_2/U(2)_\pm$ by complex conjugation. Since $\rho_j(W) = \rho_j(\bar{W})$ for every coassociative subspace W , and $j \in \{1, 2\}$, then both maps are \mathbb{Z}_2 -equivariant, with \mathbb{Z}_2 acting trivial on $G_2/SO(4)$. This means that we can ask for the Fadell-Husseini index of such bundles. Before we prove our first main result, we fix the notation for the cohomology of the group \mathbb{Z}_2 as

$$H^*(B\mathbb{Z}_2; \mathbb{F}_2) = \mathbb{F}_2[t],$$

with $\deg(t) = 1$.

Theorem 3.5. *The Fadell-Husseini index of ρ_1 and ρ_2 with respect to the introduced \mathbb{Z}_2 -action is given by*

$$\text{Index}_{\mathbb{Z}_2}^{G_2/SO(4)}(\rho_1; \mathbb{F}_2) = \text{Index}_{\mathbb{Z}_2}^{G_2/SO(4)}(\rho_2; \mathbb{F}_2) = \langle t^3 + u_2 t + u_3 \rangle.$$

Consequently, the Borel cohomology of $G_2/U(2)_\pm$ is given by

$$H^*(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm; \mathbb{F}_2) = H^*(B\mathbb{Z}_2 \times G_2/SO(4); \mathbb{F}_2) / \langle t^3 + u_2 t + u_3 \rangle.$$

Proof. Consider the Borel construction of the bundle ρ_i , for $i \in \{1, 2\}$,

$$\begin{array}{ccc} S^2 \hookrightarrow & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm & \\ & \downarrow \rho_i & \\ & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/SO(4), & \end{array}$$

where the base space can be written as $B\mathbb{Z}_2 \times G_2/SO(4)$. Since the fundamental group

$$\pi_1(B\mathbb{Z}_2 \times G_2/SO(4)) \cong \pi_1(B\mathbb{Z}_2) \times \pi_1(G_2/SO(4)) \cong \pi_1(B\mathbb{Z}_2) \cong \mathbb{Z}_2$$

acts trivially on the cohomology of the fiber, then the E_2 -terms of the associated Serre spectral sequence looks as follows:

$$E_2^{p,q} = H^p(B\mathbb{Z}_2 \times G_2/SO(4); H^q(S^2; \mathbb{F}_2)).$$

Using proposition 3.1 and lemma 3.3, we get that the transgression map $d_3^{0,2}$ is given by

$$d_3^{0,2}(z) = t^3 + u_2t + u_3,$$

where $H^2(S^2; \mathbb{F}_2) = \langle z \rangle$. Finally, by the Leibniz rule, $d_3^{0,2}$ determines the complete cohomology of $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_{\pm}$ and the Fadell-Husseini index of ρ_1 and ρ_2 with coefficients in \mathbb{F}_2 . \square

3.4 The Fadell-Husseini index of the pullback bundle ζ_n

Let us consider the sphere bundle associated the tautological bundle γ over $\tilde{G}_4(\mathbb{R}^7)$

$$s\gamma = (E(s\gamma), \tilde{G}_4(\mathbb{R}^7), E(s\gamma) \xrightarrow{s\pi} \tilde{G}_4(\mathbb{R}^7), S^3),$$

and the map $f = c \circ i$, where i denotes the embedding of $G_2/SO(4)$ on $\tilde{G}_3(\mathbb{R}^7)$ and $c: \tilde{G}_3(\mathbb{R}^7) \rightarrow \tilde{G}_4(\mathbb{R}^7)$ assigns to every linear subspace its orthogonal complement. Then, using the map f and the fiber bundle ρ_j defined in 2.3 and 2.4, we construct the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{S}_\gamma^1 & \longrightarrow & E(f^*(s\gamma)) & \longrightarrow & E(S\gamma) \\ \phi_1 \downarrow & & \downarrow & & \downarrow s\pi \\ G_2/U(2)_{\pm} & \xrightarrow{\rho_j} & G_2/SO(4) & \xrightarrow{f} & \tilde{G}_4(\mathbb{R}^7), \end{array}$$

where the map on the left is the pullback bundle of $s\gamma$ along the map $f \circ \rho_j$. We will denote consider the bundle

$$\zeta_1 = (E(\zeta_1) = \mathcal{S}_\gamma^1, G_2/U(2)_{-}, \mathcal{S}_\gamma^1 \xrightarrow{\phi_1} G_2/U(2)_{-}, S^3).$$

To be more precise, the total space of ζ_1 is given by

$$\mathcal{S}_\gamma^1 = \{(W; \xi^\perp, v) \mid W \in G_2/U(2)_{\pm}, \rho_j(W) = \xi, v \in \xi^\perp \text{ and } \deg(v_1) = 1\}.$$

The construction of the bundle ζ_1 can be easily generalized as follows. Consider the the n -fold product bundle $(s\gamma)^n$, and the pullback along

the diagonal map $\Delta_n : \tilde{G}_4(\mathbb{R}^7) \rightarrow \tilde{G}_4(\mathbb{R}^7)^n$:

$$\begin{array}{ccc} E(\Delta_n^*(s\gamma^n)) & \longrightarrow & E(s\gamma)^n \\ \downarrow & & \downarrow (s\pi)^n \\ \tilde{G}_4(\mathbb{R}^7) & \xrightarrow{\Delta_n} & \tilde{G}_4(\mathbb{R}^7)^n. \end{array}$$

Applying the last construction to the fiber bundle $\Delta_n^*(s\gamma^n)$, we get a new bundle

$$\zeta_n = (E(\zeta_n) = \mathcal{S}_\gamma^n, G_2/U(2)_\pm, \mathcal{S}_\gamma^n \xrightarrow{\phi_n} G_2/U(2)_\pm, (S^3)^n),$$

where

$$\begin{aligned} \mathcal{S}_\gamma^n = \{ & (W; \xi^\perp, v_1, \dots, v_n) \mid W \in G_2/U(2)_\pm, \rho_j(W) = \xi, \\ & v_1, \dots, v_n \in \xi^\perp \text{ and } \deg(v_1) = \dots = \deg(v_n) = 1\}. \end{aligned}$$

To describe the Fadell-Husseini index of \mathcal{S}_γ^n , the first step is to calculate the Stiefel-Whitney classes of ζ_1 . Notice that since

$$H^1(G_2/U(2)_\pm; \mathbb{F}_2) = H^3(G_2/U(2)_\pm; \mathbb{F}_2) = 0,$$

we are just looking for $w_2(\zeta_1)$ and $w_4(\zeta_1)$. Then we have the following result.

Lemma 3.6. *The Stiefel-Whitney classes of the pullback bundle ζ_1 are non-zero only in the dimensions 0, 2 and 4. Moreover, $w_2(\zeta_1) = y$ and $w_4(\zeta_1) = y^2$.*

Proof. Let $p: \tilde{G}_4(\mathbb{R}^7) \rightarrow G_4(\mathbb{R}^7)$ be the map that forgets the orientation of the 4-plane in \mathbb{R}^7 . It is well known that p is a 2-fold covering map, and that the induced homomorphism in cohomology satisfies that

$$p^*(w_i) = w_i(\tilde{\gamma}) = \tilde{w}_i,$$

where \tilde{w}_i are Stiefel-Whitney classes of $\tilde{G}_4(\mathbb{R}^7)$. In this way, $\text{im}(p^*)$ is a subalgebra of $H^*(\tilde{G}_4(\mathbb{R}^7); \mathbb{F}_2)$ generated by $\{\tilde{w}_2, \tilde{w}_3, \tilde{w}_4\}$, except by \tilde{w}_1 which is trivial since $\tilde{G}_4(\mathbb{R}^7)$ is simply connected.

By the naturality of the Stiefel-Whitney classes it is enough to describe the image of \tilde{w}_2 and \tilde{w}_4 along the composition

$$G_2/U(2)_\pm \xrightarrow{\rho_j} G_2/SO(4) \xrightarrow{f} \tilde{G}_4(\mathbb{R}^7).$$

Akbulut and Kalafat in [1, Theorem 2.11] proved that the embedding $f : G_2/SO(4) \rightarrow \tilde{G}_4(\mathbb{R}^7)$ maps the Euler class of γ , which reduces to \tilde{w}_4 , to the integral generator in dimension 4, which reduces to u_2^2 . Notice that, since $\tilde{w}_1 = 0$, there is a unique generator of the mod-2-cohomology in degree 4, and hence $\tilde{w}_4 = \tilde{w}_2^2$. In this way, f^* maps \tilde{w}_2 to u_2 .

On the other hand, by lemma 2.11, $\rho_j^*(u_2) = y$ and $\rho_j^*(u_2^2) = y^2$. This means that

$$w_2(\zeta_1) = (f \circ \rho_j)^*(\tilde{w}_2) = y$$

and

$$w_4(\zeta_1) = (f \circ \rho_j)^*(\tilde{w}_4) = y^2,$$

for $j \in \{1, 2\}$. □

There is an action of \mathbb{Z}_2^n on \mathcal{S}_γ^n defined as follows:

$$(\beta_1, \dots, \beta_n) \cdot (W; \xi^\perp, v_1, \dots, v_n) = (W; \xi^\perp, (-1)^{\beta_1} v_1, \dots, (-1)^{\beta_n} v_n),$$

for $(\beta_1, \dots, \beta_n) \in \mathbb{Z}_2^n$ and $(W; \xi^\perp, v_1, \dots, v_n) \in \mathcal{S}_\gamma^n$. Notice that the action of \mathbb{Z}_2^n on $G_2/U(2)_\pm$ is trivial. Also, with this action, the projection map ϕ_n is \mathbb{Z}_2^n -equivariant and we can ask for its Fadell-Husseini index. We will start then with the case $n = 1$.

For the cohomology of $(S^3)^n$ and $B\mathbb{Z}_2^n$ we fix the following notation:

$$\begin{aligned} H^*((S^3)^n; \mathbb{F}_2) &\cong H^*(S^3; \mathbb{F}_2)^{\otimes n} \\ &\cong \mathbb{F}_2[z_1]/\langle z_1^2 \rangle \otimes \cdots \otimes \mathbb{F}_2[z_n]/\langle z_n^2 \rangle \\ &\cong \mathbb{F}_2[z_1, \dots, z_n]/\langle z_1^2, \dots, z_n^2 \rangle, \end{aligned}$$

where $\deg(z_1) = \cdots = \deg(z_n) = 1$, and

$$H^*(B\mathbb{Z}_2^n; \mathbb{F}_2) = \mathbb{F}_2[t_1, \dots, t_n],$$

where $\deg(t_1) = \cdots = \deg(t_n) = 1$.

Proposition 3.7. *Consider the action of \mathbb{Z}_2 on \mathcal{S}_γ^1 introduced before. Then the Fadell-Husseini index of $\phi_1 : \mathcal{S}_\gamma^1 \rightarrow G_2/U(2)_\pm$ is given by*

$$\text{Index}_{\mathbb{Z}_2}^{G_2/U(2)_\pm}(\phi_1; \mathbb{F}_2) = \langle y^2 + yt_1^2 + t_1^4 \rangle.$$

Consequently, the Borel cohomology of \mathcal{S}_γ^1 is described as follows:

$$H^*(EZ_2 \times_{\mathbb{Z}_2} \mathcal{S}_\gamma^1; \mathbb{F}_2) = H^*(EZ_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm; \mathbb{F}_2) / \langle y^2 + yt_1^2 + t_1^4 \rangle.$$

Proof. Consider the Serre spectral sequence associated to the bundle

$$S^3 \hookrightarrow E\mathbb{Z}_2 \times_{\mathbb{Z}_2} \mathcal{S}_\gamma^1 \xrightarrow{id \times_{\mathbb{Z}_2} \phi_1} E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm,$$

with E_2 -term given by

$$\begin{aligned} E_2^{p,q} &\cong H^p(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} G_2/U(2)_\pm; \mathcal{H}^q(S^3; \mathbb{F}_2)) \\ &\cong H^p(B\mathbb{Z}_2 \times G_2/U(2)_\pm; \mathcal{H}^q(S^3; \mathbb{F}_2)). \end{aligned}$$

Since the fundamental group $\pi_1(B\mathbb{Z}_2 \times G_2/U(2)_\pm) \cong \mathbb{Z}_2$ acts trivially on the cohomology of the fiber, then the E_2 -term simplifies and it looks as follows:

$$E_2^{p,q} = H^p(B\mathbb{Z}_2 \times G_2/U(2)_\pm; \mathbb{F}_2) \otimes H^q(S^3; \mathbb{F}_2).$$

Notice that the first differential of the spectral sequence associated to $id \times_{\mathbb{Z}_2} \phi_1$ appears on the E_4 -term. Then, by proposition 3.1 and lemma 3.6, it is given by

$$d_4^{0,3}(\xi) = y^2 + yt_1^2 + t_1^4.$$

For an illustration of the E_4 -term see Figure 4. This describes the Fadell-Husseini index of $\phi_1: \mathcal{S}_\gamma^1 \rightarrow G_2/U(2)_\pm$, and by the Leibniz rule, also the cohomology of $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} \mathcal{S}_\gamma^1$. \square

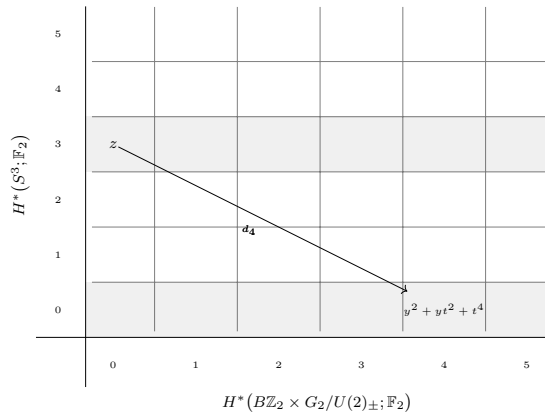


Figure 4: $E_4^{p,q} = E_2^{p,q} = H^p(B\mathbb{Z}_2 \times G_2/U(2)_\pm; H^q(S^3; \mathbb{F}_2))$

Using the projection of \mathcal{S}_γ^n on \mathcal{S}_γ^1 , and proposition 3.7, we can prove now our second main result.

Theorem 3.8. *Consider the action of \mathbb{Z}_2^n on \mathcal{S}_γ^n introduced before. Then the Fadell-Husseini index of $\phi_n: \mathcal{S}_\gamma^n \rightarrow G_2/U(2)_\pm$ is given by*

$$\text{Index}_{\mathbb{Z}_2^n}^{G_2/U(2)_\pm}(\phi_n; \mathbb{F}_2) = \langle y^2 + yt_1^2 + t_1^4, \dots, y^2 + yt_n^2 + t_n^4 \rangle.$$

Proof. Let $i_k: \mathbb{Z}_2 \rightarrow \mathbb{Z}_2^n$ be the inclusion into the k th summand, with $1 \leq k \leq n$. The map i_k induces a morphism between Borel constructions

$$\begin{array}{ccc} EZ_2^n \times_{\mathbb{Z}_2^n} \mathcal{S}_\gamma^1 & \longleftarrow & EZ_2 \times_{\mathbb{Z}_2} \mathcal{S}_\gamma^1 \\ \text{id} \times_{\mathbb{Z}_2^n} \phi_1 \downarrow & & \downarrow \text{id} \times_{\mathbb{Z}_2} \phi_1 \\ BZ_2^n \times G_2/U(2)_\pm & \longleftarrow & BZ_2 \times G_2/U(2)_\pm \end{array}$$

where \mathbb{Z}_2^n acts on \mathcal{S}_γ^1 by letting the k -th summand acts antipodally on the unitary element. We will refer to this action as the one induced by i_k . This morphism of Borel constructions induces a morphism between the corresponding Serre spectral sequences, which on the zero column of the E_2 -term is an isomorphism. Then, by proposition 3.7 and the commutativity of the differentials with the morphism between spectral sequences, the transgression map $d_4^{0,3}$ of the spectral sequence associated to $\text{id} \times_{\mathbb{Z}_2^n} \phi_1$ is given by

$$d_4^{0,3}(z) = y^2 + yt_k^2 + t_k^4,$$

for every $1 \leq k \leq n$.

Consider now the projection $p_k: \mathcal{S}_\gamma^n \rightarrow \mathcal{S}_\gamma^1$ given by

$$p_k(W; \xi^\perp, v_1, \dots, v_n) = (W; \xi^\perp, v_k),$$

for $1 \leq k \leq n$. With respect to the \mathbb{Z}_2^n -action on \mathcal{S}_γ^1 induced by i_k , the map p_k induces a morphism of Borel constructions

$$\begin{array}{ccc} EZ_2^n \times_{\mathbb{Z}_2^n} \mathcal{S}_\gamma^n & \longrightarrow & EZ_2^n \times_{\mathbb{Z}_2^n} \mathcal{S}_\gamma^1 \\ \text{id} \times_{\mathbb{Z}_2^n} \phi_n \downarrow & & \downarrow \text{id} \times_{\mathbb{Z}_2^n} \phi_1 \\ BZ_2^n \times G_2/U(2)_\pm & \longrightarrow & BZ_2^n \times G_2/U(2)_\pm. \end{array}$$

This morphism of Borel constructions, in fact, induces a morphism between the corresponding Serre spectral sequences, which on the zero column of the E_2 -term is a monomorphism.

For an illustration of the morphism between the corresponding Serre spectral sequences see Figure 5. Then, by the commutativity of the differentials with respect to the spectral sequence, the map $d_4^{0,3}$ of the spectral sequences associated to $id \times_{\mathbb{Z}_2^n} \phi_n$ is given by

$$d_4^{0,3}(z_k) = d_4^{0,3}(p_k^*(z)) = id(d_4^{0,3}(z)) = y^2 + yt_k^2 + t_k^4,$$

for every generator z_k in $H^3((S^3)^n; \mathbb{F}_2)$, with $1 \leq k \leq n$. Then, by the Leibnitz rule, this describes the Fadell-Husseini index of $\phi_n: \mathcal{S}_{\gamma^\perp}^n \rightarrow G_2/U(2)_\pm$. \square

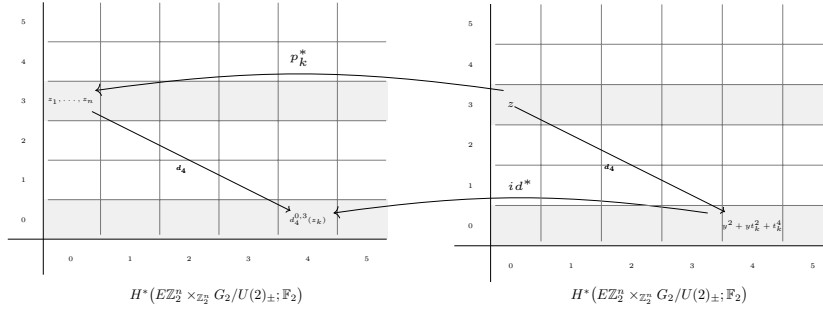


Figure 5: Morphism of spectral sequences induced by p_k .

4 Application: associative subspaces in discrete geometry.

Once we studied the associative and coassociative subspaces of $\text{Im}(\mathbb{O})$, as well as the Fadell-Husseini index of the fibrations defined in the previous sections, in this section we will show how this kind of subspaces appears within a classical problem in discrete geometry known as *the Grünbaum mass partition problem* [17].

From now on, a mass is assumed to be a finite Borel measure on a Euclidean space which vanishes on each affine hyperplane. The general problem considers a mass μ in \mathbb{R}^n , and looks for a collection of oriented affine hyperplanes H_1, H_2, \dots, H_n in \mathbb{R}^n , also called n -arrangement, such

that each of the 2^n resulting orthants contains the same amount of the mass μ , i.e.,

$$\mu(H_1^{\sigma_1} \cap \cdots \cap H_n^{\sigma_n}) = \frac{1}{2^n} \mu(\mathbb{R}^n)$$

for every $(\sigma_1, \dots, \sigma_n) \in \mathbb{Z}_2^n$, where $H_i^{\sigma_i}$ denotes the appropriate closed halfspace determined by H_i . In that case, we say that the collection of hyperplanes equiparts the mass μ .

While the case $n = 2$ is the well-known ham sandwich theorem, the positive answer for the case $n = 3$ was given by Hadwiger in [18]. On the other hand, in [2], Avis showed that in every dimension $n \geq 5$ there is a mass which cannot be equiparted by an n -arrangement. The problem becomes more complicated in the case $n = 4$, which, to this very day, is still open. The best approach to the solution was given by Dimitrijević Blagojević in [12], where she considers an almost equipartition. To be more precise, given a 4-arrangement $\{H_1, H_2, H_3, H_4\}$ in \mathbb{R}^4 , and consequently sixteen orthants $\mathbf{O}_{\sigma_1\sigma_2\sigma_3\sigma_4} = H_1^{\sigma_1} \cap H_2^{\sigma_2} \cap H_3^{\sigma_3} \cap H_4^{\sigma_4}$, an almost equipartition means that

$$\begin{aligned} \mu(\mathbf{O}_{0000}) &= \mu(\mathbf{O}_{0010}) = \mu(\mathbf{O}_{0101}) = \mu(\mathbf{O}_{0111}) \\ &= \mu(\mathbf{O}_{1000}) = \mu(\mathbf{O}_{1010}) = \mu(\mathbf{O}_{1101}) = \mu(\mathbf{O}_{1111}) \end{aligned}$$

$$\begin{aligned} \mu(\mathbf{O}_{0001}) &= \mu(\mathbf{O}_{0011}) = \mu(\mathbf{O}_{0100}) = \mu(\mathbf{O}_{0110}) \\ &= \mu(\mathbf{O}_{1001}) = \mu(\mathbf{O}_{1011}) = \mu(\mathbf{O}_{1100}) = \mu(\mathbf{O}_{1110}). \end{aligned}$$

For more details on the history and discussion of solution methods regarding Grünbaum's mass partition problem consult [9], and for the currently best known results see [8].

In this section, motivated by the computations presented by Blagojević, Calles Loperena, Crabb, and Dimitrijević Blagojević in [7], we will show that the solutions provided by Dimitrijević Blagojević in [12] can be considered, in a sense, as the orthogonal complement of an associative subspace in $\text{Im}(\mathbb{O})$. More precisely, we prove the following result

Theorem 4.1. *Let μ be a mass assignment on the Grassmann manifold $G_4(\mathbb{R}^7)$. Then there exist a vector subspace $L \in G_4(\mathbb{R}^7)$, which is the orthogonal complement of an associative subspace in $\text{Im}(\mathbb{O})$, and a 4-arrangement (H_1, H_2, H_3, H_4) in L which almost equipart the mass μ^L .*

Here we define a mass assignment on a Grassmann manifold as a cross section of the fibre bundle

$$(2) \quad M'_+(\mathbb{R}^\ell) \longrightarrow \mathcal{M}'_+(\ell, d) \xrightarrow{\rho} G_\ell(\mathbb{R}^d),$$

where $M'_+(\mathbb{R}^\ell)$ denotes the subspace of all masses on \mathbb{R}^ℓ , the total space is given by

$$\mathcal{M}'_+(\ell, d) := \{(L, \nu) \mid L \in G_\ell(\mathbb{R}^d) \text{ and } \nu \in M'_+(L)\},$$

and the map ρ is give by $(L, \nu) \mapsto L$. This means that μ assigns to each subspace L a mass μ^L in L . For more details about mass assignments see [22], [3] and [7].

4.1 Proof (theorem 4.1)

The proof is based on the *configuration space test map scheme (CS/TM - scheme)*, developed in numerous paper and formalized by Živaljević in [26, 27]. The main idea is to reduce the problem to the question about the non-existence of a particular equivariant map. We will consider a map $f: \mathcal{C} \rightarrow Z$, where \mathcal{C} is the space of all candidates to be a solution of our problem, also called *the configuration space*, Z is called *the test space*, f is called *the test map*, and there is a subspace $V \subseteq Z$ such that $x \in \mathcal{C}$ is a solution of the problem if and only if $f(x) \in V$. In addition there is a group G , of symmetries of the problem, which acts on \mathcal{C} and Z , keeping the subspace V G -invariant. Moreover, the test map f is equivariant. In this way, if we prove that $f: \mathcal{C} \rightarrow Z \setminus V$ does not exists, we guarantee that there is a solution of our problem.

In order to apply topological methods and derive an appropriate configuration test map scheme for our problem we make an additional assumption on the mass assignment. We assume that for every linear subspace $L \in G_4(\mathbb{R}^7)$ the associated mass μ^L has compact and connected support. This means that for every direction in L there exists a unique oriented affine hyperplane orthogonal to the direction equiparting μ^L .

4.1.1 Configuration space

In order to construct the equivariant map of the test map scheme, let us start considering a coassociative subspace $W \in G_2/U(2)_\pm$ such that $\rho_i(W) = \xi$, with $\xi \in G_2/SO(4)$ an associative subspace of $\text{Im}(\mathbb{O})$. Since

W corresponds to an orthogonal almost complex structure J of ξ^\perp , given two unit vectors $v_1, v_2 \in \xi^\perp$, we obtain a collection $\{v_1, J(v_1), v_2, J(v_2)\}$, which by the extra assumption over the mass assignment μ , induces a 4-arrangement of oriented affine hyperplanes

$$\{H_{v_1}, H_{J(v_1)}, H_{v_2}, H_{J(v_2)}\}$$

equiparting the mass μ^{ξ^\perp} in ξ^\perp . In that way, \mathcal{S}_γ^2 , the total space of the pullback bundle ϕ_2 defined in section 3.4, represent the space of all candidates to be a solutions of our problem.

4.1.2 Test map

Consider now the bundle map $f: \mathcal{S}_\gamma^2 \rightarrow G_2/U(2)_\pm \times \mathbb{R}^{16}$, between ϕ_2 and the projection into the first factor π , given by

$$(W; \xi^\perp, v_1, v_2) \mapsto \left(W; (\mu^{\xi^\perp} (H_{v_1}^{\sigma_1} \cap H_{J(v_1)}^{\sigma_2}) \cap H_{v_2}^{\sigma_3} \cap H_{J(v_2)}^{\sigma_4}) - \frac{1}{16} \mu^{\xi^\perp} (\xi^\perp) \right)_{(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{Z}_2^4},$$

where \mathbb{R}^{16} is indexed by the elements of the group \mathbb{Z}_2^4 . Notice that, by the extra assumption over the mass assignment μ , the codomain of the map f is a subspace \mathcal{U} of \mathbb{R}^{16} defined by the following equalities:

$$\begin{aligned} \sum x_{0\sigma_1\sigma_2\sigma_3} &= \sum x_{1\sigma_1\sigma_2\sigma_3}; & \sum x_{\sigma_1 0\sigma_2\sigma_3} &= \sum x_{\sigma_1 1\sigma_2\sigma_3} \\ \sum x_{\sigma_1\sigma_2 0\sigma_3} &= \sum x_{\sigma_1\sigma_2 1\sigma_3}; & \sum x_{\sigma_1\sigma_2\sigma_3 0} &= \sum x_{\sigma_1\sigma_2\sigma_3 1} \\ \sum x_{\sigma_1\sigma_2\sigma_3\sigma_4} &= 0, \end{aligned}$$

for $(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{Z}_2^3$ and $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{Z}_2^4$.

As shown in section 3.4, the group \mathbb{Z}_2^2 acts antipodally on the unit vectors of \mathcal{S}_γ^2 . There is also an action of \mathbb{Z}_2^2 on \mathbb{R}^{16} given by permuting the elements in the following way:

$$(\beta_1, \beta_2) \cdot x_{\sigma_1\sigma_2\sigma_3\sigma_4} = x_{\beta_1+\sigma_1 \beta_1+\sigma_2 \beta_2+\sigma_3 \beta_2+\sigma_4},$$

where $(\beta_1, \beta_2) \in \mathbb{Z}_2^2$ and $x_{\sigma_1\sigma_2\sigma_3\sigma_4} \in \mathbb{R}^{16}$. All this makes the bundle map f \mathbb{Z}_2^2 -equivariant over the codomain $G_2/U(2)_\pm \times \mathcal{U}$.

4.1.3 Test space

Let us consider the \mathbb{Z}_2^2 -invariant subspace L of \mathbb{R}^{16} generated by the equalities

$$\begin{aligned} x_{0000} &= x_{0010} = x_{0100} = x_{0110} = x_{1001} = x_{1011} = x_{1101} = x_{1111}, \\ x_{0001} &= x_{0011} = x_{0101} = x_{0111} = x_{1000} = x_{1010} = x_{1100} = x_{1110}. \end{aligned}$$

Notice that the previous equations determine the condition of ‘‘almost equipartition’’ of our 4-arrangement. In this way, the test space of our problem is the subspace $\mathcal{U} \cap L$ of codimension 10 inside \mathcal{U} .

All of this leads us to the following result, which relates our mass partition problem with a problem about the existence of an equivariant map (the main idea of the CS/TM scheme).

Proposition 4.2. *If there is no \mathbb{Z}_2^2 -equivariant map*

$$\mathcal{S}_\gamma^2 \rightarrow G_2/U(2)_\pm \times (\mathbb{U} \setminus L),$$

with the already defined actions, then our problem stated in Theorem 4.1 has a solution.

To make the upcoming computations more manageable, we consider a \mathbb{Z}_2^2 -deformation retraction between $\mathbb{U} \setminus L$ and $S((\mathbb{U} \cap L)^\perp)$, where $S((\mathbb{U} \cap L)^\perp)$ is a sphere of dimension 9.

4.1.4 The non-existence of the equivariant map.

Because of the previously mentioned reasons, the problem reduces to the non-existence of the bundle map

$$f: \mathcal{S}_\gamma^2 \rightarrow G_2/U(2)_\pm \times S((\mathbb{U} \cap L)^\perp)$$

between ϕ_2 and the corresponding sphere bundle $S\pi$. If we suppose that such map exists, by the monotonicity of the index

$$\text{Index}_{\mathbb{Z}_2^n}^{G_2/U(2)_\pm}(\phi_2; \mathbb{F}_2) \supseteq \text{Index}_{\mathbb{Z}_2^n}^{G_2/U(2)_\pm}(S\pi; \mathbb{F}_2),$$

which by theorem 3.8, and the computations made in [12, Section 3.2], as well as [15, Proposition 3.1], simplifies to

$$\langle y^2 + yt_1^2 + t_1^4, y^2 + yt_2^2 + t_2^4 \rangle \supseteq H^*(G_2/U(2)_\pm; \mathbb{F}_2) \otimes \langle t_1^5 t_2^2 + t_1^4 t_2^3 + t_1^3 t_2^4 + t_1^2 t_2^5 \rangle.$$

For a more details about the index of a projection see [7, Section 3.2]. Finally, using a computational package like *SINGULAR* or *SAGE* we confirm that

$$t_1^5 t_2^2 + t_1^4 t_2^3 + t_1^3 t_2^4 + t_1^2 t_2^5 \notin \langle y^2 + yt_1^2 + t_1^4, y^2 + yt_2^2 + t_2^4 \rangle,$$

which means that the \mathbb{Z}_2^2 -equivariant bundle map f does not exist. As a consequence, there exist an associative subspace ξ , and a 4-arrangement (H_1, H_2, H_3, H_4) in ξ^\perp which almost equipart the mass μ^{ξ^\perp} in ξ^\perp .

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